NONVANISHING OF KERNEL FUNCTIONS FOR JACOBI FORMS

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Abstract. Y. Martin introduced a set of kernel functions for the Jacobi group to study $2m$ Dirichlet series associated with a Jacobi form of weight k and index m. We study nonvanishing of these kernel functions and also study nonvanishing of $2m$ Dirichlet series associated with Jacobi form of weight k and index m.

1. Introduction and Preliminaries

Let S_k be the space of cusp forms of weight k for the full modular group $\Gamma = SL_2(\mathbb{Z})$ with the usual Petersson inner product $\langle , \rangle$. For a cusp form $f(z) \in S_k$ with Fourier expansion $f(z) = \sum_{n\geq 1} a_n q^n$ we associate the Hecke L-function $L(f, s) := \sum_{n\geq 1} \frac{a_n}{n^s}$ for $\sigma = Re(s) > \frac{k+1}{2}$. The completed L-function $L^*(f, s) := (2\pi)^{-s}\Gamma(s)L(f, s)$ has a holomorphic continuation to $\mathbb{C}$ and satisfies the functional equation $L^*(f, k - s) = (-1)^{k/2}L^*(f, s)$. It is well-known that zeroes of $L^*(f, s)$ can occur only inside the critical strip $(k-1)/2 < Re(s) < (k+1)/2$, and according to the generalized Riemann hypothesis all the zeroes should lie on the line $Re(s) = k/2$. In this direction Kohnen [7] proved the following nonvanishing results for L-functions on average:

Theorem 1.1. [7] *Let $\{f_1, f_2, \cdots , f_{dim\ S_k}\}$ be the basis of normalized Hecke eigenforms of S_k. Let $t_o \in \mathbb{R}$ and $\epsilon > 0$. Then there exists a constant $C(t_0, \epsilon) > 0$ depending only on t_0 and ϵ such that for $k > C(t_0, \epsilon)$, the function*

$$\sum_{i=1}^{dim\ S_k} \frac{1}{\langle f_i, f_i\rangle} L^*(f_i, s)$$

does not vanish at any point $s = \sigma + it$ with $t = t_0$, $(k - 1)/2 < \sigma < k/2 - \epsilon$, $k/2 + \epsilon < \sigma < (k + 1)/2$.

Corollary 1.2. [7] *Let $t_o \in \mathbb{R}$ and $\epsilon > 0$. For $k > C(t_0, \epsilon)$, and any $s = \sigma + it$ with $t = t_0$, $(k - 1)/2 < \sigma < k/2 - \epsilon$, $k/2 + \epsilon < \sigma < (k + 1)/2$, there exists a Hecke eigenform $f \in S_k$ such that $L^*(f, s) \neq 0$.*

For the complex numbers z and s with $z \neq 0$ we set $z^s = \exp(s \log z)$ with $\log z = \log |z| + i \arg z$ and $-\pi < \arg z \leq \pi$. We fix the notation $e(x) := \exp(2\pi ix)$, $e^m(x) := \exp(2\pi imx)$.

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Kohnen considered the following functions (called kernel functions) in S_k to prove the nonvanishing of L-function. For $z \in \mathbb{H}$ and $s = \sigma + it \in \mathbb{C}$ with $1 < \sigma < k-1$, define the function $R_{k,s}$ by

$$R_{k,s}(z) = \gamma_k(s) \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma} (cz+d)^{-k} \left(\frac{az+b}{cz+d} \right)^{-s}, \tag{1}$$

where $\gamma_k(s) := \frac{1}{2} e^{\pi i \frac{s}{2}} \Gamma(s) \Gamma(k-s)$. These kernel functions dual w.r.t the Petersson inner product gives the values $L^*(f,s)$ upto a constant. More precisely,

$$\langle f, R_{k,s} \rangle = \frac{(-1)^{k/2} \pi (k-2)!}{2^{k-2}} L^*(f,s), \tag{2}$$

for all cusp forms $f \in S_k$. Kohnen computed the Fourier coefficients of these kernel functions explicitly and estimate the first Fourier coefficient in an appropriate way to conclude the nonvanishing of L-functions on the average.

Theorem 1.3. [7] *The function $R_{k,s}(z)$ has the Fourier expansion*

$$R_{k,s}(z) = \sum_{n \geq 1} r_{k,s}(n) q^n,$$

where

$$\begin{aligned} r_{k,s}(n) =& (2\pi)^s \Gamma(k-s) n^{s-1} + (-1)^{\frac{k}{2}} (2\pi)^{k-s} \Gamma(s) n^{k-s-1} + \frac{1}{2} (-1)^{\frac{k}{2}} (2\pi)^k n^{k-1} \\ & \times \sum_{\substack{(a,c) \in \mathbb{Z}^2, ac>0 \\ g.c.d(a,c)=1}} c^{-k} \left(\frac{c}{a} \right)^s \Big[e^{2\pi i n a'/c} e^{\pi i \frac{s}{2}} \; {}_1f_1(s,k; -2\pi i n/ac) \\ & + e^{-2\pi i n a'/c} e^{-\pi i \frac{s}{2}} \; {}_1f_1(s,k; 2\pi i n/ac) \Big], \end{aligned} \tag{3}$$

where $a' \in \mathbb{Z}$ is an inverse of a modulo c and

$${}_1f_1(\alpha, \beta; z) = \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)} \; {}_1F_1(\alpha, \beta; z), \tag{4}$$

and ${}_1F_1$ is Kummer's degenerate hypergeometric function.

The work of Kohnen by constructing kernel functions and proving nonvanishing of L-functions has motivated many authors to work in other automorphic forms like Siegel modular forms [8], Hilbert modular forms [6]. Recently the authors in [10] proved the nonvanishing of the kernel function $R_{k,s}$ at certain real point by analytic estimate and deduced the nonvanishing of L-function associated with a Hecke eigenform.

