RANKIN-COHEN BRACKETS ON JACOBI FORMS AND THE ADJOINT OF SOME LINEAR MAPS

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ABSTRACT. Given a fixed Jacobi cusp form, we consider a family of linear maps between the spaces of Jacobi cusp forms by using the Rankin-Cohen brackets and then we compute the adjoint maps of these linear maps with respect to the Petersson scalar product. The Fourier coefficients of the Jacobi cusp forms constructed using this method involve special values of certain Dirichlet series associated to Jacobi cusp forms. This is a generalization of the work due to W. Kohnen (Math. Z., 207, 657-660 [1991]) and S. D. Herrero (Ramanujan J., [2014]) in case of elliptic modular forms to the case of Jacobi cusp forms which is also considered earlier by H. Sakata (Proc. Japan Acad. Ser. A, Math. Sc. 74 [1998]) for a special case.

1. INTRODUCTION

Using the existence of adjoint linear maps and properties of Poincaŕe series in [9] W. Kohnen constructed certain linear maps between spaces of modular forms with the property that the Fourier coefficients of image of a form involve special values of certain Dirichlet series attached to these forms. In fact, Kohnen constructed the adjoint map with respect to the usual Petersson scalar product of the product map by a fixed cusp form. This result has been generalized by several authors to other automorphic forms (see the list [10, 11, 15]). In particular, Choie, Kim and Knopp [4] and Sakata [14] have analogous results for Jacobi forms.

There are many interesting connections between differential operators and modular forms and many interesting results have been found. In [12, 13], Rankin gave a general description of the differential operators which send modular forms to modular forms. In [5], H. Cohen constructed bilinear operators and obtained elliptic modular forms with interesting Fourier coefficients. In [16, 17], Zagier studied the algebraic properties of these bilinear operators and called them Rankin–Cohen brackets.

Recently the work of Kohnen in [9] has been generalized by S. D. Herrero in [8], where the author constructed the adjoint map using the Rankin-Cohen brackets by a fixed cusp form instead of product map. Rankin–Cohen brackets for Jacobi forms were studied by Choie

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[1, 2] by using the heat operator. We generalize the work of S. D. Herrero to the case of Jacobi forms. We follow the same exposition as given in [14].

2. Preliminaries on Jacobi forms over $\mathcal{H} \times \mathbb{C}$

Let $\mathcal{H}$ and $\mathbb{C}$ be the complex upper half-plane and the complex plane, respectively. The Jacobi group $\Gamma^J = SL_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$ acts on $\mathcal{H} \times \mathbb{C}$ in the usual way by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right).(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

Let k, m be fixed positive integers. If $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in \Gamma^J$ and ϕ be a complex valued function on $\mathcal{H} \times \mathbb{C}$, then define

$$\phi|_{k,m}\gamma := (c\tau + d)^{-k} e^{2\pi i m(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z)} \phi(\gamma.(\tau, z)).$$

Let $J_{k,m}$ be the space of Jacobi forms of weight k and index m on Γ^J, i.e., the space of holomorphic functions $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying $\phi|_{k,m}\gamma = \phi, \ \forall \gamma \in \Gamma^J$ and having a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, \\ 4nm - r^2 \geq 0}} c(n, r) q^n \zeta^r \qquad (q = e^{2\pi i \tau}, \zeta = e^{2\pi i z}).$$

Further, we say ϕ is a cusp form if and only if $c(n, r) \neq 0 \implies n > r^2/4m$. We denote the space of all Jacobi cusp forms by $J_{k,m}^{cusp}$. We define the Petersson scalar product on $J_{k,m}^{cusp}$

$$\langle \phi, \psi \rangle = \int_{\Gamma^J \backslash \mathbb{H} \times \mathbb{C}} \phi(\tau, z) \overline{\psi(\tau, z)} v^k e^{\frac{-4\pi m y^2}{v}} dV_J,$$

where $\tau = u + iv, z = x + iy$ and $dV_J = \dfrac{du dv dx dy}{v^3}$ is an invariant measure under the action on Γ^J on $\mathcal{H} \times \mathbb{C}$. The space $(J_{k,m}^{cusp}, \langle, \rangle)$ is a finite dimensional Hilbert space. For more details on the theory of Jacobi forms, we refer to [6]. The following lemma tells about the growth of the Fourier coefficients of a Jacobi form.

