IDENTITIES FOR THE RAMANUJAN TAU FUNCTION AND CERTAIN CONVOLUTION SUM IDENTITIES FOR THE DIVISOR FUNCTIONS

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Dedicated to Professor R. Balasubramanian on the occasion of his 60th birthday

Abstract. We give a list of (van der Pol type) identities for the Ramanujan tau function. As consequences, we obtain congruences for the tau function and further derive convolution sums and congruence relations involving the divisor functions.

1. Introduction

In 1916, S. Ramanujan [14] studied the function

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n,\ q=e^{2\pi iz},\ z\in\mathbb{C},\ \mathrm{Im}(z)>0,$$

which is the normalized cusp form of weight 12 on $SL_2(\mathbb{Z})$. Its n-th Fourier coefficient $\tau(n)$ is called the Ramanujan tau function and it is an important arithmetic function having many applications in number theory. Ramanujan observed that the logarithmic derivative of $\Delta(z)$ gives the Eisenstein series $E_2(z)$, which is a quasi-modular form of weight 2 on $SL_2(\mathbb{Z})$. In fact he showed that

$$(1-n)\tau(n) = 24(\sigma(1)\tau(n-1)+\sigma(2)\tau(n-2)+\cdots+\sigma(n-1)\tau(1)) = 24\sum_{m=1}^{n-1}\sigma(m)\tau(n-m), \tag{1}$$

where $\sigma(n)$ is the number of divisors of n. Ramanujan also gave a recurrence formula for $\tau(n)$ from which he calculated the first 30 values of $\tau(n)$. In the literature, there are various works on finding a formula for $\tau(n)$ which involves the convolution sum of the divisor functions. In 1951, B. van der Pol [12] derived identities relating $\tau(n)$ to convolution sum of divisor functions by using differential equations satisfied by $\Delta(z)$. For example, he obtained the following formula:

$$\tau(n) = n^2\sigma_7(n) - 540\sum_{m=1}^{n-1}m(n-m)\sigma_3(m)\sigma_3(n-m), \tag{2}$$

where $\sigma_k(n) = \sum_{d|n}d^k$ is the sum of the k-th powers of divisors of n and we write $\sigma(n)$ instead of $\sigma_1(n)$. In 1975, D. Niebur [11] used logarithmic derivatives of $\Delta(z)$

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to derive a formula for $\tau(n)$ similar to (2), but has the feature that only the divisor function $\sigma(n)$ appears in the formula (see (xi) of Theorem 2.1 below). In 2004, D. Lanphier [8] used differential operators studied by H. Maass [9] to prove identities for $\tau(n)$ including (2). The identity of Niebur did not occur in the work of Lanphier. As a consequence to these identities of $\tau(n)$, Lanphier deduced certain congruences satisfied by $\tau(n)$. In [13], we used the Rankin-Cohen brackets (for both modular and quasimodular forms) to prove all the identities derived by Lanphier and Niber's identity. Our method also gave rise to some more new identities for $\tau(n)$.

In this article, we show that holomorphic differential operators are sufficient to get the identities. In fact, we show that in Lanphier's proof one can replace the Maass operator δ_k by the usual differential operator $D := \frac{1}{2\pi i}\frac{d}{dz}$. Using the basic theory of modular forms, it can be shown that they are cusp forms of weight 12 on $SL_2(\mathbb{Z})$ and hence should be a multiple of $\Delta(z)$. We list all the identities of $\tau(n)$ as one theorem and prove them using elementary methods. For each identity we give a linear combination involving the Eisenstein series E_k ($k \geq 2$) and its derivatives and show that the linear combination is a cusp form of weight 12 on $SL_2(\mathbb{Z})$. The corresponding identity for the $\tau(n)$ would then follow by equating the n-th Fourier coefficient of the particular linear combination with $\tau(n)$. For some of the identities, we use the fact that there is no cusp form of weight 14 on $SL_2(\mathbb{Z})$.

Our second theorem uses the identities of $\tau(n)$ obtained in Theorem 2.1 to derive formulas for the convolution sums of the type $\sum_{m=1}^{n-1} m^k\sigma_r(m)\sigma_s(n-m)$. Such sums have a long history, having been studied as early as the 19th century (see [1], [4]). Several identities involving convolution sums were found by Ramanujan [14]. Explicit evaluations of similar sums have been found by several authors (see for example [5, 18] and the references therein).