Jacobi forms are natural generalization of modular forms, the classical Jacobi theta function is an example of Jacobi form. The Fourier-Jacobi expansion of Siegel modular forms over the sympletic group $Sp_2(\mathbb{Z})$ are natural examples of Jacobi forms. Consider the Jacobi group $\Gamma^J := SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ consisting elements of the type (M,X) where $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$ and $X=(\lambda,\nu)\in\mathbb{Z}^2$ with the group law

$$(M,X)(M',X')=(MM',XM'+X').$$

The group Γ^J acts on $\mathbb{H}\times\mathbb{C}$ via

$$\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda,\nu)\right)\cdot(\tau,z):=\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\nu}{c\tau+d}\right).$$

For positive integers k and m, consider the automorphic factor

$$j_{k,m}(h,\tau,z):=(c\tau+d)^{-k}e^{m}\left(\frac{-c(z+\lambda\tau+\nu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\nu\right),$$

where $h=\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda,\nu)\right)$. Now we define an action of Γ^J on the collection of holomorphic functions $\phi:\mathbb{H}\times\mathbb{C}\to\mathbb{C}$ via $\phi\mapsto\phi|_{k,m}h$, where

$$\phi|_{k,m}h(\tau,z):=j_{k,m}(h,\tau,z)\phi(h.(\tau,z)).$$

Let k and m be positive integers. A Jacobi form of weight k and index m over the Jacobi group Γ^J is any holomorphic function $\phi:\mathbb{H}\times\mathbb{C}\to\mathbb{C}$ which satisfies $\phi|_{k,m}h=\phi$ for all $h\in\Gamma^J$, and has a Fourier series expression of the form

$$\phi(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z},\\ 4mn\geq r^2}}c(n,r)q^n\zeta^r,\ \ (q=e(\tau),\zeta=e(z)).$$

Furthermore, a Jacobi form ϕ is said to be a cusp form if it has a Fourier series expression of the form

$$\phi(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z},\\ 4mn> r^2}}c(n,r)q^n\zeta^r.$$

Let $J_{k,m}$ be the set of all Jacobi forms of weight k and index m which is a finite dimensional vector space over $\mathbb{C}$. Let $J_{k,m}^{cusp}$ be the subspace of Jacobi cusp forms of weight k and index m which is a finite dimensional Hilbert space w.r.t the Petersson inner product

$$\langle\phi,\psi\rangle:=\int_{\Gamma^J\backslash\mathbb{H}\times\mathbb{C}}\phi(\tau,z)\overline{\psi(\tau,z)}v^ke^{-4\pi my^2/v}dV,$$

where $dV:=v^{-3}dx\ dy\ du\ dv,(\tau=u+iv,z=x+iy)$. For details on Jacobi forms we refer [4].

The Fourier coefficients of Jacobi form satisfy $c(n,r)=c(n',r')$ whenever $r'\equiv r \pmod{2m}$ and $4n'm-r'^2=4nm-r^2$, i.e. $c(n,r)$ depends only on $4nm-r^2$ and on $r \pmod{2m}$. Set $c_r(D):=c(n,r)$ if $D=4nm-r^2$, else $c_r(D)=0$. A Jacobi form of weight k and index m can be written as

$$\phi(\tau,z)=\sum_{\mu=1}^{2m}h_\mu(\tau)\Theta_{m,\mu}(\tau,z)$$

where $h_\mu(\tau) = \sum_{D=1}^{\infty} c_\mu(D) q^{D/4m}$ and $\Theta_{m,\mu}(\tau, z) = \sum_{\substack{r\in\mathbb{Z},\\ r\equiv\mu \pmod{2m}}} q^{r^2/4m}\zeta^r$. The above representation is called the theta decomposition of Jacobi form ϕ.

To any Jacobi cusp form ϕ with the theta decomposition $\phi(\tau, z) = \sum_{\mu=1}^{2m} h_\mu(\tau)\Theta_{m,\mu}(\tau, z)$, we associate the function $\bar{\phi}(\tau, z) := \overline{\phi(-\bar{\tau}, -\bar{z})}$ which is also a Jacobi cusp form with the associated coefficients $\overline{c_\mu(D)}$ instead of $c_\mu(D)$ in the corresponding theta decomposition.

To any Jacobi cusp form ϕ with the theta decomposition $\phi(\tau, z) = \sum_{\mu=1}^{2m} h_\mu(\tau)\Theta_{m,\mu}(\tau, z)$, Berndt [2] associated the $2m-$tuple Dirichlet series

$$L_\mu(\phi, s) := \sum_{D=1}^{\infty} c_\mu(D) \left(\frac{D}{4m}\right)^{-s}$$

for $\mu = 1, 2, \cdots, 2m$. We also set $\Lambda_\mu(\phi, s) := (2\pi)^{-s}\Gamma(s)L_\mu(\phi, s)$. These series are uniformly convergent on compact subsets of the half plane $Re(s) > k/2 + 1$. Berndt [2] established following analytic properties using a variation of the Mellin transformation (also see [11] for another proof).

Theorem 1.4. [11] *Let $k > 9, m$ be positive integers and $\phi \in J_{k,m}^{cusp}$. Then every completed Dirichlet series $\Lambda_\beta(\phi, s)$, with $\beta = 1, 2, \cdots, 2m$, admits an analytic continuation to the whole complex plane, and they satisfy the set of $2m$ functional equations*

$$\Lambda_\beta(\overline{\phi}, s) = \frac{i^k}{\sqrt{2m}} \sum_{\mu=1}^{2m} exp(\pi i\mu\beta/m)\Lambda_\mu(\bar{\phi}, k - s - 1/2). \tag{5}$$

For $s \in \mathbb{C}$ with $1 < Re(s) < k - 3$ and $t_0 \in (2m)^{-1}\mathbb{Z}$, define the function

$$\Omega_{t_0,s}^{k,m}(\tau, z) = \sum_{h\in\mathcal{H}^J\backslash\Gamma^J} \phi_{t_0,s}|_{k,m}h(\tau, z), \tag{6}$$

where $\phi_{t_0,s}(\tau, z) = \dfrac{1}{\tau^s} e^m\left(\dfrac{-(z - t_0)^2}{\tau}\right)$, $\mathcal{H}^J = \{(Id, (\lambda, 0))|\lambda \in \mathbb{Z}\}$. A collection of coset representatives for the elements in $\mathcal{H}^J\backslash\Gamma^J$ is given by $\{(I, (0, \nu))(M, (0, 0))\,|M \in \Gamma = SL_2(\mathbb{Z}), \nu \in \mathbb{Z}\}$. Martin [11] proved the following result for Jacobi forms analogous to (2).

