RANKIN-COHEN BRACKETS ON JACOBI FORMS OF SEVERAL VARIABLES AND SPECIAL VALUES OF CERTAIN DIRICHLET SERIES

ABHASH KUMAR JHA AND BRUNDABAN SAHU

Abstract. We construct certain Jacobi cusp forms of several variables by computing the adjoint of linear maps constructed using Rankin-Cohen type differential operators with respect to the Petersson scalar product. We express the Fourier coefficients of the Jacobi cusp forms constructed, in terms of special values of the shifted convolution of Dirichlet series of Rankin-Selberg type. This is a generalization of an earlier work of the authors on Jacobi forms to the case of Jacobi forms of several variables.

1. Introduction

Kohnen [9] constructed elliptic cusp forms by computing the adjoint of the product map by a fixed cusp form with respect to the Petersson scalar product. The Fourier coefficients of cusp form constructed using this method involves special value of the shifted convolution of certain Dirichlet series of Rankin type associated to modular forms. Rankin-Cohen brackets of two modular forms f and g is again a modular form constructed by taking a certain combination of the derivatives of f and g. Rankin-Cohen bracket is a generalization of the usual product. Herrero [6] generalized the work of Kohnen [9] by computing the adjoint of linear maps constructed using Rankin-Cohen brackets instead of the product by a fixed cusp form.

The work of Kohnen [9] has been extended to the case of Jacobi forms by Choie, Kim and Knopp [4] and Sakata [11]. Rankin-Cohen brackets for Jacobi forms were studied by Choie [2] by using heat operators. Recently, the work of Herrero [6] has been extended to the case of Jacobi forms [7] and Siegel modular forms [8] by the authors. We explicitly computed the adjoint of linear maps constructed using Rankin-Cohen brackets with respect to the Petersson scalar product and expressed the Fourier coefficient of image of a Jacobi (or Siegel) cusp form under the adjoint map in terms of special values of the shifted convolution of Dirichlet series of Rankin type associated to Jacobi (or Siegel) cusp forms.

In this article we generalize our work [7] on Jacobi forms to the case of Jacobi forms of several variables. More precisely, we compute the adjoint of certain linear maps constructed using Rankin-Cohen brackets on Jacobi forms of several variables. In our main theorem, we express the Fourier coefficients of the image of a form under adjoint map in terms of special values of certain Dirichlet series of Rankin-Selberg type associated to Jacobi forms of several variables.

In section 2, we briefly recall some basic definitions and Rankin-Cohen brackets on Jacobi forms of several variables, then we state our main result (Theorem 3.1) in section 3. In

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section 4, we give a proof of Theorem 3.1. We also give some remarks on Rankin's method for Jacobi forms of several variables.

2. Preliminaries on Jacobi forms over $\mathcal{H} \times \mathbb{C}^{(g\times 1)}$

Fix a positive integer g. The Jacobi group $\Gamma_{1,g}^J := \Gamma_1 \ltimes \left(\mathbb{Z}^{(g\times 1)} \times \mathbb{Z}^{(g\times 1)}\right)$ acts on $\mathcal{H} \times \mathbb{C}^{(g\times 1)}$ as follows;

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right),$$

where $\Gamma_1 = SL_2(\mathbb{Z})$ is the full modular group. Let $k \in \mathbb{Z}$ and M be a positive definite, symmetric, half-integral $g \times g$ matrix. For $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma_{1,g}^J$ and ϕ a complex-valued function on $\mathcal{H} \times \mathbb{C}^{(g\times 1)}$, we define

$$\left(\phi|_{k,M}\ \gamma\right)(\tau, z) := (c\tau + d)^{-k} e\left(\frac{-c}{c\tau + d} M[z + \lambda\tau + \mu] + M[\lambda]\tau + 2\lambda^t M z\right) \phi(\gamma \cdot (\tau, z)),$$

where $A[B] := B^t A B$, where A and B are matrices of appropriate size, B^t denotes the transpose of the matrix B and $e(a) := exp(2\pi i a)$, for a complex number a. Let $J_{k,\ M}$ be the space of Jacobi forms of weight k and index M on $\Gamma_{1,g}^J$, i.e., the space of holomorphic functions $\phi : \mathcal{H} \times \mathbb{C}^{(g\times 1)} \to \mathbb{C}$ satisfying $\phi|_{k,M}\ \gamma = \phi$, $\forall \gamma \in \Gamma_{1,g}^J$ and having a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z},\ r\in\mathbb{Z}^g, \\ 4n \geq M^{-1}[r^t]}} c_\phi(n, r)\ e(n\tau + rz).$$

