RANKIN-COHEN BRACKETS AND VAN DER POL-TYPE IDENTITIES FOR THE RAMANUJAN'S TAU FUNCTION

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ABSTRACT. We use Rankin-Cohen brackets for modular forms and quasimodular forms to give a different proof of the results obtained by D. Lanphier [5] and D. Niebur [9] on the van der Pol type identities for the Ramanujan's tau function $\tau(n)$. We also obtain new identities for $\tau(n)$ and as consequences we obtain convolution sums and congruence relations involving the divisor functions.

1. INTRODUCTION

Let

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$$

be the Ramanujan's cusp form of weight 12 for $SL_2(\mathbb{Z})$, where $q = e^{2\pi iz}$, z belongs to the upper half-plane $\mathcal{H}$. The function $\tau(n)$ is called the Ramanujan's tau-function. Using differential equations satisfied by $\Delta(z)$, B. van der Pol [10] derived identities relating $\tau(n)$ to sum-of-divisors functions. For example,

$$\tau(n) = n^2\sigma_3(n) + 60\sum_{m=1}^{n-1}(2n-3m)(n-3m)\sigma_3(m)\sigma_3(n-m), \tag{1}$$

where $\sigma_k(n) = \sum_{d|n} d^k$. Using the relation between $\sigma_3(n)$ and $\sigma_7(n)$ (see Theorem 3.1 (i) below), this is equivalent to the following identity:

$$\tau(n) = n^2\sigma_7(n) - 540\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m). \tag{2}$$

In [5], D. Lanphier used differential operators studied by Maass [6] to prove the above van der Pol identity (2). He also obtained several van der Pol-type identities using the Maass operators and thereby obtained new congruences for the Ramanujan's tau-function.

In [9], D. Niebur derived a formula for $\tau(n)$ similar to the classical ones of Ramanujan and van der Pol, but has the feature that higher divisor sums do not appear

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(see Theorem 2.5 (i) below). Niebur proved the formula by expressing $\Delta(z)$ in terms of the logarithmic derivatives of $\Delta(z)$.

In this paper we show that the identities for the Ramanujan function $\tau(n)$ proved by Lanphier in [5] can be obtained using the Rankin-Cohen bracket for modular forms and the basic relations among the Eisenstein series. By our method we also obtain new identities for $\tau(n)$ which are not proved in [5]. Next we show that the theory of quasimodular forms can be used to prove Niebur's formula for $\tau(n)$. The method of using quasimodular forms gives new formulas for $\tau(n)$. Though one can obtain many identities for $\tau(n)$, here we restrict only those identities in which only the convolution of divisor functions appear. Finally, we observe that the identities of $\tau(n)$ give rise to various identities for the convolution of the divisor functions. As a consequence, we also present some congruences involving the divisor functions.

2. Statement of results

Theorem 2.1.

$$\begin{aligned}
\text{(i)}\quad \tau(n) &= n^2\sigma_7(n) - 540\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m),\\
\text{(ii)}\quad \tau(n) &= -\frac{5}{4}n^2\sigma_7(n) + \frac{9}{4}n^2\sigma_3(n) + 540\sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_3(n-m),\\
\text{(iii)}\quad \tau(n) &= n^2\sigma_7(n) - \frac{1080}{n}\sum_{m=1}^{n-1} m^2(n-m)\sigma_3(m)\sigma_3(n-m),\\
\text{(iv)}\quad \tau(n) &= -\frac{1}{2}n^2\sigma_7(n) + \frac{3}{2}n^2\sigma_3(n) + \frac{360}{n}\sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_3(n-m).
\end{aligned}$$

Theorem 2.2.