Theorem 1.5. [11] *Let k and m be positive integers with $k > 6$ and $t_0 \in (2m)^{-1}\mathbb{Z}$. If $s \in \mathbb{C}$ with $1 < Re(s) < k - 3$, then the series $\Omega_{t_0,s}^{k,m}$ defines a Jacobi cusp form in $J_{k,m}^{cusp}$. Moreover*

$$< \Omega_{t_0,s}^{k,m}, \phi >= \frac{\pi}{2^{k-2}e^{\pi is/2}} \frac{\Gamma(k - 3/2)}{\Gamma(s - 1/2)\Gamma(k - s)} \frac{1}{2m} \sum_{\mu=1}^{2m} exp(-2\pi i\mu t_0)\Lambda_\mu(\bar{\phi}, k - s), \tag{7}$$

for all $\phi \in J_{k,m}^{cusp}$ and all $s \in \mathbb{C}$ with $\frac{3}{2} < Re(s) < \frac{k}{2} - 2$.

Following the work on Kohnen, we explicitly compute the Fourier coefficients of the kernel functions $\Omega_{t_0,s}^{k,m}$ and establish the nonvanishing of any general (n,r)-coefficient of the kernel functions $\Omega_{t_0,s}^{k,m}$ for large k. Further as applications we deduce nonvanishing of the Dirichlet series $\Lambda_\mu(\phi,s)$ for a Jacobi form ϕ and a result on nonvanishing of Jacobi Poincaré series.

2. Statement of results

Theorem 2.1. *Let k and m be positive integers with $k>6$ and $t_0\in(2m)^{-1}\mathbb{Z}$. If $s\in\mathbb{C}$ with $1<Re(s)<k-3$, then $\Omega_{t_0,s}^{k,m}$ has the Fourier expansion*

$$\Omega_{t_0,s}^{k,m}(\tau,z)=\sum_{4nm>r^2}\omega(n,r)q^n\zeta^r,$$

where

$$\begin{aligned}\omega(n,r)=&\alpha_s m^{1-s}D^{s-\frac{3}{2}}(e(-rt_0)+(-1)^k e(rt_0))\\&+(-1)^s(2im)^{\frac{1}{2}}\alpha_{k-s+\frac{1}{2}}m^{\frac{1}{2}+s-k}D^{k-s-1}(1+(-1)^k)\\&+(-1)^{\frac{k-1}{2}}(2\pi)^{k-\frac{1}{2}}2^{\frac{5}{2}-2k}m^{1-k}(-D)^{k-\frac{3}{2}}\frac{1}{\Gamma(k-\frac{1}{2})}\sum_{ac>0,(a,c)=1}\left(\frac{a}{c}\right)^{k-s}a^{-k}\\&\sum_{b(a*),\nu(a)}e\left(\frac{r(\nu-t_0)}{a}\right)\left[e^m\left(\frac{-(\nu-t_0)^2c}{a}\right)e\left(\frac{nc'}{a}\right)\ {}_1F_1\left(k-s,k-\frac{1}{2};\frac{2\pi iD}{4mac}\right)\right.\\&\left.+e^m\left(\frac{(\nu-t_0)^2c}{a}\right)e\left(\frac{-nc'}{a}\right)\ {}_1F_1\left(k-s,k-\frac{1}{2};\frac{-2\pi iD}{4mac}\right)\right],\end{aligned}\tag{8}$$

where $D=4nm-r^2$.

Theorem 2.2. *Let $\omega(n,r)$ be the (n,r)- Fourier coefficient of $\Omega_{t_0,s}^{k,m}$ as above. Given any positive integer m,n,r such that $m\nmid 2r$, there exist k_0 such that $\omega(n,r)\neq 0$ for $k>k_0$.*

Theorem 2.3. *Let m be any positive integer, $t'\in\mathbb{H}$ and $\epsilon>0$. Then there exists a constant $k_0=k_0(t',\epsilon)$ such that for $k>k_0$ and any $s=\sigma+it'$ with $\frac{k}{2}-\frac{3}{4}<\sigma<\frac{k}{2}-\frac{1}{4}-\epsilon$, $\frac{k}{2}-\frac{1}{4}+\epsilon<\sigma<\frac{k}{2}+\frac{1}{4}$ there exist a Hecke eigenform $\phi\in J_{k,m}^{cusp}$ such that vector valued function $\Lambda(\bar{\phi},s):=(\Lambda_1(\bar{\phi},s),\Lambda_2(\bar{\phi},s),...,\Lambda_{2m}(\bar{\phi},s))\neq 0$.*

3. Proofs

Proof of Theorem 2.1. By definition of $\Omega_{t_0,s}^{k,m}$ we have

$$\Omega_{t_0,s}^{k,m}(\tau,z)=\sum_{\nu\in\mathbb{Z},\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma}(c\tau+d)^{-k}e^m\left(\frac{-cz^2}{c\tau+d}\right)\left(\frac{a\tau+b}{c\tau+d}\right)^{-s}e^m\left(\frac{-(\frac{z}{c\tau+d}+\nu-t_0)^2}{\frac{a\tau+b}{c\tau+d}}\right).\tag{9}$$

We split the sum into three parts corresponding to $c = 0, a = 0$ and $ac \neq 0$. Contribution of the sum corresponding to $c = 0$ in (9) is due to matrices $\pm\begin{pmatrix}1 & l\\ 0 & 1\end{pmatrix}$; $l \in \mathbb{Z}$, which we denote by C_0. Then

$$\begin{aligned}C_0 &= \sum_{l,\nu\in\mathbb{Z}}\left[(\tau+l)^{-s}e^m\left(\frac{-(z+\nu-t_0)^2}{\tau+l}\right)+(-1)^k(\tau+l)^{-s}e^m\left(\frac{-(-z+\nu-t_0)^2}{\tau+l}\right)\right]\\ &= \sum_{l,\nu\in\mathbb{Z}}\left[(\tau+l)^{-s}e^m\left(\frac{-(z+\nu-t_0)^2}{\tau+l}\right)+(-1)^k(\tau+l)^{-s}e^m\left(\frac{-(z+\nu+t_0)^2}{\tau+l}\right)\right].\end{aligned}$$