Here, $r \in \mathbb{Z}^g$, denotes a row vector. Further, we say ϕ is a cusp form if and only if $c_\phi(n, r) = 0$ whenever $4n = M^{-1}[r^t]$. We denote the space of all Jacobi cusp forms by $J_{k,\ M}^{cusp}$. Let $\phi, \psi \in J_{k,\ M}$ be such that at least one of them is cusp form, the Petersson scalar product of ϕ and ψ is defined by

$$\langle \phi, \psi \rangle := \int_{\Gamma_{1,g}^J \backslash \mathcal{H} \times \mathbb{C}^{(g\times 1)}} \phi(\tau, z)\ \overline{\psi(\tau, z)}\ v^k\ exp(-4\pi M[y] v^{-1}) dV_g^J,$$

where $\tau = u + iv \in \mathcal{H}, z = x + iy \in \mathbb{C}^{(g\times 1)}$ and $dV_g^J = \dfrac{dudvdxdy}{v^{g+2}}$ is an invariant measure under the action on $\Gamma_{1,g}^J$ on $\mathcal{H} \times \mathbb{C}^{(g\times 1)}$. The space $(J_{k,\ M}^{cusp}, \langle .\ ,\ . \rangle)$ is a finite dimensional Hilbert space. For more details on the theory of Jacobi forms of several variables, we refer to [12]. The following lemma describes the growth of the Fourier coefficients of a Jacobi form.

Lemma 2.1 (Lemma 3.6, [10]). *Suppose that $k > g + 2$ and $\phi \in J_{k,\ M}$ with Fourier coefficients $c_\phi(n, r)$, then*

$$c_\phi(n, r) \ll |D|^{k - \frac{g}{2} - 1},$$

if $D < 0$. Moreover, if ϕ is a cusp form, then

$$c_\phi(n, r) \ll |D|^{\frac{k}{2} - \frac{g}{2}},$$

where $D = \widetilde{M}[r^t] - 4n|M|$ and $\widetilde{M}$ denotes the matrix of cofactors of the matrix M.

2.1. **Poincaré series.** Let M be a fixed positive definite, symmetric, half-integral $g \times g$ matrix, and $n \in \mathbb{Z},\ r \in \mathbb{Z}^g$ be such that $4n > M^{-1}[r^t]$. The (n, r)-th Poincaré series of weight k and index M is defined by

$$P_{k,M;(n,r)}(\tau, z) := \sum_{\gamma \in \Gamma^J_{1,g,\infty} \backslash \Gamma^J_{1,g}} e(n\tau + rz)|_{k,M}\ \gamma, \tag{2.1}$$

where

$$\Gamma^J_{1,g,\infty} := \left\{ \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) : t \in \mathbb{Z}, \mu \in \mathbb{Z}^g \right\}$$

is the stabilizer of $e(n\tau + rz)$ in $\Gamma^J_{1,g}$. It is well-known that $P_{k,M;(n,r)} \in J^{cusp}_{k,\ M}$ for $k > g + 2$. This series has the following property:

Lemma 2.2. [1] *Let $\phi \in J^{cusp}_{k,\ M}$ with Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n \in \mathbb{Z},\ r \in \mathbb{Z}^g \\ 4n > M^{-1}[r^t]}} c_\phi(n, r) e(n\tau + rz).$$