Finally, as a consequence to our theorems, we get certain congruences for the $\tau(n)$ and also for the divisor functions (Corollary 2.7 and Corollary 2.11 respectively).

2. Statement of results

Theorem 2.1.

$$\text{(i)} \quad \tau(n) = n^2\sigma_7(n) - 540\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m),$$

$$\text{(ii)} \quad \tau(n) = -\frac{5}{4}n^2\sigma_7(n) + \frac{9}{4}n^2\sigma_3(n) + 540\sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_3(n-m),$$

$$\text{(iii)} \quad \tau(n) = -\frac{11}{24}n\sigma_9(n) + \frac{35}{24}n\sigma_5(n) + 350\sum_{m=1}^{n-1} (n-m)\sigma_3(m)\sigma_5(n-m),$$

$$\text{(iv)} \quad \tau(n) = \frac{11}{36}n\sigma_9(n) + \frac{25}{36}n\sigma_3(n) - 350\sum_{m=1}^{n-1} m\sigma_3(m)\sigma_5(n-m),$$

$$
\begin{aligned}
\text{(v)} \quad \tau(n) &= \frac{1}{6}n\sigma_9(n) + \frac{5}{6}n\sigma_3(n) - \frac{420}{n}\sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_5(n-m),\\
\text{(vi)} \quad \tau(n) &= \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{1382}{3n}\sum_{m=1}^{n-1} m\sigma_5(m)\sigma_5(n-m),\\
\text{(vii)} \quad \tau(n) &= -\frac{91}{600}\sigma_{11}(n) + \frac{691}{600}\sigma_3(n) + \frac{2764}{5n}\sum_{m=1}^{n-1} m\sigma_3(m)\sigma_7(n-m),\\
\text{(viii)} \quad \tau(n) &= -\frac{91}{600}\sigma_{11}(n) + \frac{691}{600}\sigma_7(n) + \frac{1382}{5n}\sum_{m=1}^{n-1} (n-m)\sigma_3(m)\sigma_7(n-m),\\
\text{(ix)} \quad \tau(n) &= \frac{5}{12}n\sigma_3(n) + \frac{7}{12}n\sigma_5(n) + 70\sum_{m=1}^{n-1} (2n-5m)\sigma_3(m)\sigma_5(n-m),\\
\text{(x)} \quad \tau(n) &= n^2\sigma_3(n) + 60\sum_{m=1}^{n-1} (9m^2-5mn)\sigma_3(m)\sigma_3(n-m),\\
\text{(xi)} \quad \tau(n) &= n^4\sigma(n) - 24\sum_{m=1}^{n-1} (35m^4-52m^3n+18m^2n^2)\sigma(m)\sigma(n-m),\\
\text{(xii)} \quad \tau(n) &= n^4(7\sigma(n)-6\sigma_3(n)) - 168\sum_{m=1}^{n-1} (5m^4-4m^3n)\sigma(m)\sigma(n-m),\\
\text{(xiii)} \quad \tau(n) &= \frac{15}{32}n\sigma(n) - \frac{33}{32}n\sigma_9(n) + \frac{25}{16}n^2\sigma_7(n) + 225\sum_{m=1}^{n-1} m\sigma(m)\sigma_7(n-m),\\
\text{(xiv)} \quad \tau(n) &= \frac{6}{7}n^2\sigma(n) - \frac{9}{7}n^3\sigma_5(n) + \frac{10}{7}n^2\sigma_7(n) - 432\sum_{m=1}^{n-1} m^2\sigma(m)\sigma_5(n-m),\\
\text{(xv)} \quad \tau(n) &= \frac{14}{5}n^3\sigma(n) + \frac{12}{5}n^4\sigma_3(n) - \frac{21}{5}n^3\sigma_5(n) + 672\sum_{m=1}^{n-1} m^3\sigma(m)\sigma_3(n-m),\\
\text{(xvi)} \quad \tau(n) &= \frac{5}{12}n\sigma(n) + \frac{25}{24}n\sigma_7(n) - \frac{11}{24}n\sigma_9(n) + 25\sum_{m=1}^{n-1} (9m-n)\sigma(m)\sigma_7(n-m),\\
\text{(xvii)} \quad \tau(n) &= \frac{9}{14}n^2\sigma(n) + \frac{5}{14}n^2\sigma_7(n) - 108\sum_{m=1}^{n-1} (4m^2-mn)\sigma(m)\sigma_5(n-m),\\
\text{(xviii)} \quad \tau(n) &= \frac{8}{5}n^3\sigma(n) - \frac{3}{5}n^3\sigma_5(n) + 96\sum_{m=1}^{n-1} (7m^3-3m^2n)\sigma(m)\sigma_3(n-m),\\
\text{(xix)} \quad \tau(n) &= \frac{1}{2}n^2\sigma(n) + \frac{1}{2}n^2\sigma_5(n) - 12\sum_{m=1}^{n-1} (36m^2-16mn+n^2)\sigma(m)\sigma_5(n-m),\\
\text{(xx)} \quad \tau(n) &= n^3\sigma(n) - 24\sum_{m=1}^{n-1} (21m^2n-28m^3-3mn^2)\sigma(m)\sigma_3(n-m).
\end{aligned}
$$