The contribution of first part of above sum to the (n,r)-Fourier coefficient, which we denote by $C_{01}(n,r)$, is

$$\begin{aligned}C_{01}(n,r) &= \int_{iC_1-\infty}^{iC_1+\infty}\tau^{-s}\left(\int_{iC_2-\infty}^{iC_2+\infty}e^m\left(-\frac{(z-t_0)^2}{\tau}\right)e(-rz)\,dz\right)e(-n\tau)\,d\tau,\ (C_1>0, C_2\in\mathbb{R})\\ &= e(-rt_0)\int_{iC_1-\infty}^{iC_1+\infty}\tau^{-s}\left(\int_{iC_2-\infty}^{iC_2+\infty}e^m\left(-\frac{z^2}{\tau}-rz\right)dz\right)e(-n\tau)\,d\tau\\ &= e(-rt_0)\int_{iC_1-\infty}^{iC_1+\infty}\tau^{-s}\left(\frac{\tau}{2im}\right)^{\frac{1}{2}}e\left(\frac{r^2\tau}{4m}\right)e(-n\tau)\,d\tau.\end{aligned}$$

If $r^2 \geq 4nm$ then we can deform the integral at $C_1 = \infty$ to get $C_{01}(n,r) = 0$. If $r^2 < 4nm$ then we have $C_{01}(n,r) = e(-rt_0)\alpha_s m^{1-s}D^{s-\frac{3}{2}}$, where $\alpha_s = \dfrac{(-1)^{\frac{s}{2}}\pi^{s-\frac{1}{2}}}{2^{s-2}\Gamma(s-\frac{1}{2})}$. Similarly we can compute the contribution corresponding to the second part of C_0. Hence we get the contribution of C_0 in (n,r)-Fourier coefficient, which we denote by $C_0(n,r)$.

$$C_0(n,r) = \alpha_s m^{1-s}D^{s-\frac{3}{2}}(e(-rt_0)+(-1)^k e(rt_0)). \tag{10}$$

Contribution of the sum corresponding to $a = 0$ in (9) is due to matrices $\pm\begin{pmatrix}0 & -1\\ 1 & l\end{pmatrix}$, $l \in \mathbb{Z}$, which we denote by A_0. Then

$$\begin{aligned}A_0 = \sum_{l,\nu\in\mathbb{Z}}&\left[(\tau+l)^{-k}\left(\frac{-1}{\tau+l}\right)^{-s}e^m\left(\frac{-z^2}{\tau+l}\right)e^m\left(\frac{-(\frac{z}{\tau+l}+\nu-t_0)^2}{\frac{-1}{\tau+l}}\right)\right.\\ &\left.+(-1)^k(\tau+l)^{-k}\left(\frac{-1}{\tau+l}\right)^{-s}e^m\left(\frac{-z^2}{\tau+l}\right)e^m\left(\frac{-(-\frac{z}{\tau+l}+\nu-t_0)^2}{\frac{-1}{\tau+l}}\right)\right]\\ =(-1)^{-s}&\sum_{l,\nu\in\mathbb{Z}}\left[(\tau+l)^{-k+s}e^m\Big((\nu-t_0)^2(\tau+l)+2z(\nu-t_0)\Big)\right.\\ &\left.+(-1)^k(\tau+l)^{-k+s}e^m\Big((\nu-t_0)^2(\tau+l)+2z(t_0-\nu)\Big)\right].\end{aligned}$$

Calculating as before we get the contribution of A_0 to (n,r)-Fourier coefficient, which we denote by $A_0(n,r)$

$$A_0(n,r)=0,\ r^2\geq 4nm,$$

and if $r^2<4nm$ then we get

$$A_0(n,r)=(-1)^{-s}(2im)^{\frac{1}{2}}\alpha_{k-s+\frac{1}{2}}m^{\frac{1}{2}+s-k}D^{k-s-1}(1+(-1)^k). \tag{11}$$

Now assume $ac\neq 0$. The contribution of the sum correspoding to terms $ac\neq 0$ in (9), which we denote by B_0, is

$$\begin{aligned}
B_0 &= \sum_{\substack{ac\neq 0\\ (a,c)=1,\nu\in\mathbb{Z}}} (c\tau+d)^{-k}e^m\left(\frac{-cz^2}{c\tau+d}\right)\left(\frac{a\tau+b}{c\tau+d}\right)^{-s}e^m\left(\frac{-(\frac{z}{c\tau+d}+\nu-t_0)^2}{\frac{a\tau+b}{c\tau+d}}\right)\\
&= \sum_{\substack{ac\neq 0\\ (a,c)=1,\nu\in\mathbb{Z}}} \left(\frac{c\tau+d}{a\tau+b}\right)^{-k+s}(a\tau+b)^{-k}e^{-m}\left(\frac{a}{a\tau+b}\left(z+\frac{\nu-t_0}{a}\right)^2\right)e^{-m}\left((\nu-t_0)^2\frac{c}{a}\right)\\
&= \sum_{\substack{ac\neq 0\\ (a,c)=1,\nu\in\mathbb{Z}}} a^{-k}\left(\frac{c}{a}-\frac{1}{a^2(\tau+\frac{b}{a})}\right)^{-k+s}\left(\tau+\frac{b}{a}\right)^{-k}e^{-m}\left(\frac{\left(z+\frac{\nu-t_0}{a}\right)^2}{\tau+\frac{b}{a}}\right)e^{-m}\left((\nu-t_0)^2\frac{c}{a}\right)\\
&= \sum_{\substack{ac\neq 0 (a,c)=1\\ \alpha,\beta\in\mathbb{Z}\\ \nu(a),b(a*)}} a^{-k}\left(\frac{c}{a}-\frac{1}{a^2(\tau+\beta+\frac{b}{a})}\right)^{-k+s}\left(\tau+\beta+\frac{b}{a}\right)^{-k}e^{-m}\left(\frac{\left(z+\alpha+\frac{\nu-t_0}{a}\right)^2}{\tau+\beta+\frac{b}{a}}\right)\\
&\quad\times e^{-m}\left((\nu-t_0)^2\frac{c}{a}\right)\\
&= \sum_{\substack{ac\neq 0 (a,c)=1\\ \nu(a),b(a*)}} a^{-k}F_{c,a}\left(\tau+\frac{b}{a},z+\frac{\nu-t_0}{a}\right)e^{-m}\left((\nu-t_0)^2\frac{c}{a}\right)
\end{aligned}$$