Then

$$\langle \phi, P_{k,M;(n,r)} \rangle = \lambda_{\kappa,M,D}\ c_\phi(n, r), \tag{2.2}$$

where

$$\lambda_{\kappa,M,D} = 2^{\kappa(g-1)-g}\Gamma(\kappa)\pi^{-\kappa}(det\ M)^{\kappa-\frac{1}{2}}D^{-\kappa},$$

with

$$\kappa = k - \frac{g}{2} - 1,\ \ D = det(T),\ \ T = \begin{pmatrix} 2n & r \\ r^t & 2M \end{pmatrix}.$$

2.2. **Differential operators on Jacobi forms.** For a positive definite, symmetric, half-integral $g \times g$ matrix M, we define the heat operator by

$$L_M := \frac{1}{(2\pi i)^2}\left(8\pi i |M|\ \frac{\partial}{\partial \tau} - \sum_{1 \le i,j \le g} M_{ij} \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \right),$$

where $\tau \in \mathcal{H}$ and $z^t = (z_1, z_2, \cdots, z_g) \in \mathbb{C}^g$ and M_{ij} is the (i, j)-th cofactor of the matrix M.

Remark 2.1. Note that for $g = 1$, the above heat operator is the classical heat operator studied in [5].

Remark 2.2. We note that the action of L_M on $e(n\tau + rz)$ is given by

$$L_M(e(n\tau + rz)) = (4n|M| - \widetilde{M}[r^t])e(n\tau + rz),$$

where $\widetilde{M}$ denotes the matrix of cofactors M_{ij} of the matrix M.

We now define Rankin-Cohen brackets for complex-valued functions on $\mathcal{H} \times \mathbb{C}^{(g\times 1)}$.

Definition 2.3. *Suppose that k_1 and k_2 are positive integers, and M_1, M_2 are positive definite, symmetric, half-integral $g \times g$ matrices. Let ϕ and ψ be two complex-valued holomorphic* *functions on* $\mathcal{H} \times \mathbb{C}^{(g,1)}$. *Then for a non-negative integer* ν, *the* ν*-th Rankin-Cohen bracket of* ϕ *and* ψ *is defined by*

$$[\phi, \psi]_\nu := \sum_{l=0}^{\nu} A(k_1, k_2, M_1, M_2, g, \nu; l)\ L^l_{M_1}(\phi)\ L^{\nu-l}_{M_2}(\psi),$$

where

$$A(k_1, k_2, M_1, M_2, g, \nu; l) = (-1)^l \binom{\kappa_1 + \nu}{\nu - l} \binom{\kappa_2 + \nu}{l}\ |M_1|^{\nu-l}\ |M_2|^l$$

with $\kappa_j = k_j - g/2 - 1$ *for* $j = 1, 2$.

Remark 2.3. Using the action of heat operator, one can verify that

$$[\phi|_{k_1,\ M_1}\ \gamma, \psi|_{k_2,\ M_2}\ \gamma]_\nu = [\phi, \psi]_\nu|_{k_1+k_2+2\nu,\ M_1+M_2}\ \gamma,\ \forall \gamma \in \Gamma^J_{1,g}. \tag{2.3}$$

Theorem 2.4. [10] *Let* $\phi \in J_{k_1,\ M_1}$, $\psi \in J_{k_2,\ M_2}$, *where* k_1, k_2 *are positive integers and* M_1, M_2 *are positive definite, symmetric, half-integral* $g \times g$ *matrices. Then for a non-negative integer* ν, $[\phi, \psi]_\nu \in J_{k_1+k_2+2\nu,\ M_1+M_2}$. *Moreover, if* $\nu > 0$, *then* $[\phi, \psi]_\nu \in J^{cusp}_{k_1+k_2+2\nu,\ M_1+M_2}$.

