EVALUATION OF CONVOLUTION SUMS AND SOME REMARKS ON CUSP FORMS OF WEIGHT 4 AND LEVEL 12.

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Abstract. In this note, we evaluate certain convolution sums and make some remarks about the Fourier coefficients of cusp forms of weight 4 for $\Gamma_0(12)$. We express the normalized newform of weight 4 on $\Gamma_0(12)$ as a linear combination of the (quasimodular) Eisenstein series (of weight 2) $E_2(dz)$, $d|12$ and their derivatives. Now, by comparing the work of Alaca-Alaca-Williams [1] with our results, as a consequence, we express the coefficients $c_{1,12}(n)$ and $c_{3,4}(n)$ that appear in [1, Eqs.(2.7) and (2.12)] in terms of linear combination of the Fourier coefficients of newforms of weight 4 on $\Gamma_0(6)$ and $\Gamma_0(12)$. The properties of $c_{1,12}(n)$ and $c_{3,4}(n)$ that are derived in [1] now follow from the properties of the Fourier coefficients of the newforms mentioned above. We also express the newforms as a linear combination of certain eta-quotients and obtain an identity involving eta-quotients.

1. Introduction

Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ be the set of positive integers, integers, rational numbers respectively. For $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we define $\sigma_k(n)$ to be the sum of the kth powers of the positive divisors of n. We write $\sigma(n)$ for $\sigma_1(n)$ and we assume that $\sigma_k(x) = 0$ for a rational number x which is not an integer. Following [14, 11], for $n, N \in \mathbb{N}$, we define $W_N(n)$ as follows.

$$W_N(n) = \sum_{m \in \mathbb{N}, m < n/N} \sigma(m)\sigma(n - Nm). \tag{1}$$

Also, following [1], we define $W_{a,b}(n)$ for $a, b \in \mathbb{N}$ by

$$W_{a,b}(n) := \sum_{\substack{l,m \in \mathbb{N} \\ al+bm=n}} \sigma(l)\sigma(m). \tag{2}$$

Note that $W_{1,N}(n) = W_{N,1}(n) = W_N(n)$. These type of sums were evaluated as early as the 19th century. For example, the sum $W_1(n)$ was evaluated by Besge, Glaisher and Ramanujan [4, 5, 10].

In [11], E. Royer used the structure of the space of quasimodular forms to evaluate the convolution sums $W_N(n)$ for $1 \le N \le 14$, $N \ne 12$. In [1], A. Alaca, S. Alaca and K. S. Williams have evaluated W_{12} using a different method and in [9], we evaluated the convolution sum W_{15} using the theory of quasimodular forms. Recently, there have been many works on the evalution of these sums starting with the work of Huard et. al [6] and we refer to [15, Table 1] for a recent update. We also refer to the book of Williams [13], which gives a beautiful history about the problem and its elementary methods.

Date: March 26, 2015.
2010 *Mathematics Subject Classification.* Primary 11F11, 11A25; Secondary 11E20, 11E25, 11F20.