Lemma 2.1. *If $\phi \in J_{k,m}$ and $k > 3$ with Fourier coefficients $c(n, r)$, then*

$$c(n, r) << |r^2 - 4nm|^{k - \frac{3}{2}},$$

and moreover, if ϕ is a Jacobi cusp form, then

$$c(n, r) << |r^2 - 4nm|^{\frac{k}{2} - \frac{1}{2}}.$$

For a proof, we refer to [3].

2.1. **Poincaŕe series.** We define the Jacobi Poincaŕe series.

Definition 2.2. *Let m, n and r be fixed integers with $r^2 < 4mn$.*

$$P_{k,m;(n,r)}(\tau, z) := \sum_{\gamma \in \Gamma^J_\infty \backslash \Gamma^J} e^{2\pi i(n\tau + rz)}|_{k,m}\gamma \tag{1}$$

be the (n, r)-th Poincaŕe series of weight k and index m. Here $\Gamma^J_\infty := \left\{ \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) | t, \mu \in \mathbb{Z} \right\}$ is the stabilizer of $q^n \zeta^r$ in Γ^J. It is well known that $P_{k,m;(n,r)} \in J^{cusp}_{k,m}$ for $k > 2$ (see [7]).

This series has the following property.

Lemma 2.3. *Let $\phi \in J^{cusp}_{k,m}$ with Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, \\ 4nm - r^2 > 0}} c(n, r) q^n \zeta^r.$$

Then

$$\langle \phi, P_{k,m;(n,r)} \rangle = \alpha_{k,m} (4mn - r^2)^{-k+\frac{3}{2}} c(n, r), \tag{2}$$

where

$$\alpha_{k,m} = \frac{m^{k-2}\Gamma(k - \frac{3}{2})}{2\pi^{k-\frac{3}{2}}}.$$

One can get explicit Fourier expansion of $P_{k,m;(n,r)}$, for details we refer to [7].

2.2. **Rankin-Cohen Brackets for Jacobi forms.** For an integer m, we define the heat operator

$$L_m := \frac{1}{(2\pi i)^2} \left(8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} \right).$$

Let k_1, k_2, m_1 and m_2 be positive integers and $\nu \geq 0$ be an integer. Let ϕ and ψ be two complex valued holomorphic functions on $\mathcal{H} \times \mathbb{C}$. Define the ν-th Rankin-Cohen bracket of ϕ and ψ by

$$[\phi, \psi]_\nu := \sum_{l=0}^{\nu} (-1)^l \binom{k_1 + \nu - \frac{3}{2}}{\nu - l} \binom{k_2 + \nu - \frac{3}{2}}{l} m_1^{\nu-l} m_2^l L^l_{m_1}(\phi) L^{\nu-l}_{m_2}(\psi). \tag{3}$$

We note that here $x! = \Gamma(x + 1)$. Using the action of heat operator (Lemma 3.1 in [1]), one can easily verify that

$$[\phi|_{k_1,m_1}\gamma, \psi|_{k_2,m_2}\gamma]_\nu = [\phi, \psi]|_{k_1+k_2+2\nu, m_1+m_2}\gamma, \ \forall \gamma \in \Gamma^J. \tag{4}$$

Theorem 2.4. [1] *Let $\nu \geq 0$ be an integer and $\phi \in J_{k_1,m_1}$ and $\psi \in J_{k_2,m_2}$ where k_1, k_2, m_1 and m_2 are positive integers. Then $[\phi, \psi]_\nu \in J_{k_1+k_2+2\nu, m_1+m_2}$. If $\nu > 0$, then $[\phi, \psi]_\nu \in J^{cusp}_{k_1+k_2+2\nu, m_1+m_2}$. If $\phi \in J^{cusp}_{k_1,m_1}$ and $\psi \in J^{cusp}_{k_2,m_2}$, then $[\phi, \psi]_\nu \in J^{cusp}_{k_1+k_2+2\nu, m_1+m_2}$ for any ν. In fact, $[\ ,\ ]_\nu$ is a bilinear map from $J_{k_1,m_1} \times J_{k_2,m_2}$ to $J_{k_1+k_2+2\nu, m_1+m_2}$.*

For a proof, we refer to [1].