$$\begin{aligned}
\text{(i)}\quad \tau(n) &= -\frac{11}{24}n\sigma_9(n) + \frac{35}{24}n\sigma_5(n) + 350\sum_{m=1}^{n-1} (n-m)\sigma_3(m)\sigma_5(n-m),\\
\text{(ii)}\quad \tau(n) &= \frac{11}{36}n\sigma_9(n) + \frac{25}{36}n\sigma_3(n) - 350\sum_{m=1}^{n-1} m\sigma_3(m)\sigma_5(n-m),\\
\text{(iii)}\quad \tau(n) &= \frac{1}{6}n\sigma_9(n) + \frac{5}{6}n\sigma_3(n) - \frac{420}{n}\sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_5(n-m),\\
\text{(iv)}\quad \tau(n) &= n\sigma_9(n) - \frac{2100}{n}\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_5(n-m),\\
\text{(v)}\quad \tau(n) &= -\frac{1}{4}n\sigma_9(n) + \frac{5}{4}n\sigma_5(n) + \frac{300}{n}\sum_{m=1}^{n-1} (n-m)^2\sigma_3(m)\sigma_5(n-m).
\end{aligned}$$

Theorem 2.3.

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{2\cdot 691}{3n}\sum_{m=1}^{n-1} m\sigma_5(m)\sigma_5(n-m).$$

Theorem 2.4.

$$\text{(i)}\quad \tau(n) = -\frac{91}{600}\sigma_{11}(n) + \frac{691}{600}\sigma_3(n) + \frac{4\cdot 691}{5n}\sum_{m=1}^{n-1} m\sigma_3(m)\sigma_7(n-m),$$

$$\text{(ii)}\quad \tau(n) = -\frac{91}{600}\sigma_{11}(n) + \frac{691}{600}\sigma_7(n) + \frac{2\cdot 691}{5n}\sum_{m=1}^{n-1} (n-m)\sigma_3(m)\sigma_7(n-m).$$

Remark 2.1. Theorems 2.1 to 2.4 are exactly the same as Theorems 1 to 4 of [5].

The following theorem gives identities for $\tau(n)$ in which the convolution part contains only the divisor function $\sigma(n)(=\sigma_1(n))$.

Theorem 2.5.

$$\text{(i)}\quad \tau(n) = n^4\sigma(n) - 24\sum_{m=1}^{n-1}(35m^4 - 52m^3n + 18m^2n^2)\sigma(m)\sigma(n-m),$$

$$\text{(ii)}\quad \tau(n) = n^4(7\sigma(n) - 6\sigma_3(n)) - 168\sum_{m=1}^{n-1}(5m^4 - 4m^3n)\sigma(m)\sigma(n-m),$$

$$\text{(iii)}\quad \tau(n) = n^4\sigma_3(n) - 168\sum_{m=1}^{n-1}(5m^4 - 8m^3n + 3m^2n^2)\sigma(m)\sigma(n-m),$$

$$\text{(iv)}\quad \tau(n) = \frac{n^4}{3}(7\sigma(n) - 4\sigma_3(n)) - 56\sum_{m=1}^{n-1}(15m^4 - 20m^3n + 6m^2n^2)\sigma(m)\sigma(n-m).$$

Remark 2.2. In Theorem 2.5, formula (i) was proved by Niebur [9]. The rest of the formulas are obtained while proving (i) by using quasimodular forms.

In the above theorems we presented results that are already proved by Niebur and Lanphier and a few more identities like Niebur's. We now state some more new identities using our method. As mentioned in the introduction we restrict ourselves to the identities which involve only the convolution of the divisor functions. We list these identities in two theorems, one uses the theory of modular forms and the other uses the theory of quasimodular forms.

Theorem 2.6.

$$\text{(i)}\quad \tau(n) = \frac{5}{12}n\sigma_3(n) + \frac{7}{12}n\sigma_5(n) + 70\sum_{m=1}^{n-1}(2n - 5m)\sigma_3(m)\sigma_5(n-m),$$

$$\text{(ii)}\quad \tau(n) = n^2\sigma_3(n) + 60\sum_{m=1}^{n-1}(4n^2 - 13mn + 9m^2)\sigma_3(m)\sigma_3(n-m).$$