In this note we evaluate the convolution sums $W_{12}(n)$ and $W_{3,4}(n)$ using the theory of quasimodular forms. Though these convolution sums have been evaluated earlier by Alaca-Alaca-Williams [1] by using a different method, the purpose of our evaluation is that we obtain certain interesting properties of the Fourier coefficients of certain cusp forms when we compare the two results. More precisely, in [1], they expressed $W_{12}(n)$ and $W_{3,4}(n)$ in terms of the divisor functions and coefficients of certain q-series given by $c_{1,12}(n)$ and $c_{3,4}(n)$ (see [1, Eqs. (2,7) and (2.12)]). They observed some properties of $c_{1,12}(n)$ and $c_{3,4}(n)$ which implies that the sums $W_{12}(n)$ and $W_{3,4}(n)$ have elementary evaluation for $n = 2^r 3^s$. Our method in this note gives the expressions for $W_{12}(n)$ and $W_{3,4}(n)$ in terms of divisor functions and Fourier coefficients of newforms of weight 4 on $\Gamma_0(6)$ and $\Gamma_0(12)$. Thus, by comparing our results with the result of Alaca-Alaca-Williams as mentioned above, we get expressions for $c_{1,12}(n)$ and $c_{3,4}(n)$ in terms of the Fourier coefficients $\tau_{4,6}(n)$ and $\tau_{4,12}(n)$ of the newforms of weight 4 with levels 6 and 12 respectively. Since the newforms are Hecke eigenforms with respect the Hecke operators U_p for the primes dividing the level (here $p = 2, 3$), the required properties (elementary evaluation when $n = 2^r 3^s$) of $c_{1,12}(n)$ and $c_{3,4}(n)$ follow from the properties of $\tau_{4,6}(n)$ and $\tau_{4,12}(n)$ (see Remark 3.3). For the basic theory of newforms we refer to [3]. We also get several other consequences, which are listed below.
(i) In Remark 3.1, we express the newform $\Delta_{4,12}(z)$ as a linear combination of the (quasimodular) Eisenstein series (of weight 2) $E_2(dz)$, $d|12$ and their derivatives.
(ii) In Remark 3.2 we express the newform $\Delta_{4,12}(z)$ and the oldform $\Delta_{4,6}(z) + 4\Delta_{4,6}(2z)$ as a linear combination of certain eta-quotients and also obtain an identity involving some eta-quotients. We express $c_{1,12}(n)$ and $c_{3,4}(n)$ in terms of the Fourier coefficients of the newforms $\Delta_{4,6}(z)$ and $\Delta_{4,12}(z)$.
(iii) In Remark 3.4, we give a formula for $N_4(n)$, the number of representations of a natural number n by the quadratic form $x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+4(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)$ in terms of the divisor functions and $\tau_{4,6}(n)$, which gives the fact that $N_4(n)$ has elementary evaluation for $n = 2^r 3^s$.

2. Certain convolution sums

Theorem 2.1. *Let $n \in \mathbb{N}$. Then,*

$$\begin{aligned}
W_{12}(n) &= \frac{1}{480}\sigma_3(n) + \frac{1}{160}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{160}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{4}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{12}\right)\\
&\quad + \left(\frac{1}{24} - \frac{n}{48}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{12}\right) - \frac{1}{80}\tau_{4,6}(n) - \frac{1}{20}\tau_{4,6}\left(\frac{n}{2}\right) - \frac{1}{96}\tau_{4,12}(n),\\
W_{3,4}(n) &= \frac{1}{480}\sigma_3(n) + \frac{1}{160}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{160}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{4}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{12}\right)\\
&\quad + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma\left(\frac{n}{3}\right) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{4}\right) - \frac{1}{80}\tau_{4,6}(n) - \frac{1}{20}\tau_{4,6}\left(\frac{n}{2}\right) + \frac{1}{96}\tau_{4,12}(n),
\end{aligned}$$

where $\tau_{4,6}(n)$ and $\tau_{4,12}(n)$ are the nth Fourier coefficients of the cusp forms $\Delta_{4,6}(z)$ and $\Delta_{4,12}(z)$, which are the unique normalised newforms of weight 4 on $\Gamma_0(6)$ and $\Gamma_0(12)$ respectively. In fact,

$$\Delta_{4,6}(z) = \eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z) := \sum_{n=0}^{\infty} \tau_{4,6}(n)q^n \tag{3}$$