Remark 2.1. We note that the 0-th Rankin-Cohen bracket is the usual product of Jacobi forms i.e., $[\phi, \psi]_0 = \phi\psi$.

3. Statement of the Theorem

For a fixed $\psi \in J^{cusp}_{k_2,m_2}$ and an integer $\nu \geq 0$, we define the map

$$T_{\psi,\nu} : J^{cusp}_{k_1,m_1} \to J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$$

defined by $T_{\psi,\nu}(\phi) = [\phi,\psi]_\nu$. $T_{\psi,\nu}$ is a $\mathbb{C}$-linear map of finite dimensional Hilbert spaces and therefore has an adjoint map $T^*_{\psi,\nu} : J^{cusp}_{k_1+k_2+2\nu,m_1+m_2} \to J^{cusp}_{k_1,m_1}$ such that

$$\langle \phi, T_{\psi,\nu}(\omega)\rangle = \langle T^*_{\psi,\nu}(\phi),\omega\rangle \ \forall \phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2} \text{ and } \omega \in J^{cusp}_{k_1,m_1}.$$

In the main result we exhibit the Fourier coefficients of $T^*_{\psi,\nu}(\phi)$ for $\phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$. These involve special values of certain Dirichlet series associated to ϕ and ψ. Now we shall state the main theorem of this paper.

Theorem 3.1. *Let $k_1 > 4, k_2 > 3, m_1$ and m_2 be natural numbers. Let $\psi \in J^{cusp}_{k_2,m_2}$ with Fourier expansion*

$$\psi(\tau,z) = \sum_{\substack{n_1,r_1\in\mathbb{Z},\\ 4m_2n_1-r_1^2>0}} a(n_1,r_1)q^{n_1}\zeta^{r_1}.$$

Then the image of any cusp form $\phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$ with Fourier expansion

$$\phi(\tau,z) = \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}} b(n_2,r_2)q^{n_2}\zeta^{r_2}$$

under $T^*_{\psi,\nu}$ is given by

$$T^*_{\psi,\nu}(\phi)(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z},\\ 4m_1n-r^2>0}} c_\nu(n,r)q^n\zeta^r,$$

where

$$c_\nu(n,r) = \frac{(4m_1n-r^2)^{k_1-\frac{3}{2}}}{\pi^{k_2+2\nu}}\frac{(m_1+m_2)^{k_1+k_2+2\nu-2}}{m_1^{k_1-2}}\frac{\Gamma(k_1+k_2+2\nu-\frac{3}{2})}{\Gamma(k_1-\frac{3}{2})}$$

$$\times\sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu)(4m_1n-r^2)^l \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0\\ 4(m_1+m_2)(n+n_1)-(r+r_1)^2>0}} \frac{(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{(4(n+n_1)(m_1+m_2)-(r+r_1)^2)^{k_1+k_2+2\nu-\frac{3}{2}}},$$

and

$$A_l(k_1,m_1,k_2,m_2;\nu) \ = \ (-1)^l\binom{k_1+\nu-\frac{3}{2}}{\nu-l}\binom{k_2+\nu-\frac{3}{2}}{l}m_1^{\nu-l}m_2^l.$$

Remark 3.1. Using the Lemma 2.1 (as given in the remark 3.1 in [14]) one can show that the inner sum of the series converges for $k_1 > 4$ and $k_2 > 3$.

Remark 3.2. Fixing $\psi \in J^{cusp}_{k_2,m_2}$ and suppose that $J^{cusp}_{k_1,m_1}$ is one dimensional space generated by $f(\tau,z)$, then applying the above theorem we get $T^*_{\psi,\nu}(\phi)(\tau,z) = \alpha_\phi f(\tau,z)$ for some constant α_ϕ and for all $\phi \in J^{cusp}_{k_1+k_2+2\nu,m_1+m_2}$. Now equating the (n,r)-th Fourier coefficients both the sides, we get relation among the special values of Rankin-Selberg type convolution of the Jacobi forms ϕ and ψ with the Fourier coefficients of $f(\tau,z)$.