Remark 2.2. All the identities in Theorem 2.1 also appeared in [13, Theorems 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7], where we use Rankin-Cohen brackets of modular and quasimodular forms to prove these identities.

Remark 2.3. In the above theorem, identity (i) was proved by B. van der Pol [12], identity (xi) was proved by D. Niebur [11] and identities (i) to (viii) were obtained by D. Lanphier [8].

Remark 2.4. In [3, Proposition 6.8] A. M. El Gradechi presents a general approach using the Lie theoretic structure on spaces of modular forms of integral weight that arises through the Rankin-Cohen bracket. However, only the identities (i), (ii) and (ix) follow from this approach. In fact, to get (i) and (ii) one uses the bracket $[E_4, E_4]_2$ and Theorem 3.1 (i). To get (ix), one uses the bracket $[E_4, E_6]_1$. The other identity comes from the bracket $[E_6, E_6]_0$ and is given in Theorem 3.1 (iv). (In [3, Table 3] these are given by (A5), (A4) and (A3) respectively.)

Remark 2.5. We note that the identity (x) in Theorem 2.1 has other possible expressions like in equation (1) in [13]:

$$\tau(n) = n^2\sigma_3(n) + 60\sum_{m=1}^{n-1}(2n-3m)(n-3m)\sigma_3(m)\sigma_3(n-m),$$

and as given by Resnikoff [16] or in [3, p. 428]:

$$\tau(n) = n^2\sigma_3(n) + 60\sum_{m=1}^{n-1}(4n^2-13mn+9m^2)\sigma_3(m)\sigma_3(n-m).$$

We see that the differences of the convolution sums on the right hand side of the above three identities are of the form $\sum_{m=1}^{n-1} p(n,m)\sigma_3(m)\sigma_3(n-m)$ where $p(x,y)$ is a polynomial in two variables with the property that $p(n,n-m) = -p(n,m)$ and with this condition, one concludes that $\sum_{m=1}^{n-1} p(n,m)\sigma_3(m)\sigma_3(n-m) = 0$. Therefore, all the three identities are equivalent. In fact, we observe that

$$\begin{aligned}\sum_{m=1}^{n-1}(9m^2-5mn)\sigma_3(m)\sigma_3(n-m) &= \sum_{m=1}^{n-1}(2n-3m)(n-3m)\sigma_3(m)\sigma_3(n-m)\\ &= \sum_{m=1}^{n-1}(4n^2-13mn+9m^2)\sigma_3(m)\sigma_3(n-m).\end{aligned}$$

The difference of the first two identities is equal to

$$2n\sum_{m=1}^{n-1}(n-2m)\sigma_3(m)\sigma_3(n-m) = 0.$$

Remark 2.6. In [8], Lanphier also derived the following identities for $\tau(n)$. However, these identities are not new as they can be obtained using some of the identities of Theorem 2.1.

$$
\begin{aligned}
\text{(a)}\quad \tau(n) &= n^2\sigma_7(n) - \frac{1080}{n}\sum_{m=1}^{n-1} m^2(n-m)\sigma_3(m)\sigma_3(n-m),\\
\text{(b)}\quad \tau(n) &= -\frac{1}{2}n^2\sigma_7(n) + \frac{3}{2}n^2\sigma_3(n) + \frac{360}{n}\sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_3(n-m),\\
\text{(c)}\quad \tau(n) &= n\sigma_9(n) - \frac{2100}{n}\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_5(n-m),\\
\text{(d)}\quad \tau(n) &= -\frac{1}{4}n\sigma_9(n) + \frac{5}{4}n\sigma_5(n) + \frac{300}{n}\sum_{m=1}^{n-1} (n-m)^2\sigma_3(m)\sigma_5(n-m).
\end{aligned}
$$