where

$$F_{(c,a)}(\tau,z)=\sum_{\alpha,\beta\in\mathbb{Z}}\left(\frac{c}{a}-\frac{1}{a^2(\tau+\beta)}\right)^{-k+s}(\tau+\beta)^{-k}e^{-m}\left(\frac{(z+\alpha)^2}{\tau+\beta}\right).$$

Contribution of the terms with $ac > 0$ in $F_{c,a}$ to (n,r)-Fourier coefficients, which we denote by $F^+_{c,a}(n,r)$, is given by

$$\begin{aligned} F^+_{c,a}(n,r) &= \int_{iC_1-\infty}^{iC_1+\infty} \left(\frac{c}{a} - \frac{1}{a^2\tau}\right)^{-k+s} \tau^{-k} \left(\int_{iC_2-\infty}^{iC_2+\infty} e^{-m}\left(\frac{z^2}{\tau}\right) e(-rz)\, dz\right) e(-n\tau)\, d\tau \\ &= \int_{iC_1-\infty}^{iC_1+\infty} \left(\frac{c}{a}\tau - \frac{1}{a^2}\right)^{-k+s} \tau^{-s} \int_{iC_2-\infty}^{iC_2+\infty} \left(e^{m}\left(-\frac{z^2}{\tau} - rz\right) dz\right) e(-n\tau)\, d\tau \\ &= \int_{iC_1-\infty}^{iC_1+\infty} \left(\frac{c}{a}\tau - \frac{1}{a^2}\right)^{-k+s} \tau^{-s} \left(\frac{\tau}{2im}\right)^{\frac{1}{2}} e\left(\frac{-D}{4m}\right) \tau\, d\tau \end{aligned}$$

If $r^2 \geq 4nm$ then again above integral becomes 0. If $r^2 < 4nm$ then we make the change of variable $\tau \mapsto \frac{a}{c} it$ to get

$$\begin{aligned} F^+_{c,a}(n,r) &= \int_{iC_1-\infty}^{iC_1+\infty} \left(it - \frac{1}{a^2}\right)^{-k+s} (\frac{a}{c}it)^{-s} \left(\frac{at}{2cm}\right)^{\frac{1}{2}} e^{2\pi i}\left(\left(\frac{-D}{4m}\right)\frac{a}{c}it\right) \frac{a}{c} idt \\ &= (-1)^{\frac{-k+1}{2}} \left(\frac{a}{c}\right)^{-s+\frac{3}{2}} \frac{1}{\sqrt{2m}} \int_{C-i\infty}^{C+i\infty} \left(t + \frac{i}{a^2}\right)^{-k+s} t^{-s+\frac{1}{2}} e^{-2\pi}\left(\left(\frac{-D}{4m}\right)\frac{a}{c}t\right) dt. \end{aligned}$$

Using the fact

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} (t+\alpha)^{-\mu}(t+\beta)^{-\nu} e^{pt}\, dt = \frac{1}{\Gamma(\mu+\nu)} p^{\mu+\nu-1} e^{-\beta p}\ {}_1F_1(\mu, \mu+\nu; (\beta-\alpha)p)$$

for $Re(\mu,\nu) > 0, p \in \mathbb{C}$, we have

$$F^+_{c,a}(n,r) = \frac{(2\pi)^{k-\frac{1}{2}}(-1)^{\frac{k-1}{2}}(\frac{a}{c})^{k-s} 2^{5/2-k} m^{1-k}(-D)^{k-\frac{3}{2}}}{\Gamma(k-\frac{1}{2})}\ {}_1F_1\left(k-s, k-\frac{1}{2}; 2\pi i\frac{(D)}{4mac}\right).$$

Similarly for $ac < 0$ we replace (a,c) by $(a,-c)$. and perform the similar calculations. Finally we have contribution of B_0 to (n,r)-Fourier coefficient which we denote by $B_0(n,r)$, is given by

$$\begin{aligned} B_0(n,r) = \frac{(2\pi)^{k-\frac{1}{2}}(-1)^{\frac{k-1}{2}} 2^{5/2-k} m^{1-k}(-D)^{k-\frac{3}{2}}}{\Gamma(k-\frac{1}{2})} \sum_{(a,c)=1, ac>0} \left(\frac{a}{c}\right)^{k-s} a^{-k} \\ \sum_{cc'\equiv 1 \pmod a), \nu(a)} e\left(r\frac{\nu-t_0}{a}\right)\Big[e^{-m}\left(\frac{(\nu-t_0)^2c}{a}\right) e\left(\frac{nc'}{a}\right)\ {}_1F_1\left(k-s, k-\frac{1}{2}; \frac{2\pi iD}{4mac}\right) \\ + e^{m}\left(\frac{(\nu-t_0)^2c}{a}\right) e\left(-\frac{nc'}{a}\right)\ {}_1F_1\left(k-s, k-\frac{1}{2}; \frac{-2\pi iD)}{4mac}\right)\Big]. \end{aligned} \tag{12}$$

Now the theorem follows from (10), (11) and (12). □

Proof of Theorem 2.2. Given positive inetger m, n and r we show that $\omega(n,r)$ is non zero for large k. On contrary let $\omega(n,r)=0$ i.e.