For example, taking $\psi = \phi_{10,1} = \frac{1}{144}(E_6E_{4,1} - E_4E_{6,1}) \in J_{10,1}^{cusp}$ and $k_1 = 12, m_1 = 1$ ($J_{12,1}^{cusp}$ is one dimensional space generated by $\phi_{12,1} := \frac{1}{144}(E_4^2E_{4,1} - E_6E_{6,1})$), we get the following relation:

$$\sum_{l=0}^{\nu} A_l(12,1,10,1;\nu)(4n-r^2)^l \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4n_1-r_1^2>0\\ 8(n+n_1)-(r+r_1)^2>0}} \frac{(4n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{(8(n+n_1)-(r+r_1)^2)^{22+2\nu-\frac{3}{2}}} = \alpha_\phi c(n,r)$$

for all $n, r \in \mathbb{Z}$ such that $4n-r^2 > 0$, where $a(p,q), b(p,q)$ and $c(p,q)$ are the (p,q)-th Fourier coefficients of $\phi_{10,1}, \phi$ and $\phi_{12,1}$ respectively.

Remark 3.3. In particular taking $\nu = 0$ in the above example, we get the special value of Rankin-Selberg type convolution of $\phi_{10,1}$ and ϕ in terms of Fourier coefficients of $\phi_{12,1}$, i.e.,

$$\sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4n_1-r_1^2>0\\ 8(n+n_1)-(r+r_1)^2>0}} \frac{\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{(8(n+n_1)-(r+r_1)^2)^{\frac{41}{2}}} = \alpha_\phi c(n,r).$$

4. Proof of Theorem 3.1

We need the following lemma to proof the main theorem.

Lemma 4.1. *Using the same notation in Theorem 3.1, we have*

$$\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J} \int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \mid \phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)}\mid_{k_1,m_1}\gamma,\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\frac{-4\pi(m_1+m_2)y^2}{v}} \mid dV_J$$

converges.

Proof. Changing the variable (τ,z) to $\gamma^{-1}.(\tau,z)$ and using (4), the sum equals to

$$\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J} \int_{\gamma.\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \mid \phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)},\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\frac{-4\pi(m_1+m_2)y^2}{v}} \mid dV_J$$

and using Rankin unfolding argument, the last sum equals to

$$\int_{\Gamma_\infty^J\backslash\mathcal{H}\times\mathbb{C}} \mid \phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)},\psi]_\nu} v^{k_1+k_2+2\nu} e^{\frac{-4\pi(m_1+m_2)y^2}{v}} \mid dV_J.$$

Now replacing ϕ and ψ with their Fourier expansions and using the definition of Rankin-Cohen brackets, the last integral is majorized by

$$\sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu)\mathcal{I}_l(k_1,k_2,m_1,m_2,\nu;n,r),$$

where

$$\mathcal{I}_l(k_1,k_2,m_1,m_2,\nu;n,r) = \int_{\Gamma_\infty^J\backslash\mathbb{H}\times\mathbb{C}} \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}} \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0}} \mid (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}$$

$$\times\ a(n_1,r_1)b(n_2,r_2)e^{2\pi i((n+n_1+n_2)\tau+(r+r_1+r_2)z)}\mid v^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)y^2}{v}}dV_J.$$

Now it suffices to show that the integral $\mathcal{I}_l(k_1,k_2,m_1,m_2,\nu;n,r)$ is finite for each l. We choose a fundamental domain for the action of Γ_∞^J on $\mathbb{H}\times\mathbb{C}$ which is given by $([0,1]\times[0,\infty])\times([0,1]\times\mathbb{R})$ and integrating over it, we have

$$\mathcal{I}_l(k_1,k_2,m_1,m_2,\nu;n,r)\leq\frac{4(m_1+m_2)^{k_1+k_2+2\nu-2}\Gamma(k_1+k_2+2\nu-3/2)}{\pi^{k_1+k_2+2\nu-3/2}}$$

$$\times\sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}}\ \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0}}\frac{(4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}|a(n_1,r_1)b(n_2,r_2)|}{(8(m_1+m_2)(n+n_1+n_2)-(r+r_1+r_2)^2)^{k_1+k_2+2\nu-3/2}}.$$