Taking derivative of the modular form identity giving (i) (see the proof of Theorem 2.1) and then comparing the n-th Fourier coefficients, we obtain (a). Similarly, taking the derivative of the modular form identity for (ii) and then using (a), we get (b). Multiplying identity (v) by 5/6 and subtracting it from identity (iv), we get (c). Identity (d) follows from the fact that $4D\Delta - \frac{1}{264}D^2E_{10} + \frac{5}{504}D^2E_6E_4$ is a cusp form of weight 14 on $SL_2(\mathbb{Z})$ and hence it is zero. To prove this fact, we use identity (iii). We also get the following identities by using identities (xi) and (xii).

$$
\begin{aligned}
\text{(e)}\quad \tau(n) &= n^4\sigma_3(n) - 168\sum_{m=1}^{n-1}(5m^4 - 8m^3n + 3m^2n^2)\sigma(m)\sigma(n-m),\\
\text{(f)}\quad \tau(n) &= \frac{n^4}{3}(7\sigma(n) - 4\sigma_3(n)) - 56\sum_{m=1}^{n-1}(15m^4 - 20m^3n + 6m^2n^2)\sigma(m)\sigma(n-m).
\end{aligned}
$$

To get (e), we multiply (xi) by 7 and subtract (xii) from it. Identity (f) is obtained by adding 7 times (xi) and 2 times (xii).

In the following we derive certain congruences for $\tau(n)$, which are consequences of the identities given in Theorem 2.1.

Corollary 2.7.

$$
\begin{aligned}
\text{(i)}\quad \tau(n) &\equiv n^3\sigma(n) \equiv n^4\sigma(n) \pmod{2^3\cdot 3},\\
\text{(ii)}\quad \tau(n) &\equiv n^2\sigma_3(n) \pmod{2^2\cdot 3\cdot 5},\\
\text{(iii)}\quad \tau(n) &\equiv n^2\sigma(n) \pmod{2^2\cdot 3},\\
\text{(iv)}\quad 5\tau(n) &\equiv n^3(8\sigma(n) - 3\sigma_5(n)) \pmod{2^5\cdot 3},\\
\text{(v)}\quad \tau(n) &\equiv n^4(7\sigma(n) - 6\sigma_3(n)) \pmod{2^3\cdot 3\cdot 7}.
\end{aligned}
$$

Proof. Since the identities (i) to (viii) are already known due to Lanphier (and van der Pol), we use only the identities (ix) to (xx) of Theorem 2.1. Below we give the corresponding congruence relation which follows from each of the identity.

$$\begin{array}{rrcl}
(a) & 12\tau(n) & \equiv & 5n\sigma_3(n) + 7n\sigma_5(n) \pmod{2^3\cdot 3\cdot 5\cdot 7},\\
(b) & \tau(n) & \equiv & n^2\sigma_3(n) \pmod{2^2\cdot 3\cdot 5},\\
(c) & \tau(n) & \equiv & n^4\sigma(n) \pmod{2^3\cdot 3},\\
(d) & \tau(n) & \equiv & n^4(7\sigma(n) - 6\sigma_3(n)) \pmod{2^3\cdot 3\cdot 7},\\
(e) & 32\tau(n) & \equiv & 15n\sigma(n) + 50n^2\sigma_7(n) - 33n\sigma_9(n) \pmod{2^5\cdot 3^2\cdot 5^2},\\
(f) & 7\tau(n) & \equiv & 6n^2\sigma(n) - 9n^3\sigma_5(n) + 10n^2\sigma_7(n) \pmod{2^4\cdot 3^3\cdot 7},\\
(g) & 5\tau(n) & \equiv & 14n^3\sigma(n) + 12n^4\sigma_3(n) - 21n^3\sigma_5(n) \pmod{2^5\cdot 3\cdot 5\cdot 7},\\
(h) & 24\tau(n) & \equiv & 10n\sigma(n) + 25n\sigma_7(n) - 11n\sigma_9(n) \pmod{2^3\cdot 3\cdot 5^2},\\
(i) & 14\tau(n) & \equiv & 9n^2\sigma(n) + 5n^2\sigma_7(n) \pmod{2^3\cdot 3^3\cdot 7},\\
(j) & 5\tau(n) & \equiv & 8n^3\sigma(n) - 3n^3\sigma_5(n) \pmod{2^5\cdot 3\cdot 5},\\
(k) & 2\tau(n) & \equiv & n^2(\sigma(n) + \sigma_5(n)) \pmod{2^3\cdot 3},\\
(l) & \tau(n) & \equiv & n^3\sigma(n) \pmod{2^3\cdot 3}.
\end{array}$$