and

$$\Delta_{4,12}(z) \;:=\; \sum_{n=0}^{\infty} \tau_{4,12}(n) q^n, \tag{4}$$

which is a linear combination of certain eta-quotients (see Eq.(14)*).*

Proof. As mentioned in the introduction, we make use of the theory of quasimodular forms to obtain the convolution sums. Since the method is exactly the same as in [11, 9], we will be brief in giving the proof and leave the details to the reader. Let $M_k(\Gamma_0(N))$ denote the $\mathbb{C}$-vector space of modular forms of weight k on the congruence subgroup $\Gamma_0(N)$ and let $M_k^{\le k/2}(\Gamma_0(N))$ denote the space of quasimodular forms of weight k, depth $\le k/2$ on $\Gamma_0(N)$. The following structure theorem is the key ingredient in the proof (see [7, 11]). For an even integer k with $k \ge 2$, we have

$$\tilde{M}_k^{\le k/2}(\Gamma_0(N)) = \bigoplus_{j=0}^{k/2-1} D^j M_{k-2j}(\Gamma_0(N)) \oplus \mathbb{C} D^{k/2-1} E_2, \tag{5}$$

where the differential operator D is defined by $D := \frac{1}{2\pi i}\frac{d}{dz}$.

Let $E_k(z)$ denote the normalized Eisenstein series of weight k on $SL_2(\mathbb{Z})$. For $k \ge 4$ it is a modular form of weight k and for $k = 2$, it is a quasimodular form of weight 2 and is given by

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n. \tag{6}$$

When $k = 4$ and $N = 12$, Eq. (5) gives

$$\tilde{M}_4^{\le 2}(\Gamma_0(12)) = M_4(\Gamma_0(12)) \oplus D M_2(\Gamma_0(12)) \oplus \mathbb{C} D E_2. \tag{7}$$

Using the dimension formula (see for example [8, 12]), it follows that the set $\{E_4(z), E_4(2z), E_4(3z), E_4(4z), E_4(6z), E_4(12z), \Delta_{4,6}(z), \Delta_{4,6}(2z), \Delta_{4,12}(z)\}$ forms a basis for the space $M_4(\Gamma_0(12))$ and the set $\{\Phi_{1,2}(z), \Phi_{1,3}(z), \Phi_{1,4}(z), \Phi_{1,6}(z), \Phi_{1,12}(z)\}$ forms a basis for the space $M_2(\Gamma_0(12))$, where

$$\Phi_{a,b}(z) := \frac{1}{b-a}(bE_2(bz) - aE_2(az)). \tag{8}$$

To compute the convolution sum $W_{12}(n)$, we consider the quasimodular form $E_2(z)E_2(12z)$ in $\tilde{M}_4^{\le 2}(\Gamma_0(12))$. Therefore, using the structure (7) and the bases mentioned above, we have

$$\begin{aligned} E_2(z)E_2(12z) \;\; = \;\; & \frac{1}{200}E_4(z) + \frac{3}{200}E_4(2z) + \frac{9}{200}E_4(3z) + \frac{2}{25}E_4(4z) + \frac{27}{200}E_4(6z) + \frac{18}{25}E_4(12z) \\ & - \frac{36}{5}\Delta_{4,6}(z) - \frac{144}{5}\Delta_{4,6}(2z) - 6\Delta_{4,12}(z) + \frac{11}{2}D\Phi_{1,12}(z) + DE_2(z). \end{aligned}$$

In a similar way, to get the convolution sum $W_{3,4}(n)$, we consider the quasimodular form $E_2(3z)E_2(4z)$ and get the following expression

$$\begin{aligned} E_2(3z)E_2(4z) \;\; = \;\; & \frac{1}{200}E_4(z) + \frac{3}{200}E_4(2z) + \frac{9}{200}E_4(3z) + \frac{2}{25}E_4(4z) + \frac{27}{200}E_4(6z) + \frac{18}{25}E_4(12z) \\ & - \frac{36}{5}\Delta_{4,6}(z) - \frac{144}{5}\Delta_{4,6}(2z) + 6\Delta_{4,12}(z) + D\Phi_{1,3}(z) + \frac{3}{2}D\Phi_{1,4}(z) + DE_2(z). \end{aligned}$$

By comparing the n-th Fourier coefficients, we get the required the convolution sums. $\square$

3. Remarks on cusp forms of weight 4 and level 12

In this section we make several remarks which are consequences of the comparison of our theorem with the corresponding results of [1].