Using the growth of Fourier coefficients given in Lemma 2.1, the above series converges absolutely. □

Now we give a proof of Theorem 3.1. Put

$$T^*_{\psi,\nu}(\phi)(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z},\\ 4m_1n-r^2>0}}c_\nu(n,r)q^n\zeta^r.$$

Now, we consider the (n,r)-th Poincaŕe series of weight k_1 and index m_1 as given in (1). Then using the Lemma 2.3, we have

$$\langle T^*_{\psi,\nu}\phi,P_{k_1,m_1;(n,r)}\rangle=\alpha_{k_1,m_1}(4m_1n-r^2)^{\frac{3}{2}-k_1}c_\nu(n,r),$$

where

$$\alpha_{k_1,m_1}=\frac{m_1^{k_1-2}\Gamma(k_1-\frac{3}{2})}{2\pi^{k_1-\frac{3}{2}}}.$$

On the other hand, by definition of the adjoint map we have

$$\langle T^*_{\psi,\nu}\phi,P_{k_1,m_1;(n,r)}\rangle=\langle\phi,T_{\psi,\nu}(P_{k_1,m_1;(n,r)})\rangle=\langle\phi,[P_{k_1,m_1;(n,r)},\psi]_\nu\rangle.$$

Hence we get

$$c_\nu(n,r)=\frac{(4m_1n-r^2)^{k_1-\frac{3}{2}}}{\alpha_{k_1,m_1}}\langle\phi,[P_{k_1,m_1;(n,r)},\psi]_\nu\rangle. \tag{5}$$

By definition,

$$\begin{aligned}\langle\phi,[P_{k_1,m_1;(n,r)},\psi]_\nu\rangle &= \int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}}\phi(\tau,z)\overline{\left[P_{k_1,m_1;(n,r)},\psi\right]_\nu}v^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)y^2}{v}}dV_J\\ &= \int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}}\phi(\tau,z)\overline{[\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma_1^J}e^{2\pi i(n\tau+rz)}\ |_{k_1,m_1}\ \gamma,\ \psi]_\nu}v^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)y^2}{v}}dV_J\\ &= \int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}}\phi(\tau,z)\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}\overline{[e^{2\pi i(n\tau+rz)}\ |_{k_1,m_1}\ \gamma,\ \psi]_\nu}v^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)y^2}{v}}dV_J\\ &= \int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}}\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}\phi(\tau,z)\overline{[e^{2\pi i(n\tau+rz)}\ |_{k_1,m_1}\ \gamma,\ \psi]_\nu}v^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)y^2}{v}}dV_J.\end{aligned}$$

By Lemma 4.1, we can interchange the sum and integration in $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$. Hence we get,

$$\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle = \sum_{\gamma \in \Gamma^J_\infty \backslash \Gamma^J} \int_{\Gamma^J \backslash \mathcal{H} \times \mathbb{C}} \phi(\tau, z) \overline{[e^{2\pi i(n\tau + rz)} \mid_{k_1,m_1} \gamma,\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\frac{-4\pi(m_1+m_2)y^2}{v}} dV_J.$$

Using the change of variable (τ, z) to $\gamma^{-1}.(\tau, z)$ and using (4), we get

$$\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle = \sum_{\gamma \in \Gamma^J_\infty \backslash \Gamma^J} \int_{\gamma.\Gamma^J \backslash \mathcal{H} \times \mathbb{C}} \phi(\tau, z) \overline{[e^{2\pi i(n\tau + rz)},\ \psi]_\nu} v^{k_1+k_2+2\nu} e^{\frac{-4\pi(m_1+m_2)y^2}{v}} dV_J.$$