3. Statement of the Theorem

Let $\psi \in J^{cusp}_{k_2,\ M_2}$ and $\nu \geq 0$ be an integer. We define the map

$$T_{\psi,\nu} : J^{cusp}_{k_1,\ M_1} \to J^{cusp}_{k_1+k_2+2\nu,\ M_1+M_2}$$

by

$$T_{\psi,\nu}(\phi) = [\phi, \psi]_\nu.$$

$T_{\psi,\nu}$ is a $\mathbb{C}$-linear map between finite dimensional Hilbert spaces and therefore there exists a unique adjoint map

$$T^*_{\psi,\nu} : J^{cusp}_{k_1+k_2+2\nu,\ M_1+M_2} \to J^{cusp}_{k_1,\ M_1}$$

such that

$$\langle \phi, T_{\psi,\nu}(\omega) \rangle = \langle T^*_{\psi,\nu}(\phi), \omega \rangle,\ \forall \phi \in J^{cusp}_{k_1+k_2+2\nu,\ M_1+M_2} \text{ and } \omega \in J^{cusp}_{k_1,\ M_1}.$$

In Theorem 3.1, we exhibit the Fourier coefficients of $T^*_{\psi,\nu}(\phi)$ for $\phi \in J^{cusp}_{k_1+k_2+2\nu,\ M_1+M_2}$ for any ν. These Fourier coefficients involve special values of certain shifted convolution of Dirichlet series of Rankin-Selberg type associated to ϕ and ψ.

Theorem 3.1. *Let* $\nu \geqslant 0, k_1, k_2 > g + 2$ *be natural numbers and let* M_1, M_2 *be positive definite, symmetric, half-integral* $g \times g$ *matrices. Write* $k = k_1 + k_2 + 2\nu$ *and* $M = M_1 + M_2$. *Let* $\psi \in J_{k_2,\ M_2}$ *with Fourier expansion*

$$\psi(\tau, z) = \sum_{\substack{n_1 \in \mathbb{Z},\ r_1 \in \mathbb{Z}^g \\ 4n_1 > M_2^{-1}[r_1^t]}} c_\psi(n_1, r_1)\ e(n_1\tau + r_1 z).$$

Suppose that either (a) ψ *is a cusp form or (b)* ψ *is not a cusp form and* $k_1 > k_2 - g + 2$. *Then the image of any cusp form* $\phi \in J^{cusp}_{k,M}$ *with Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n_2 \in \mathbb{Z},\ r_2 \in \mathbb{Z}^g \\ 4n_2 > M^{-1}[r_2^t]}} c_\phi(n_2, r_2)\ e(n_2\tau + r_2 z)$$

under $T^*_{\psi,\nu}$ is given by

$$T^*_{\psi,\nu}(\phi)(\tau,z) = \sum_{\substack{n\in\mathbb{Z},\ r\in\mathbb{Z}^g \\ 4n>M_1^{-1}[r^t]}} c(n,r)\ e(n\tau+rz),$$

where

$$\begin{aligned} c(n,r) \ &= \frac{|M|^{k_1+k_2+2\nu-g-1}(4n|M_1|-\widetilde{M_1}[r^t])^{k_1-g/2-1}\Gamma(k-g/2-1)}{2^{(k_1-g/2-1)(g-1)}\pi^{k_2+2\nu}\Gamma(k_1-g/2-1)|M_1|^{k_1-g/2-3/2}} \\ &\times \ \sum_{l=0}^{\nu} A(k_1,k_2,M_1,M_2,g,\nu;l)\left(4n|M_1|-\widetilde{M_1}[r^t]\right)^l \\ &\times \sum_{\substack{n_1\in\mathbb{Z},r_1\in\mathbb{Z}^g \\ 4n_1>M_2^{-1}[r_1^t] \\ 4(n+n_1)>M^{-1}[(r+r_1)^t]}} \frac{\left(4n_1|M_2|-\widetilde{M_2}[r_1^t]\right)^{\nu-l}\overline{c_\psi(n_1,r_1)}c_\phi(n+n_1,r+r_1)}{\left(4(n+n_1)|M|-(\widetilde{M_1+M_2})[(r+r_1)^t]\right)^{k-g/2-1}}. \end{aligned}$$

4. Proofs

We follow the same exposition as given in the proof of Theorem 3.1 in [7].