Some of the above congruences are already known (for example (f) and (g) are corresponding to (7.15) and (5.6) of [7]) and some are new. We now prove the congruences of the corollary using the above list of congruences. The congruences (c) and (l) give the congruence (i) and (b) gives the congruence (ii). The rest of the congruences (iii), (iv), (v) follow from (h), (j) and (d) respectively. From (iv) we also get the following congruence:

$$\tau(n) \equiv n^3\sigma(n) \pmod{2^4}. \tag{3}$$

□

Our next theorem gives identities for the convolution sums of the divisor functions, which is a consequence of Theorem 2.1.

Theorem 2.8.

$$\begin{array}{lrcl}
\text{(i)} & \displaystyle\sum_{m=1}^{n-1} m^2\sigma(m)\sigma_3(n-m) & = & \displaystyle -\frac{1}{240}n^2\sigma(n) - \frac{1}{120}n^3\sigma_3(n) + \frac{1}{80}n^2\sigma_5(n),\\
\text{(ii)} & \displaystyle\sum_{m=1}^{n-1} m\sigma(m)\sigma_3(n-m) & = & \displaystyle -\frac{1}{240}n\sigma(n) - \frac{1}{40}n^2\sigma_3(n) + \frac{7}{240}n\sigma_5(n),\\
\text{(iii)} & \displaystyle\sum_{m=1}^{n-1} m\sigma(m)\sigma_5(n-m) & = & \displaystyle \frac{1}{504}n\sigma(n) - \frac{1}{84}n^2\sigma_5(n) + \frac{5}{504}n\sigma_7(n),\\
\text{(iv)} & \displaystyle\sum_{m=1}^{n-1} \sigma(m)\sigma_5(n-m) & = & \displaystyle \frac{1}{504}\sigma(n) - \frac{1}{12}n\sigma_5(n) + \frac{1}{24}\sigma_5(n) + \frac{5}{126}\sigma_7(n),\\
\text{(v)} & \displaystyle\sum_{m=1}^{n-1} \sigma(m)\sigma_7(n-m) & = & \displaystyle -\frac{1}{480}\sigma(n) + \frac{1}{24}\sigma_7(n) + \frac{11}{480}\sigma_9(n) - \frac{1}{16}n\sigma_7(n),
\end{array}$$

$$\text{(vi)} \qquad \sum_{m=1}^{n-1} m\sigma_3(m)\sigma_3(n-m) = \frac{n}{240}(\sigma_7(n)-\sigma_3(n)),$$

$$\text{(vii)} \qquad \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m) = \frac{11}{5040}\sigma_9(n)+\frac{1}{504}\sigma_3(n)-\frac{1}{240}\sigma_5(n),$$

$$\text{(viii)} \qquad \sum_{m=1}^{n-1} m\sigma(m)\sigma(n-m) = \frac{1}{24}n(1-6n)\sigma(n)+\frac{5}{24}n\sigma_3(n),$$

$$\text{(ix)} \qquad \sum_{m=1}^{n-1} m^2\sigma(m)\sigma(n-m) = \frac{1}{8}n^2\sigma_3(n)-\frac{1}{24}n^2(4n-1)\sigma(n),$$

$$\text{(x)} \qquad \sum_{m=1}^{n-1} m^3\sigma(m)\sigma(n-m) = \frac{1}{12}n^3\sigma_3(n)-\frac{1}{24}n^3(3n-1)\sigma(n).$$

Remark 2.9. We note that the identities (viii) and (implicitly) (ix) were derived by J. W. L. Glaisher [4] and the identities (iv), (v), (vii) appear in the work of Ramanujan [14, Table IV], [15, p. 146]. On the other hand, P. A. MacMahon [10, p.203] obtained the identities (ii), (iv). All the identities appear in the work of D. B. Lahiri [6] (the corresponding numbers in the work of Lahiri are given by (5.8), (5.4), (7.7), (7.2), (9.2), (7.5), (9.1), (3.2), (3.4) and (3.6)) and also in the work of J. G. Huard et. al [5].