Now using Rankin unfolding argument, we have

$$\begin{aligned} \langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle &= \int_{\Gamma^J_\infty \backslash \mathcal{H} \times \mathbb{C}} \phi(\tau, z) \overline{[e^{2\pi i(n\tau + rz)}, \psi]_\nu} v^{k_1+k_2+2\nu} e^{\frac{-4\pi(m_1+m_2)y^2}{v}} dV_J \\ &= \int_{\Gamma^J_\infty \backslash \mathbb{H} \times \mathbb{C}} \phi(\tau, z) \sum_{l=0}^{\nu} (-1)^l \binom{k_1 + \nu - \frac{3}{2}}{\nu - l} \binom{k_2 + \nu - \frac{3}{2}}{l} m_1^{\nu - l} m_2^l \\ &\quad \times \overline{L^l_{m_1}(e^{2\pi i(n\tau + rz)}) L^{\nu-l}_{m_2}(\psi)} v^{k_1+k_2+2\nu} e^{\frac{-4\pi(m_1+m_2)y^2}{v}} dV_J \\ &= \sum_{l=0}^{\nu} A_l(k_1, m_1, k_2, m_2; \nu) \int_{\Gamma^J_\infty \backslash \mathbb{H} \times \mathbb{C}} \phi(\tau, z) \overline{L^l_{m_1}(e^{2\pi i(n\tau + rz)}) L^{\nu-l}_{m_2}(\psi)} \\ &\quad \times v^{k_1+k_2+2\nu} e^{\frac{-4\pi(m_1+m_2)y^2}{v}} dV_J. \end{aligned} \tag{6}$$

The repeated action of heat operator on L_{m_1} on $e^{2\pi i(n\tau + rz)}$ gives

$$L^l_{m_1}(e^{2\pi i(n\tau + rz)}) = (4m_1 n - r^2)^l e^{2\pi i(n\tau + rz)},$$

and similarly the repeated action of heat operator on L_{m_2} on the Fourier expansion of ψ gives

$$L^{\nu-l}_{m_2}(\psi) = \sum_{\substack{n_1, r_1 \in \mathbb{Z} \\ 4m_2 n_1 - r_1^2 > 0}} a(n_1, r_1)(4m_2 n_1 - r_1^2)^{\nu - l} e^{2\pi i(n_1\tau + r_1 z)}.$$

Now replacing ϕ and ψ by their Fourier series in (6), we have

$$\begin{aligned} \langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle &= \sum_{l=0}^{\nu} A_l(k_1, m_1, k_2, m_2; \nu) \\ &\quad \times \int_{\Gamma^J_\infty \backslash \mathbb{H} \times \mathbb{C}} \left(\sum_{\substack{n_2, r_2 \in \mathbb{Z}, \\ 4(m_1+m_2)n_2 - r_2^2 > 0}} b(n_2, r_2) e^{2\pi i(n_2\tau + r_2 z)} \right) (4m_1 n - r^2)^l \overline{e^{2\pi i(n\tau + rz)}} \\ &\quad \times \sum_{\substack{n_1, r_1 \in \mathbb{Z} \\ 4m_2 n_1 - r_1^2 > 0}} (4m_2 n_1 - r_1^2)^{\nu - l} \overline{a(n_1, r_1) e^{2\pi i(n_1\tau + r_1 z)}}\ v^{k_1+k_2+2\nu} e^{\frac{-4\pi(m_1+m_2)y^2}{v}} dV_J \end{aligned}$$

$$= \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu)$$
$$\times \int_{\Gamma_\infty^J\backslash\mathbb{H}\times\mathbb{C}} \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}} \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0}} (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}$$
$$\times\ b(n_2,r_2)e^{2\pi i(n_2\tau+r_2z)}\overline{e^{2\pi i(n_1\tau+r_1z)}e^{2\pi i(n\tau+rz)}}\ v^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)y^2}{v}}dV_J$$
$$= \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu)$$
$$\times \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}} \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0}} (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}b(n_2,r_2)$$
$$\times \int_{\Gamma_\infty^J\backslash\mathbb{H}\times\mathbb{C}} e^{2\pi i(n_2\tau+r_2z)}\overline{e^{2\pi i(n_1\tau+r_1z)}e^{2\pi i(n\tau+rz)}}\ v^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)y^2}{v}}dV_J.$$