Lemma 4.1. *Using the same notation in Theorem 3.1, the series*

$$\sum_{\gamma\in\Gamma^J_{1,g,\infty}\backslash\Gamma^J_{1,g}} \int\limits_{\Gamma^J_{1,g}\backslash\mathcal{H}\times\mathbb{C}^{(g,1)}} |\ \phi(\tau,z)\overline{[e(n\tau+rz)\ |_{k_1,M_1}\ \gamma,\ \psi]_\nu}\ v^k exp(-4\pi M[y]v^{-1})\ |\ dV^J_g$$

converges.

Proof. Proof is similar to Lemma 4.1 in [7]. □

We now give a proof of Theorem 3.1. Write

$$T^*_{\psi,\nu}(\phi)(\tau,z) = \sum_{\substack{n\in\mathbb{Z},\ r\in\mathbb{Z}^g \\ 4n>M_1^{-1}[r^t]}} c(n,r)\ e(n\tau+rz).$$

Consider the (n,r)-th Poincaré series of weight k_1 and index M_1 as defined in (2.1). Then using Lemma 2.2, we have

$$\langle T^*_{\psi,\nu}\phi, P_{k_1,M_1;(n,r)}\rangle = \lambda_{\kappa_1,M_1,D_1}\ c(n,r),$$

where $\kappa_1 = k_1 - \frac{g}{2} - 1$ and $D_1 = det(T_1),\ T_1 = \begin{pmatrix} 2n & r \\ r^t & 2M_1 \end{pmatrix}$.

On the other hand, by definition of the adjoint map we have

$$\langle T^*_{\psi,\nu}\ \phi, P_{k_1,M_1;(n,r)}\rangle = \langle \phi, T_{\psi,\nu}\ (P_{k_1,M_1;(n,r)})\rangle = \langle \phi, [P_{k_1,M_1;(n,r)},\psi]_\nu\rangle.$$

Hence we get

$$c(n,r) = \frac{1}{\lambda_{k_1,M_1,D_1}}\langle \phi, [P_{k_1,M_1;(n,r)},\psi]_\nu\rangle. \tag{4.1}$$

Now our aim is to compute the Petersson scalar product $\langle \phi, [P_{k_1,M_1;(n,r)}, \psi]_\nu \rangle$ in (4.1). By definition,

$$\langle \phi, [P_{k_1,M_1;(n,r)}, \psi]_\nu \rangle = \int_{\Gamma_{1,g}^J \backslash \mathcal{H} \times \mathbb{C}^{(g\times 1)}} \phi(\tau,z) \overline{\left[P_{k_1,M_1;(n,r)}, \psi\right]_\nu}\; v^k\; exp(-4\pi M[y]v^{-1})\; dV_g^J,$$

where $k = k_1 + k_2 + 2\nu$ and $M = M_1 + M_2$. Inserting the expression of Poincaré series from (2.1), we have

$$\langle \phi, [P_{k_1,M_1;(n,r)}, \psi]_\nu \rangle = \int_{\Gamma_{1,g}^J \backslash \mathcal{H} \times \mathbb{C}^{(g\times 1)}} \sum_{\gamma \in \Gamma_{1,g,\infty}^J \backslash \Gamma_{1,g}^J} \phi(\tau,z) \overline{\left[e(n\tau + rz) \mid_{k_1,M_1} \gamma,\; \psi\right]_\nu}\, v^k exp(-4\pi M[y]v^{-1})\, dV_g^J.$$

By Lemma 4.1, we can interchange the order of summation and integration in $\langle \phi, [P_{k_1,M_1;(n,r)}, \psi]_\nu \rangle$. Hence $\langle \phi, [P_{k_1,M_1;(n,r)}, \psi]_\nu \rangle$ equals

$$\sum_{\gamma \in \Gamma_{1,g,\infty}^J \backslash \Gamma_{1,g}^J} \int_{\Gamma_{1,g}^J \backslash \mathcal{H} \times \mathbb{C}^{(g\times 1)}} \phi(\tau,z) \overline{\left[e(n\tau + rz) \mid_{k_1,M_1} \gamma, \psi\right]_\nu}\; v^k exp(-4\pi M[y]v^{-1})\; dV_g^J.$$