$$
\begin{aligned}
\text{(iii)}\quad \tau(n) &= \frac{65}{756}\sigma_{11}(n) + \frac{5\cdot 691}{12\cdot 756n}\sigma_7(n) + \frac{691}{12\cdot 108n}\sigma_5(n)\\
&\quad -\frac{5\cdot 691}{54n^2}\sum_{m=1}^{n-1}(3n-7m)\sigma_5(m)\sigma_7(n-m)\\
&\quad -\frac{13\cdot 691}{9n^2}\sum_{m=1}^{n-1} m(n-m)\sigma_5(m)\sigma_5(n-m),\\
\text{(iv)}\quad \tau(n) &= \frac{65}{756}\sigma_{11}(n) + \frac{3\cdot 691}{8\cdot 441}\sigma_3(n) + \frac{5\cdot 691}{24\cdot 441}\sigma_7(n)\\
&\quad +\frac{5\cdot 691}{441n^2}\sum_{m=1}^{n-1}(91m^2-65mn+10n^2)\sigma_3(m)\sigma_7(n-m)\\
&\quad -\frac{13\cdot 691}{9n^2}\sum_{m=1}^{n-1} m(n-m)\sigma_5(m)\sigma_5(n-m),\\
\text{(v)}\quad \tau(n) &= \frac{65}{756}\sigma_{11}(n) + \frac{25\cdot 691}{36\cdot 756n}\sigma_3(n) + \frac{11\cdot 691}{36\cdot 756n}\sigma_9(n)\\
&\quad -\frac{55\cdot 691}{1134n^2}\sum_{m=1}^{n-1}(7m-2n)\sigma_3(m)\sigma_9(n-m)\\
&\quad -\frac{13\cdot 691}{9n^2}\sum_{m=1}^{n-1} m(n-m)\sigma_5(m)\sigma_5(n-m).
\end{aligned}
$$

Theorem 2.7.

$$
\begin{aligned}
\text{(i)}\quad \tau(n) &= \frac{5\cdot 691}{9504}\sigma(n) - \frac{(6n-5)\cdot 691}{864}\sigma_9(n)\\
&\quad +\frac{2275}{1584}\sigma_{11}(n) - \frac{5\cdot 691}{864}\sum_{m=1}^{n-1}\sigma(m)\sigma_9(n-m),\\
\text{(ii)}\quad \tau(n) &= \frac{15}{32}n\sigma(n) - \frac{33}{32}n\sigma_9(n) + \frac{50}{32}n^2\sigma_7(n) + 225\sum_{m=1}^{n-1} m\sigma(m)\sigma_7(n-m),\\
\text{(iii)}\quad \tau(n) &= \frac{6}{7}n^2\sigma(n) - \frac{9}{7}n^3\sigma_5(n) + \frac{10}{7}n^2\sigma_7(n) - 24\cdot 18\sum_{m=1}^{n-1} m^2\sigma(m)\sigma_5(n-m),\\
\text{(iv)}\quad \tau(n) &= \frac{14}{5}n^3\sigma(n) + \frac{12}{5}n^4\sigma_3(n) - \frac{21}{5}n^3\sigma_5(n) + 24\cdot 28\sum_{m=1}^{n-1} m^3\sigma(m)\sigma_3(n-m),\\
\text{(v)}\quad \tau(n) &= \frac{5}{12}n\sigma(n) + \frac{25}{24}n\sigma_7(n) - \frac{11}{24}n\sigma_9(n) + 25\sum_{m=1}^{n-1}(9m-n)\sigma(m)\sigma_7(n-m),
\end{aligned}
$$

$$\text{(vi)} \quad \tau(n) = \frac{9}{14}n^2\sigma(n) + \frac{5}{14}n^2\sigma_7(n) - 108\sum_{m=1}^{n-1}(4m^2 - mn)\sigma(m)\sigma_5(n-m),$$

$$\text{(vii)} \quad \tau(n) = \frac{8}{5}n^3\sigma(n) - \frac{3}{5}n^3\sigma_5(n) + 96\sum_{m=1}^{n-1}(7m^3 - 3m^2n)\sigma(m)\sigma_3(n-m),$$

$$\text{(viii)} \quad \tau(n) = \frac{1}{2}n^2\sigma(n) + \frac{1}{2}n^2\sigma_5(n) - 12\sum_{m=1}^{n-1}(36m^2 - 16mn + n^2)\sigma(m)\sigma_5(n-m),$$

$$\text{(ix)} \quad \tau(n) = n^3\sigma(n) - 24\sum_{m=1}^{n-1}(21m^2n - 28m^3 - 3mn^2)\sigma(m)\sigma_3(n-m).$$