$$\begin{aligned}
0 =&\alpha_s m^{1-s} D^{s-\frac{3}{2}}\Bigg(e(-irt_0)+(-1)^k e(rt_0)\Bigg)\\
&+(-1)^s(2im)^{\frac{1}{2}}\alpha_{k-s+\frac{1}{2}} m^{\frac{1}{2}+s-k} D^{k-s-1}\Bigg(1+(-1)^k\Bigg)\\
&+\frac{(2\pi)^{k-\frac{1}{2}}(-1)^{\frac{k-1}{2}}2^{\frac{5}{2}-2k} m^{1-k}(-D)^{k-\frac{3}{2}}}{\Gamma(k-\frac{1}{2})}\sum_{(a,c)=1,ac>0}\left(\frac{a}{c}\right)^{k-s} a^{-k}\\
&\sum_{cc'\equiv 1}\ \sum_{(\mathrm{mod}\ a),\nu(a)} e\left(\frac{r(\nu-t_0)}{a}\right)\Bigg[e^{-m}\left(\frac{(\nu-t_0)^2 c}{a}\right)e\left(\frac{nc'}{a}\right)\ {}_1F_1\left(k-s,k-\frac{1}{2};\frac{2\pi iD}{4mac}\right)\\
&+e^{m}\left(\frac{(\nu-t_0)^2 c}{a}\right)e\left(-\frac{nc'}{a}\right)\ {}_1F_1\left(k-s,k-\frac{1}{2};\frac{-2\pi iD}{4mac}\right)\Bigg].\\
-1 =&\frac{(-1)^s(2im)^{\frac{1}{2}}\alpha_{k-s+\frac{1}{2}} m^{\frac{1}{2}+s-k} D^{k-s-1}}{\alpha_s m^{1-s} D^{s-\frac{3}{2}}\Bigg(e(-rt_0)+(-1)^k e(rt_0)\Bigg)}\\
&+\frac{(2\pi)^{k-\frac{1}{2}}(-1)^{\frac{k-1}{2}}2^{\frac{5}{2}-2k} m^{1-k}(-D)^{k-\frac{3}{2}}}{\alpha_s m^{1-s} D^{s-\frac{3}{2}}\Bigg(e(-rt_0)+(-1)^k e(rt_0)\Bigg)\Gamma(k-\frac{1}{2})}\sum_{(a,c)=1,ac>0}\left(\frac{a}{c}\right)^{k-s} a^{-k}\\
&\times\sum_{cc'\equiv 1}\ \sum_{(\mathrm{mod}\ a),\nu(a)} e\left(\frac{r(\nu-t_0)}{a}\right)\Bigg[e^{-m}\left((\nu-t_0)^2\frac{c}{a}\right)e(\frac{nc'}{a})\ {}_1F_1\left(k-s,k-\frac{1}{2};\frac{2\pi iD}{4mac}\right)\\
&+e^{m}\left(\frac{(\nu-t_0)^2 c}{a}\right)e\left(-\frac{nc'}{a}\right)\ {}_1F_1\left(k-s,k-\frac{1}{2};\frac{-2\pi iD}{4mac}\right)\Bigg].
\end{aligned}$$

Taking modulus, we have

$$\begin{aligned}
1 \leq&\left|\frac{(-1)^s(2im)^{\frac{1}{2}}\alpha_{k-s+\frac{1}{2}} m^{\frac{1}{2}+s-k} D^{k-s-1}}{\alpha_s m^{1-s} D^{s-\frac{3}{2}}\Bigg(e(-rt_0)+(-1)^k e(rt_0)\Bigg)}\right|\\
&+\left|\frac{(2\pi)^{k-\frac{1}{2}}(-1)^{\frac{k-1}{2}}2^{\frac{5}{2}-2k} m^{1-k}(-D)^{k-\frac{3}{2}}}{\alpha_s m^{1-s} D^{s-\frac{3}{2}}\Bigg(e(-rt_0)+(-1)^k e(rt_0)\Bigg)\Gamma(k-\frac{1}{2})}\right|\sum_{(a,c)=1,ac>0}\left|\left(\frac{a}{c}\right)^{k-s} a^{-k}\right|\\
&\times\sum_{cc'\equiv 1}\ \sum_{(\mathrm{mod}\ a),\nu(a)}\Bigg| e\left(\frac{r(\nu-t_0)}{a}\right)\Bigg[e^{-m}\left((\nu-t_0)^2\frac{c}{a}\right)e\left(\frac{nc'}{a}\right)\ {}_1F_1\left(k-s,k-\frac{1}{2};\frac{2\pi iD}{4mac}\right)\\
&+e^{m}\left(\frac{(\nu-t_0)^2 c}{a}\right)e\left(\frac{-nc'}{a}\right)\ {}_1F_1\left(k-s,k-\frac{1}{2};\frac{-2\pi iD}{4mac}\right)\Bigg]\Bigg|.
\end{aligned}$$

For $s = \frac{k}{2} + \frac{1}{4} - \delta - it'$ we have

$$\begin{aligned} 1 \leq & \left| \frac{(2m)^{\frac{1}{2}} \alpha_{k-s+\frac{1}{2}} m^{\frac{1}{2}+s-k} D^{k-s-1}}{\alpha_s m^{1-s} D^{s-\frac{3}{2}} \left(e(-rt_0) + (-1)^k e(rt_0) \right)} \right| \\ & + \left| \frac{(2\pi)^{k-\frac{1}{2}} (-1)^{\frac{k-1}{2}} 2^{\frac{5}{2}-2k} m^{1-k} (-D)^{k-\frac{3}{2}}}{\alpha_s m^{1-s} D^{s-\frac{3}{2}} \left(e(-rt_0) + (-1)^k e(rt_0) \Gamma(k - \frac{1}{2}) \right)} \right| L, \end{aligned} \tag{13}$$

where L is a constant independent of k. We denote the first and second part of (13) as I_1 and I_2. Now we estimate I_1 and I_2 separately.