Using the change of variable (τ, z) to $\gamma^{-1} \cdot (\tau, z)$ and using (2.3), $\langle \phi, [P_{k_1,M_1;(n,r)}, \psi]_\nu \rangle$ equals

$$\sum_{\gamma \in \Gamma_{1,g,\infty}^J \backslash \Gamma_{1,g}^J} \int_{\gamma \cdot \Gamma_{1,g}^J \backslash \mathcal{H} \times \mathbb{C}^{(g\times 1)}} \phi(\tau,z) \overline{\left[e(n\tau + rz), \psi\right]_\nu}\; v^k exp(-4\pi M[y]v^{-1})\; dV_g^J.$$

Now using the Rankin's unfolding trick, $\langle \phi, [P_{k_1,M_1;(n,r)}, \psi]_\nu \rangle$ equals

$$\begin{aligned} &\int_{\Gamma_{1,g,\infty}^J \backslash \mathcal{H} \times \mathbb{C}^{(g\times 1)}} \phi(\tau,z) \overline{\left[e(n\tau + rz), \psi\right]_\nu}\; v^k exp(-4\pi M[y]v^{-1})\; dV_g^J. \\ = &\sum_{l=0}^{\nu} A(k_1,k_2,M_1,M_2,g,\nu;l) \int_{\Gamma_{1,g,\infty}^J \backslash \mathcal{H} \times \mathbb{C}^{(g\times 1)}} \phi(\tau,z) \overline{L_{M_1}^l\big(e(n\tau + rz)\big) L_{M_2}^{\nu-l}(\psi(\tau,z))}\, v^k exp(-4\pi M[y]v^{-1}) dV_g^J. \end{aligned}$$

Now replacing ϕ and ψ by their Fourier series expansion and using the iterated action of the heat operators L_{M_1} and L_{M_2} as given in the Remark 2.2, $\langle \phi, [P_{k_1,M_1;(n,r)}, \psi]_\nu \rangle$ equals

$$\sum_{l=0}^{\nu} A(k_1,k_2,M_1,M_2,g,\nu;l) \int\limits_{\Gamma^J_{1,g,\infty}\backslash \mathcal{H}\times\mathbb{C}^{(g\times 1)}} \left(\sum_{\substack{n_2\in\mathbb{Z}, r_2\in\mathbb{Z}^g\\ 4n_2>M^{-1}[r_2^t]}} c_\phi(n_2,r_2) e(n_2\tau + r_2 z) \right) \left(4n|M_1| - \widetilde{M_1}[r^t]\right)^l$$

$$\times\, \overline{e(n\tau+rz)} \left(\sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathbb{Z}^g\\ 4n_1>M_2^{-1}[r_1^t]}} \left(4n_1|M_2| - \widetilde{M_2}[r_1^t]\right)^{\nu-l} \overline{c_\psi(n_1,r_1)}\ \overline{e(n_1\tau + r_1 z)} \right) v^k exp(-4\pi M[y]v^{-1}) dV_g^J$$

$$= \sum_{l=0}^{\nu} A(k_1,k_2,M_1,M_2,g,\nu;l) \left(4n|M_1| - \widetilde{M_1}[r^t]\right)^l \sum_{\substack{n_2\in\mathbb{Z}, r_2\in\mathbb{Z}^g\\ 4n_2>M^{-1}[r_2^t]}} \sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathbb{Z}^g\\ 4n_1>M_2^{-1}[r_1^t]}} \left(4n_1|M_2| - \widetilde{M_2}[r_1^t]\right)^{\nu-l}$$

$$\times\, c_\phi(n_2,r_2)\overline{c_\psi(n_1,r_1)} \int\limits_{\Gamma^J_{1,g,\infty}\backslash \mathcal{H}\times\mathbb{C}^{(g,1)}} e(n_2\tau + r_2 z)\overline{e(n\tau+rz)}\ \overline{e(n_1\tau + r_1 z)}\ v^k exp(-4\pi M[y]v^{-1}) dV_g^J.$$