Remark 3.1. By taking the difference of the expressions for $E_2(z)E_2(12z)$ and $E_2(3z)E_2(4z)$ given in the proof of the theorem, we get an expression of the newform $\Delta_{4,12}$ of weight 4 on $\Gamma_0(12)$ in terms of the Eisenstein series $E_2(z)$ and its derivatives. In fact, subtracting the expressions and simplifying, we get

$$\Delta_{4,12}(z) = \frac{1}{12}\left(E_2(3z)E_2(4z) - E_2(z)E_2(12z) + \frac{11}{2}D\Phi_{1,12}(z) - D\Phi_{1,3}(z) - \frac{3}{2}D\Phi_{1,4}(z)\right). \quad (9)$$

Using the definition of $\Phi_{a,b}(z)$ from (8), we obtain

$$\Delta_{4,12}(z) = \frac{1}{12}\left(E_2(3z)E_2(4z) - E_2(z)E_2(12z) + \frac{1}{2}DE_2(z) - \frac{3}{2}DE_2(3z) - 2DE_2(4z) + 6DE_2(12z)\right). \quad (10)$$

Remark 3.2. As mentioned in the introduction, the convolution sums $W_{12}(n)$ and $W_{3,4}(n)$ have been evaluated in [1] by a different method, which are given below:

$$\begin{aligned}
W_{12}(n) &= \frac{1}{480}\sigma_3(n) + \frac{1}{160}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{160}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{4}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{12}\right)\\
&\quad + \left(\frac{1}{24} - \frac{n}{48}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{12}\right) - \frac{11}{480}c_{1,12}(n),\\
W_{3,4}(n) &= \frac{1}{480}\sigma_3(n) + \frac{1}{160}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{160}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{4}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{12}\right)\\
&\quad + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma\left(\frac{n}{3}\right) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{4}\right) - \frac{1}{480}c_{3,4}(n),
\end{aligned}$$

where

$$\frac{10}{11}\ \frac{\eta^2(2z)\eta^3(3z)\eta^3(4z)\eta^2(6z)}{\eta(z)\eta(12z)} + \frac{1}{11}\ \frac{\eta^8(2z)\eta^8(6z)}{\eta^2(z)\eta^2(3z)\eta^2(4z)\eta^2(12z)} := \sum_{n=0}^{\infty} c_{1,12}(n)q^n,$$

$$10\ \frac{\eta^3(z)\eta^2(2z)\eta^2(6z)\eta^3(12z)}{\eta(3z)\eta(4z)} + \frac{\eta^8(2z)\eta^8(6z)}{\eta^2(z)\eta^2(3z)\eta^2(4z)\eta^2(12z)} := \sum_{n=0}^{\infty} c_{3,4}(n)q^n.$$

Comparing the above formulas with the formulas in Theorem 2.1, we can express $c_{1,12}(n)$ and $c_{3,4}(n)$ in terms of $\tau_{4,6}(n)$ and $\tau_{4,12}(n)$:

$$c_{1,12}(n) = \frac{6}{11}\tau_{4,6}(n) + \frac{24}{11}\tau_{4,6}\left(\frac{n}{2}\right) + \frac{5}{11}\tau_{4,12}(n), \quad (11)$$

$$c_{3,4}(n) = 6\tau_{4,6}(n) + 24\tau_{4,6}\left(\frac{n}{2}\right) - 5\tau_{4,12}(n) \quad (12)$$

Consequently, we can also express $\tau_{4,12}(n)$ in terms of the coefficients $c_{1,12}(n)$ and $c_{3,4}(n)$:

$$\tau_{4,12}(n) = \frac{11}{10}c_{1,12}(n) - \frac{1}{10}c_{3,4}(n). \quad (13)$$