$$\begin{aligned} I_1 = & \left| \frac{(2m)^{\frac{1}{2}} \alpha_{k-s+\frac{1}{2}} m^{\frac{1}{2}+s-k} D^{k-s-1}}{\alpha_s m^{1-s} D^{s-\frac{3}{2}} \left(e(-rt_0) + (-1)^k e(rt_0) \right)} \right| \\ = & \frac{\sqrt{2} m^{\frac{1}{2}-2\delta} D^{2\delta}}{\left(e(-rt_0) + (-1)^k e(rt_0) \right)} \frac{\pi^{2\delta}}{2^{2\delta}} \left| \frac{\Gamma(\frac{k}{2} - \frac{1}{4} - \delta - it')}{\Gamma(\frac{k}{2} - \frac{1}{4} + \delta + it')} \right|. \end{aligned}$$

The only term depending on k is the ratio of gamma functions. By [13.2.1, [1]] we have

$$\left(\frac{k}{2} \right)^{2\delta+2it'} \frac{\Gamma(\frac{k}{2} - \frac{1}{4} - \delta - it')}{\Gamma(\frac{k}{2} - \frac{1}{4} + \delta + it')} \to 1, \text{ as } k \to \infty.$$

Hence $I_1 \to 0$ as $k \to \infty$. Now consider

$$\begin{aligned} I_2 = & \left| \frac{(2\pi)^{k-\frac{1}{2}} (-1)^{\frac{k-1}{2}} 2^{\frac{5}{2}-2k} m^{1-k} (-D)^{k-\frac{3}{2}}}{\alpha_s m^{1-s} D^{s-\frac{3}{2}} \left(e(-rt_0) + (-1)^k e(rt_0) \Gamma(k - \frac{1}{2}) \right)} \right| L \\ I_2 \leq & \frac{(2\pi)^{k-\frac{1}{2}} 2^{\frac{5}{2}-2k} m^{\frac{1}{4}-\frac{k}{2}-\delta} D^{\frac{k}{2}-\frac{1}{4}+\delta} L}{\left(e(-rt_0) + (-1)^k e(rt_0) \Gamma(k - \frac{1}{2}) \right)} \frac{2^{\frac{k}{2}-\frac{7}{4}-\delta}}{\pi^{\frac{k}{2}-\delta-\frac{1}{4}}} \left| \frac{\Gamma(\frac{k}{2} - \frac{1}{4} - \delta - it')}{\Gamma(k - \frac{1}{2})} \right| \\ \leq & \frac{(2\pi)^{k-\frac{1}{2}} 2^{\frac{5}{2}-2k} m^{\frac{1}{4}-\frac{k}{2}-\delta} D^{\frac{k}{2}-\frac{1}{4}+\delta} L}{\left(e(-rt_0) + (-1)^k e(rt_0) \Gamma(k - \frac{1}{2}) \right)} \frac{2^{\frac{k}{2}-\frac{7}{4}-\delta}}{\pi^{\frac{k}{2}-\delta-\frac{1}{4}}} \\ & \times \frac{1}{(k-\frac{3}{2})(k-\frac{5}{2})...(\left[\frac{k}{2}\right]+\frac{1}{2})} \left| \frac{\Gamma(\frac{k}{2} - \frac{1}{4} - \delta - it')}{\Gamma(\left[\frac{k}{2}\right]+\frac{1}{2})} \right|. \end{aligned}$$

The first term tends to zero, as before the ratio of gamma functions $\left| \frac{\Gamma(\frac{k}{2} - \frac{1}{4} - \delta - it')}{\Gamma(\left[\frac{k}{2}\right]+\frac{1}{2})} \right|$ $\to 0$ as $k \to \infty$. Hence $I_2 \to 0$ as $k \to \infty$. This gives contradiction to (13) and the theorem follows. □

Remark 3.1. In the above proof one can calculate k_0 explicitly such that $I_1 < \frac{1}{2}$ and $I_2 < \frac{1}{2}$ for $k > k_0$.

Proof of Theorem 2.3. Fist we prove statement of Theorem 2.3 for the region on the right of line of symmetry $\frac{k}{2}-\frac{1}{4}$. Let $s=\frac{k}{2}+\frac{1}{4}-\delta-it'$ with $0<\delta<\frac{1}{2},\ t'\in\mathbb{R}$ and $\mathfrak{B}_k$ be a basis of Hecke eigenforms. Then

$$\Omega_{t_0,s}^{k,m}(\tau,z)=\sum_{\phi_i\in\mathfrak{B}_k}\frac{<\Omega_{t_0,s}^{k,m},\phi_i>}{<\phi_i,\phi_i>}\phi_i(\tau,z).$$

Comparing the (n,r)-Fourier coefficient of both sides and using Theorem 2.2 for given s, there exist k_0 such that for $k>k_0$, $\omega(n,r)\neq 0$. Hence there exists a Hecke eigenform $\phi_{i_0}\in J_{k,m}^{cusp}$ such that

$$<\Omega_{t_0,s}^{k,m},\phi_{i_0}>=\frac{\pi}{2^{k-2}e^{\pi is/2}}\frac{\Gamma(k-3/2)}{\Gamma(s-1/2)\Gamma(k-s)}\frac{1}{2m}\sum_{\mu=1}^{2m}exp(-2\pi i\mu t_0)\Lambda_\mu(\bar{\phi_{i_0}},k-s)\neq 0.$$

Hence there exists a Hecke egienform ϕ_{i_0} and $\mu\in\{1,2,...,2m\}$ such that $\Lambda_\mu(\bar{\phi_{i_0}},\frac{k}{2}-\frac{1}{4}+\delta+it')\neq 0$. Now using the functional equation given in Theorem 1.4, there exists $\beta\in\{1,2,...,2m\}$ such that $\Lambda_\beta(\bar{\phi_{i_0}},\frac{k}{2}-\frac{1}{4}-\delta-it')\neq 0$. Hence the theorem follows. □

4. Nonvanishing of Jacobi Poincaré Series

Let $P_{n,r}^{k,m}$ be the (n,r)-th Poincaré series of weight k and index m (of exponential type) defined by

$$P_{n,r}^{k,m}(\tau,z):=\sum_{\gamma\in\Gamma^J\backslash\Gamma_\infty^J}e(n\tau+rz)|_{k,m}\gamma(\tau,z),$$

where

$$\Gamma_\infty^J:=\left\{\left(\begin{pmatrix}1&n\\0&1\end{pmatrix},(0,\nu)\right)|n\in\mathbb{Z},\nu\in\mathbb{Z}\right\}$$

with $p_{n,r}^{k,m}(n',r')$ its (n',r') coefficients. For the explicit expression for $p_{n,r}^{k,m}(n',r')$ one may refer to [5].