Using the convolution identity (x) of Theorem 2.8 in Theorem 2.1 (xii), we get another identity for $\tau(n)$, which is given in the following corollary.

Corollary 2.10.

$$\tau(n) = 50n^4\sigma_3(n) - 7n^4(12n-5)\sigma(n) - 840\sum_{m=1}^{n-1} m^4\sigma(m)\sigma(n-m). \tag{4}$$

We end this section by stating some congruence relations among the divisor functions. Each congruence is obtained corresponding to each of the identity in Theorem 2.8. We observe that all these congruences were derived by Lahiri [6].

Corollary 2.11.

$$\text{(i)} \qquad 3\sigma_5(n) \equiv \sigma(n)+2n\sigma_3(n) \pmod{2^4\cdot 3\cdot 5},\ \gcd(n,30)=1,$$

$$\text{(ii)} \qquad 7\sigma_5(n) \equiv \sigma(n)+6n\sigma_3(n) \pmod{2^4\cdot 3\cdot 5},\ \gcd(n,30)=1,$$

$$\text{(iii)} \qquad 6n\sigma_5(n) \equiv \sigma(n)+5\sigma_7(n) \pmod{2^3\cdot 3^2\cdot 7},\ \gcd(n,42)=1,$$

$$\text{(iv)} \qquad 21(2n-1)\sigma_5(n) \equiv \sigma(n)+20\sigma_7(n) \pmod{2^3\cdot 3^2\cdot 7},\ \gcd(n,42)=1,$$

$$\text{(v)} \qquad 11\sigma_9(n) \equiv \sigma(n)+10(3n-2)\sigma_7(n) \pmod{2^5\cdot 3\cdot 5},\ \gcd(n,30)=1,$$

$$\begin{array}{rrcl}
\text{(vi)} & \sigma_3(n) & \equiv & \sigma_7(n) \pmod{2^4\cdot 3\cdot 5},\ \gcd(n,30)=1,\\
\text{(vii)} & 21\sigma_5(n) & \equiv & 11\sigma_9(n)+10\sigma_3(n) \pmod{2^4\cdot 3^2\cdot 5\cdot 7},\ \gcd(n,210)=1,\\
\text{(viii)} & (6n-1)\sigma(n) & \equiv & 5\sigma_3(n) \pmod{2^3\cdot 3},\ \gcd(n,6)=1,\\
\text{(ix)} & (4n-1)\sigma(n) & \equiv & 3\sigma_3(n) \pmod{2^3\cdot 3},\ \gcd(n,6)=1,\\
\text{(x)} & (3n-1)\sigma(n) & \equiv & 2\sigma_3(n) \pmod{2^3\cdot 3},\ \gcd(n,6)=1.
\end{array}$$

Remark 2.12. The congruences (i) and (ii) of the above corollary give the following.

$$n\sigma_3(n) \equiv \sigma_5(n) \pmod{60}, \text{ if } \gcd(n,30)=1. \tag{5}$$

Similarly, congruences (iii) and (iv) give the congruence

$$(12n-7)\sigma_5(n) \equiv 5\sigma_7(n) \pmod{168}, \text{ if } \gcd(n,42)=1. \tag{6}$$

From (ix) and (x), we see that

$$n\sigma(n) \equiv \sigma_3(n) \pmod{24}, \text{ if } \gcd(n,6)=1, \tag{7}$$

and using this in the congruence (viii) we get

$$(n-1)\sigma(n) \equiv 0 \pmod{24}, \text{ if } \gcd(n,6)=1. \tag{8}$$

3. Preliminaries

In this section we shall provide some well-known facts about modular forms which are needed to prove the results. For basic details of the theory of modular forms and quasimodular forms, we refer to [17, 2].