Since $c_{1,12}(n)$ and $c_{3,4}(n)$ are nth Fourier coefficients of certain linear combination of eta-quotients (as given above), from the above equation we get an expression for the newform $\Delta_{4,12}(z)$ as a linear combination of eta-quotients:

$$\Delta_{4,12}(z) = \frac{\eta^2(2z)\eta^3(3z)\eta^3(4z)\eta^2(6z)}{\eta(z)\eta(12z)} - \frac{\eta^3(z)\eta^2(2z)\eta^2(6z)\eta^3(12z)}{\eta(3z)\eta(4z)}. \tag{14}$$

Next, eliminating $\tau_{4,12}(n)$ from the equations (11) and (12), we get the following.

$$\tau_{4,6}(n) + 4\tau_{4,6}(n/2) \ = \ \frac{1}{12}\left(11c_{1,12}(n) + c_{3,4}(n)\right). \tag{15}$$

In other words,

$$\begin{aligned}\Delta_{4,6}(z) + 4\Delta_{4,6}(2z) = \frac{5}{6}\ \frac{\eta^2(2z)\eta^3(3z)\eta^3(4z)\eta^2(6z)}{\eta(z)\eta(12z)} + \frac{1}{6}\ \frac{\eta^8(2z)\eta^8(6z)}{\eta^2(z)\eta^2(3z)\eta^2(4z)\eta^2(12z)} \\ + \frac{5}{6}\ \frac{\eta^3(z)\eta^2(2z)\eta^2(6z)\eta^3(12z)}{\eta(3z)\eta(4z)}.\end{aligned} \tag{16}$$

Note that $\Delta_{4,6}(z) + 4\Delta_{4,6}(2z)$ is an oldform of weight 4 on $\Gamma_0(12)$ and we have expressed it in terms of a linear combination of eta-quotients. Moreover, if we use the expression for $\Delta_{4,6}(z)$ in terms of eta-product from equation (3), we get the following identity (after cancelling the factor $\eta^2(2z)\eta^2(6z)$):

$$\eta^2(z)\eta^2(3z) + 4\eta^2(4z)\eta^2(12z) = \frac{5}{6}\frac{\eta^3(3z)\eta^3(4z)}{\eta(z)\eta(12z)} + \frac{1}{6}\frac{\eta^6(2z)\eta^6(6z)}{\eta^2(z)\eta^2(3z)\eta^2(4z)\eta^2(12z)} + \frac{5}{6}\frac{\eta^3(z)\eta^3(12z)}{\eta(3z)\eta(4z)}. \tag{17}$$

Remark 3.3. In [1, §5] (see also [2, §5]), the authors proved some properties of the functions $c_{1,12}(n)$ and $c_{3,4}(n)$. They showed that

$$c_{1,12}(2n) \ = \ \frac{12}{11}\tau_{4,6}(n), \qquad c_{3,4}(2n) \ = \ 12\tau_{4,6}(n), \tag{18}$$

$$c_{1,12}(3n) \ = \ \frac{-3}{11}c_{3,4}(n), \qquad c_{3,4}(3n) \ = \ -33c_{1,12}(n). \tag{19}$$

Note that in the notation of [1], $\tau_{4,6}(n) = c_{1,6}(n)$. Since $\Delta_{4,6}(z)$ and $\Delta_{4,12}(z)$ are newforms of weight 4 on $\Gamma_0(6)$ and $\Gamma_0(12)$ respectively, these properties observed in [1] follow easily. The function $\Delta_{4,6}(z)$ is an eigenfunction under the Hecke operators U_2 and U_3 with eigenvalues -2 and -3. By the theory of newforms, it is known that the function has eigenvalues ± 2 and ± 3 with respect to the Hecke operators U_2 and U_3 respectively and the Fourier coefficient identity is given by