As a fundamental domain for the action of $\Gamma^J_{1,g,\infty}$ on $\mathcal{H}\times\mathbb{C}^{(g,1)}$ we choose the set

$$\left\{(\tau,z)\in\mathcal{H}\times\mathbb{C}^{(g,1)} : 0\leqslant u\leqslant 1, v>0, 0\leqslant x_i\leqslant 1, i=1,2,\cdots g; y\in\mathbb{R}^g\right\},$$

where $\tau = u+iv, z = x+iy$ with $x^t = (x_1,x_2,\cdots,x_g)$. Integrating over this region, $\langle \phi, [P_{k_1,M_1;(n,r)}, \psi]_\nu \rangle$ equals

$$\frac{|M|^{k-g-1}\Gamma(k-g/2-1)}{2^g\ \pi^{k-g/2-1}} \sum_{l=0}^{\nu} A(k_1,k_2,M_1,M_2,g,\nu;l) \left(4n|M_1| - \widetilde{M_1}[r^t]\right)^l$$

$$\times \sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathbb{Z}^g\\ 4n_1>M_2^{-1}[r_1^t]\\ 4(n+n_1)>M^{-1}[(r+r_1)^t]}} \frac{\left(4n_1|M_2| - \widetilde{M_2}[r_1^t]\right)^{\nu-l} \overline{c_\psi(n_1,r_1)} c_\phi(n+n_1, r+r_1)}{\left(4(n+n_1)|M| - \widetilde{M}[(r+r_1)^t]\right)^{k-g/2-1}}.$$

Now substituting $\langle \phi, [P_{k_1,M_1;(n,r)}, \psi]_\nu \rangle$ in (4.1), we get the required expression for $c(n,r)$ given in Theorem 3.1.

Remark 4.1. We observe that the method of proof of Theorem 3.1 can be used to compute the adjoint of linear maps constructed using Rankin-Cohen type differential operators studied in [3].

Remark 4.2. Ramakrishnan and Sahu [10] computed the Petersson scalar product $\langle \phi, [\psi, E_{k_2,M_2}]_\nu \rangle$, where $\phi \in J^{cusp}_{k_1+k_2+2\nu,\ M_1+M_2}$, $\psi \in J^{cusp}_{k_1,\ M_1}$ and E_{k_2,M_2} is the Eisentein series of weight k_2 and index M_2. To compute the Petersson scalar product $\langle \phi, [\psi, E_{k_2,M_2}]_\nu \rangle$, they expressed the Rankin-Cohen bracket $[\psi, E_{k_2,M_2}]_\nu$ as a linear combination of Poincaré series and then used Lemma 2.2. As an application, we observe that, following the proof of Theorem 3.1 one can compute the Petersson scalar product $\langle \phi, [\psi, E_{k_2,M_2}]_\nu \rangle$, by evaluating the integral

$$\int\limits_{\Gamma^J_{1,g}\backslash\mathcal{H}\times\mathbb{C}^{(g,1)}} \phi(\tau,z)\overline{[\psi, E_{k_2,M_2}]_\nu} v^k \; exp(-4\pi M[y]v^{-1}) dV^J_g$$

explicitly using the Rankin-Selberg unfolding argument.

Remark 4.3. We remark here that the structure theorem for Jacobi forms of several variables has not been still well understood. However, one can get some arithmetical examples and application to the Theorem 3.1 similar to the case of Jacobi forms of scalar index (see Remark 3.2 in [7]).

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Department of Mathematics, Indian Institute of Science, Bangalore- 560012, India
E-mail address: `abhashjha@iisc.ac.in, abhashkumarjha@gmail.com`

School of Mathematical Sciences, National Institute of Science Education and Research, HBNI, Bhubaneswar, PO: Jatni, Khurdha, Odisha- 752 050, India
E-mail address: `brundaban.sahu@niser.ac.in`