For an even integer $k \geq 4$, let $M_k(1)$ (resp. $S_k(1)$) denote the vector space of modular forms (resp. cusp forms) of weight k for the full modular group $SL_2(\mathbb{Z})$. Let E_k be the normalized Eisenstein series of weight k in $M_k(1)$, given by

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n, \tag{9}$$

where B_k is the k-th Bernoulli number defined by $\dfrac{x}{e^x-1} = \displaystyle\sum_{m=0}^{\infty}\frac{B_m}{m!}x^m$. The first few Eisenstein series are given below.

(10)

$$E_4(z) = 1+240\sum_{n\geq 1}\sigma_3(n)q^n,\ E_6(z) = 1-504\sum_{n\geq 1}\sigma_5(n)q^n,\ E_8(z) = 1+480\sum_{n\geq 1}\sigma_7(n)q^n,$$

$$E_{10}(z) = 1-264\sum_{n\geq 1}\sigma_9(n)q^n,\ E_{12}(z) = 1+\frac{65520}{691}\sum_{n\geq 1}\sigma_{11}(n)q^n.$$

Further, the Eisenstein series $E_2(z)$ has the following Fourier expansion:

$$E_2(z) = 1-24\sum_{n\geq 1}\sigma(n)q^n.$$

Since the space $M_k(1)$ is one-dimensional for $4 \leq k \leq 10$ and for $k=14$ and $S_{12}(1)$ is one-dimensional, we have the following well-known identities:

Theorem 3.1.

(i) $E_8(z) = E_4^2(z)$, (iii) $E_{12}(z) - E_8(z)E_4(z) = \left(\frac{65520}{691} - 720\right)\Delta(z)$,

(ii) $E_{10}(z) = E_4(z)E_6(z)$, (iv) $E_{12}(z) - E_6^2(z) = \left(\frac{65520}{691} + 1008\right)\Delta(z)$.

4. PROOFS

4.1. **Proof of Theorem 2.1.** As mentioned in the Introduction, all the identities of Theorem 2.1 follow using basic theory of modular forms. In this section we present a proof using elementary methods. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have the following transformation properties of the Eisenstein series $E_k(z)$, $k \geq 2$.

(11)

$$\text{(i) } E_k(\gamma(z)) = (cz+d)^k E_k(z) \quad (k \geq 4); \quad \text{(ii) } E_2(\gamma(z)) = (cz+d)^2 E_2(z) - \frac{6ic}{\pi}(cz+d),$$

where $\gamma(z) = \frac{az+b}{cz+d}$.

The space of cusp forms of weight 12 on $SL_2(\mathbb{Z})$ is one dimensional, generated by $\Delta(z)$. A direct verification using (11)(i) shows that $D^2E_8 - \frac{9}{2}(DE_4)^2 = 480\Delta$, since the space of cusp forms of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$ is one-dimensional generated by $\Delta(z)$. Comparing the n-th Fourier coefficients of the above identity proves (i). The remaining identities follow in a similar way. In the following we list only the modular identities, whose proof uses both the transformations in (11). The respective identities would follow by comparing the n-th Fourier coefficients both the sides.

(ii) $384\Delta = -D^2E_8 + \frac{18}{5}E_4D^2E_4$,

(iii) $576\Delta = DE_{10} - \frac{5}{3}E_4DE_6$,

(iv) $864\Delta = -DE_{10} + \frac{5}{2}E_6DE_4$,

(vi) Apply the D operator on $\frac{762048}{691}\Delta = E_{12} - E_6^2$,

(vii) Apply the D operator on $-\frac{432000}{691}\Delta = E_{12} - E_4^3$ and replace $E_4^2 = E_8$,

(ix) $1728\Delta = 2E_4DE_6 - 3E_6DE_4$,

(x) $960\Delta = 4E_4D^2E_4 - 5(DE_4)^2$,

(xi) $24\Delta = -E_2D^4E_2 + 16DE_2D^3E_2 - 18(D^2E_2)^2$,

(xii) $24\Delta = -7E_2D^4E_2 + 28(D^3E_2)DE_2 - \frac{3}{5}D^4E_4$,

(xiii) $\frac{256}{5}\Delta = -DE_2E_8 + \frac{1}{5}DE_{10} + \frac{1}{6}D^2E_8,$

(xiv) $28\Delta = -E_6D^2E_2 + \frac{1}{12}D^2E_8 + \frac{1}{14}D^3E_6,$

(xv) $\frac{60}{7}\Delta = -E_4D^3E_2 + \frac{1}{14}D^3E_6 + \frac{3}{35}D^4E_4,$