$$\tau_{4,6}(np) = \pm p\,\tau_{4,6}(n) \qquad (p = 2, 3).$$

Since $\tau_{4,6}(1) = 1$, by comparing the Fourier coefficients $\tau_{4,6}(2)$ and $\tau_{4,6}(3)$ we see that the eigenvalues are -2 and -3. Therefore, we get for all $n \geq 1$,

$$\tau_{4,6}(2n) \ = \ -2\tau_{4,6}(n), \tag{20}$$

$$\tau_{4,6}(3n) \ = \ -3\tau_{4,6}(n). \tag{21}$$

In a similar way, we see that the newform $\Delta_{4,12}(z)$ is an eigenfunction under the Hecke operators U_2 and U_3 with eigenvalues 0 and 3. Thus, we have for all $n \geq 1$,

$$\tau_{4,12}(2n) = 0, \tag{22}$$

$$\tau_{4,12}(3n) = 3\tau_{4,12}(n). \tag{23}$$

Now the properties (18) and (19) follow easily using the above properties of $\tau_{4,6}(n)$ and $\tau_{4,12}(n)$ together with the identities (11) and (12). In [1, Eq(5.16)] it is observed that

$$\tau_{4,6}(2^r 3^s) = (-1)^{r+s} 2^r 3^s, \tag{24}$$

where r, s are non-negative integers. This is also a consequence of the fact that $\Delta_{4,6}(z)$ is a Hecke eigenform with eigenvalues -2 and -3 under the Hecke operators U_2 and U_3 respectively. Since $\tau_{4,12}(n)$ is zero when n is even, we see that our formulas for $W_{12}(n)$ and $W_{3,4}(n)$ have elementary evaluations when $n = 2^r 3^s$.

Remark 3.4. In [1], the authors used the convolution sums $W_{12}(n)$ and $W_{3,4}(n)$ and computed the number $N_4(n)$, which is the number of representations of a positive integer n by the quadratic form $x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 4(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)$ and showed that

$$\begin{aligned} N_4(n) &= \frac{6}{5}\sigma_3(n) + \frac{18}{5}\sigma_3\left(\frac{n}{2}\right) + \frac{54}{5}\sigma_3\left(\frac{n}{3}\right) + \frac{96}{5}\sigma_3\left(\frac{n}{4}\right) \\ &\quad + \frac{162}{5}\sigma_3\left(\frac{n}{6}\right) + \frac{864}{5}\sigma_3\left(\frac{n}{12}\right) + \frac{99}{10}c_{1,12}(n) + \frac{9}{10}c_{3,4}(n). \end{aligned} \tag{25}$$

Replacing the values $c_{1,12}(n)$ and $c_{3,4}(n)$ in terms of $\tau_{4,6}(n)$ and $\tau_{4,12}(n)$ from equations (11) and (12), we get

$$\begin{aligned} N_4(n) = \quad & \frac{6}{5}\sigma_3(n) + \frac{18}{5}\sigma_3\left(\frac{n}{2}\right) + \frac{54}{5}\sigma_3\left(\frac{n}{3}\right) + \frac{96}{5}\sigma_3\left(\frac{n}{4}\right) \\ & + \frac{162}{5}\sigma_3\left(\frac{n}{6}\right) + \frac{864}{5}\sigma_3\left(\frac{n}{12}\right) + \frac{54}{5}\tau_{4,6}(n) + \frac{216}{5}\tau_{4,6}\left(\frac{n}{2}\right). \end{aligned} \tag{26}$$

Notice that the above formula for $N_4(n)$ given in (26) depends only on the values of the divisor functions and the Fourier coefficients $\tau_{4,6}(n)$ of the newform $\Delta_{4,6}(z)$. Therefore, by using (24) it follows that our formula for $N_4(n)$ has elementary evaluations when $n = 2^r 3^s$, where r and s are non-negative integers.

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