This is well known that $P_{n,r}^{k,m}, n\in\mathbb{Z}_{\geq 0}, r\in\mathbb{Z}$ generate the space $J_{k,m}^{cusp}$. Similar to the case of modular forms, one can ask the non-vanishing of Jacobi Poincaré series $P_{n,r}^{k,m}$. In this direction, Das [3] considered the non-vanishing of a Jacobi Poincaré series (for the general Jacobi group of any genus) analogous to Rankin's result [12] for the case of modular forms. We mention here the result for the case $g=1$.

Theorem 4.1. [3] *Suppose $m|r$. Then there exists an integer k_0 and constant $B>3\ log2$ such that for all even $k\geq k_0$, the Jacobi Poincaré series $P_{n,r}^{k,m}$ does not vanish identically when (here $D=4nm-r^2$)*

$$k-3/2\leq\frac{\pi D}{2m}\leq(k-3/2)^{1+2/9}exp\left(-\frac{B\ \log(k-3/2)}{\log\ \log(k-3/2)}\right).$$

Further Das [3] gave conditions of non-vanishing of the Poincaré series $P_{n,r}^{k,m}$ independent of the weight k.

Theorem 4.2. *Suppose that $\pi D > 2m$ (where $D = 4nm - r^2$). Then $P^{k,m}_{n,r} \neq 0$ provided*

$$exp\left(-\frac{B_1 \ \log(\pi D/m)}{\log \ \log 2(\pi D/m)}\right)\sigma_0(D)D < \frac{m^{8/7}}{2^{9/4}\pi}\left(\frac{2}{6^{2/3}}+\frac{54}{2^{5/6}}+\frac{16}{2^{3/4}}\right)^{-3/2},$$

where $\sigma_0(D) = \sum_{d|D} 1$.

As a consequence of our Theorem 2.2, we deduce nonvanishing of Jacobi Poincaré series. One can express the kernel function in terms of Poincaré series.

Proposition 4.3. [11] *Let k and m be positive integers with $k > 6$ and $t_0 \in (2m)^{-1}\mathbb{Z}$. If $s \in \mathbb{C}$ with $1 < Re(s) < k-3$, then*

$$\begin{aligned}\Omega^{k,m}_{t_0,s}(\tau,z) =& \frac{(2\pi)^{s-1/2}}{e^{\pi is/2}\Gamma(s-1/2)}\frac{1}{\sqrt{2m}}\sum_{\mu=1}^{2m} exp(-2\pi i\mu t_0) \\ &\times \sum_{\substack{D=1,\\ 4m|D+\mu^2}}^{\infty}\left(\frac{D}{4m}\right)^{s-3/2} P^{k,m}_{(D+\mu^2)/4m,\mu}(\tau,z).\end{aligned} \tag{14}$$

Now comparing the (n', r') coefficients both sides of the above expression (14),

$$\omega(n',r') = \frac{(2\pi)^{s-1/2}}{e^{\pi is/2}\Gamma(s-1/2)}\frac{1}{\sqrt{2m}}\sum_{\mu=1}^{2m} exp(-2\pi i\mu t_0)\sum_{\substack{D=1\\ 4m|D+\mu^2}}^{\infty}\left(\frac{D}{4m}\right)^{s-3/2} p^{k,m}_{(D+\mu^2)/4m,\mu}(n',r'),$$

equivalently,

$$\omega(n',r') = \frac{(2\pi)^{s-1/2}}{e^{\pi is/2}\Gamma(s-1/2)}\frac{1}{\sqrt{2m}}\sum_{\mu=1}^{2m} exp(-2\pi i\mu t_0)\sum_{\substack{D=1\\ 4m|D+\mu^2}}^{\infty}\left(\frac{D}{4m}\right)^{s-3/2} p^{k,m}_{n',r'}\left(\frac{(D+\mu^2)}{4m},\mu\right).$$

As mentioned in Remark 3.1 we have $\omega(n', r') \neq 0$ for $k > k_0$ (one can take $k_0 = \max\left\{\frac{3}{2}+2\frac{(2\sqrt{2}m^{2\delta}(\pi D')^{\frac{2}{1+4\delta}})}{e^{-2\pi i r' t_0}+(-1)^k e^{2\pi i r' t_0}},\ 4+2\left(\frac{\pi}{2\sqrt{2m}}D'\right)^2 e\right\}$ for $s = k/2 - \delta$ which we got using gamma function inequalities [9]). Then there exist μ, D with $4m|D+\mu^2$ such that $p^{k,m}_{n',r'}((D+\mu^2)/4m, \mu) \neq 0$ which implies $P^{k,m}_{n',r'} \neq 0$. Hence we have the following theorem.

Theorem 4.4. *Given positive integers m, n' and r' with $m \nmid r'$, choose δ such that $\frac{1}{2} - 2\delta \in (\frac{3+\sqrt{11}}{33}, 1)$ then for $k > k_0$ the Poincare series $P^{k,m}_{n',r'} \neq 0$, where*

$$k_0 = max\left\{\frac{3}{2}+2\frac{(2\sqrt{2}m^{2\delta}(\pi D')^{\frac{2}{1+4\delta}})}{e^{-2\pi i r' t_0}+(-1)^k e^{2\pi i r' t_0}},\ 4+2\left(\frac{\pi}{2\sqrt{2m}}D'\right)^2 e\right\}.$$

Remark 4.1. Kohnen proved that $r_{k,s}(1) \neq 0$ for large k in [7]. Using similar arguments for any given n one can prove that $r_{k,s}(n) \neq 0$ for large k and deduce the nonvanishing of Poincaré Series in case of modular forms.

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