(xvi) $\frac{2304}{5}\Delta = E_2DE_8 - 8DE_2E_8 + \frac{4}{5}DE_{10},$

(xvii) $224\Delta = 2DE_2DE_6 - 6D^2E_2E_6 + \frac{1}{6}D^2E_8,$

(xviii) $60\Delta = 3D^2E_2DE_4 - 4D^3E_2E_4 + \frac{1}{14}D^3E_6,$

(xix) $1008\Delta = -E_2D^2E_6 + 14DE_2DE_6 - 21E_6D^2E_2,$

(xx) $240\Delta = -3DE_2D^2E_4 + 15D^2E_2DE_4 - 10E_4D^3E_2.$

It remains to prove (v) and (viii). First we prove (v). Differentiating identities (iii) and (iv), we get

$$576D\Delta = D^2E_{10} - \frac{5}{3}DE_4DE_6 - \frac{5}{3}E_4D^2E_6,$$
$$864D\Delta = -D^2E_{10} + \frac{5}{2}DE_4DE_6 + \frac{5}{2}E_6D^2E_4.$$

As there is no non-zero cusp form of weight 14 for $SL_2(\mathbb{Z})$, we get the following identity:

$$10E_4D^2E_6 - 35DE_4DE_6 + 21E_6D^2E_4 = 0.$$

Using this, substitute $E_4D^2E_6$ in the first identity for $D\Delta$ and eliminate the term DE_4DE_6 to get

$$3168D\Delta = -2D^2E_{10} + 11E_6D^2E_4,$$

from which (v) follows. Now we prove (viii). Note that $E_8DE_4 = \frac{1}{2}E_4DE_8$, this also follows from the fact that there is no non-zero cusp form of weight 14. Use this in the identity $-\frac{432000}{691}D\Delta = D(E_{12} - E_4E_8)$ to get (viii). This completes the proof.

4.2. **Proof of Theorem 2.8.** Subtracting identity (xviii) from (xv) in Theorem 2.1 gives the identity (i). Similarly, identities (ii) and (iii) follow using the identities (xviii), (xx) and the identities (xiv), (xvii) of Theorem 2.1 respectively. Using identities (xiv) and (xix) of Theorem 2.1 and identity (iii) of Theorem 2.8, we get (iv). To get identity (v) we use (xiii) and (xvi) of Theorem 2.1 and to get identity (vi), we use (i) and (ii) of Theorem 2.1. Identity (vii) follows using (iii) and (iv) of Theorem 2.1. It can also be derived using Theorem 3.1 (ii) (by comparing the n-th Fourier coefficients). Now we prove the identity (viii). Ramanujan discovered the following (see [2, Proposition 15]).

$$12DE_2(z) = E_2^2(z) - E_4(z). \tag{12}$$

Taking derivative of the above and comparing the n-th Fourier coefficients, we obtain the identity (viii). Observing that the sums of the type $\sum p(m,n)\sigma(m)\sigma(n-m)$ can be symmetrized by $\frac{1}{2}\sum\left(p(m,n)+p(n-m,n)\right)\sigma(m)\sigma(n-m)$, we see that

$$\sum_{m=1}^{n-1}(2m^3-3m^2n+mn^2)\sigma(m)\sigma(n-m)=0, \tag{13}$$

as $2m^3-3m^2n+mn^2=-(2(n-m)^3-3(n-m)^2n+(n-m)n^2)$. Using the identity for $\sum m\sigma(m)\sigma(n-m)$ from (viii) in equation (13), we obtain the following:

$$\sum_{m=1}^{n-1}(2m^3-3m^2n)\sigma(m)\sigma(n-m)=-\frac{1}{24}n^3\sigma(n)+\frac{1}{4}n^4\sigma(n)-\frac{5}{24}n^3\sigma_3(n). \tag{14}$$

Moreover, subtracting identity (xi) from (xii) of Theorem 2.1, we get the following:

$$\sum_{m=1}^{n-1}(4m^3-3m^2n)\sigma(m)\sigma(n-m)=\frac{1}{24}n^3\sigma(n)-\frac{1}{24}n^3\sigma_3(n). \tag{15}$$

Now the identities (ix) and (x) are derived using equations (14) and (15). The proof of Theorem 2.8 is now complete.

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