Putting $\tau=u+iv, z=x+iy$, $\langle\phi,[P_{k_1,m_1;(n,r)},\psi]_\nu\rangle$ equals

$$\sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}} \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0}} (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}b(n_2,r_2)$$
$$\times \int_{\Gamma_\infty^J\backslash\mathbb{H}\times\mathbb{C}} e^{-2\pi v(n_2+n+n_1)}e^{-2\pi y(r_2+r+r_1)}e^{2\pi i(r_2-r-r_1)x}e^{2\pi i(n_2-n-n_1)u}v^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)y^2}{v}}dV_J. \quad (7)$$

A fundamental domain for the action of Γ_∞^J on $\mathbb{H}\times\mathbb{C}$ is given by $([0,1]\times[0,\infty])\times([0,1]\times\mathbb{R})$. Integrating on this region, $\langle\phi,[P_{k_1,m_1;(n,r)},\psi]_\nu\rangle$ equals

$$\sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}} \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0}} (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}b(n_2,r_2)$$

$$\times \int_0^1\!\!\int_0^\infty\!\!\int_0^1\!\!\int_{-\infty}^\infty e^{-2\pi v(n_2+n+n_1)}e^{-2\pi y(r_2+r+r_1)}e^{2\pi i(r_2-r-r_1)x}e^{2\pi i(n_2-n-n_1)u}v^{k_1+k_2+2\nu-3}e^{\frac{-4\pi(m_1+m_2)y^2}{v}}dudvdxdy.$$

Integrating on x and u first, $\langle\phi,[P_{k_1,m_1;(n,r)},\psi]_\nu\rangle$ equals

$$\sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0\\ 4(m_1+m_2)(n+n_1)-(r+r_1)^2>0}} (4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}b(n+n_1,r+r_1)$$
$$\times \int_0^\infty\int_{-\infty}^\infty e^{-4\pi v(n+n_1)}e^{-4\pi y(r+r_1)}v^{k_1+k_2+2\nu-3}e^{\frac{-4\pi(m_1+m_2)y^2}{v}}dydv. \quad (8)$$

Integrating over y, we have

$$\int_{-\infty}^{\infty} e^{-4\pi\left((r_1+r)y+\frac{(m_1+m_2)y^2}{v}\right)} dy = \frac{\sqrt{v}e^{\pi\frac{(r+r_1)^2 v}{m_1+m_2}}}{2\sqrt{m_1+m_2}}. \tag{9}$$

Substituting the value on (9) and integrating over v, we have

$$\begin{aligned}\int_0^{\infty} & e^{-4\pi v(n+n_1)} v^{k_1+k_2+2\nu-3} \frac{\sqrt{v}e^{\pi\frac{(r_1+r)^2 v}{m_1+m_2}}}{2\sqrt{m_1+m_2}} dv \\ &= \frac{1}{2\pi^{k_1+k_2+2\nu-\frac{3}{2}}} \frac{(m_1+m_2)^{k_1+k_2+2\nu-2}\Gamma(k_1+k_2+2\nu-\frac{3}{2})}{(4(n+n_1)(m_1+m_2)-(r+r_1)^2)^{k_1+k_2+2\nu-\frac{3}{2}}}.\end{aligned} \tag{10}$$

Putting the value of integral (10) for $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$ in (8) we have

$$\begin{aligned}\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle &= \frac{(m_1+m_2)^{k_1+k_2+2\nu-2}\Gamma(k_1+k_2+2\nu-\frac{3}{2})}{2\pi^{k_1+k_2+2\nu-\frac{3}{2}}} \\ \times \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) & \sum_{\substack{n_1,r_1\in\mathbb{Z} \\ 4m_2n_1-r_1^2>0 \\ 4(m_1+m_2)(n+n_1)-(r+r_1)^2>0}} \frac{(4m_1n-r^2)^l(4m_2n_1-r_1^2)^{\nu-l}\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{(4(n+n_1)(m_1+m_2)-(r+r_1)^2)^{k_1+k_2+2\nu-\frac{3}{2}}}.\end{aligned} \tag{11}$$

Now substituting $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_\nu \rangle$ from (11) in (5), we get the required expression for $c_\nu(n,r)$ given in Theorem 3.1.

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