As a consequence of the above theorems, we get the following congruences for $\tau(n)$ in terms of the divisor functions. Some of the congruences are already known (for example (iii) and (iv) are corresponding to (7.15) and (5.6) of [4]) and some are new.

Corollary 2.8.

$$\begin{array}{lrcl}
\text{(i)} & 12\tau(n) & \equiv & 5n\sigma_3(n) + 7n\sigma_5(n) \pmod{2^3\cdot 3\cdot 5\cdot 7}, \\
\text{(ii)} & 32\tau(n) & \equiv & 15n\sigma(n) + 50n^2\sigma_7(n) - 33n\sigma_9(n) \pmod{2^5\cdot 3^2\cdot 5^2}, \\
\text{(iii)} & 7\tau(n) & \equiv & 6n^2\sigma(n) - 9n^3\sigma_5(n) + 10n^2\sigma_7(n) \pmod{2^4\cdot 3^3\cdot 7}, \\
\text{(iv)} & 5\tau(n) & \equiv & 14n^3\sigma(n) + 12n^4\sigma_3(n) - 21n^3\sigma_5(n) \pmod{2^5\cdot 3\cdot 5\cdot 7}, \\
\text{(v)} & 24\tau(n) & \equiv & 10n\sigma(n) + 25n\sigma_7(n) - 11n\sigma_9(n) \pmod{2^3\cdot 3\cdot 5^2}, \\
\text{(vi)} & 14\tau(n) & \equiv & 9n^2\sigma(n) + 5n^2\sigma_7(n) \pmod{2^3\cdot 3^3\cdot 7}, \\
\text{(vii)} & 5\tau(n) & \equiv & 8n^3\sigma(n) - 3n^3\sigma_5(n) \pmod{2^5\cdot 3\cdot 5}, \\
\text{(viii)} & 2\tau(n) & \equiv & n^2\sigma(n) + n^2\sigma_5(n) \pmod{2^3\cdot 3}.
\end{array}$$

Finally, we state some identities for certain convolution of divisor functions. Some of these identities are obtained while using our method and the rest follow from the identities for $\tau(n)$ mentioned in the above theorems. We remark that except for (vii), all other formulas are different from the ones obtained by E. Royer [12] .

Theorem 2.9.

$$\text{(i)} \quad \sum_{m=1}^{n-1} m^3\sigma(m)\sigma(n-m) = \frac{1}{12}n^3\sigma_3(n) - \frac{1}{24}n^3(3n-1)\sigma(n),$$

$$\text{(ii)} \quad \sum_{m=1}^{n-1} m^2\sigma(m)\sigma(n-m) = \frac{1}{8}n^2\sigma_3(n) - \frac{1}{24}n^2(4n-1)\sigma(n),$$

$$\begin{aligned}
\text{(iii)} \quad \sum_{m=1}^{n-1} m\sigma(m)\sigma(n-m) &= \frac{1}{24}n(1-6n)\sigma(n) + \frac{5}{24}n\sigma_3(n),\\
\text{(iv)} \quad \sum_{m=1}^{n-1} m^2\sigma(m)\sigma_3(n-m) &= -\frac{1}{240}n^2\sigma(n) - \frac{1}{120}n^2\sigma_3(n) + \frac{1}{80}n^2\sigma_5(n),\\
\text{(v)} \quad \sum_{m=1}^{n-1} m\sigma(m)\sigma_3(n-m) &= -\frac{1}{240}n\sigma(n) - \frac{1}{40}n^2\sigma_3(n) + \frac{7}{240}n\sigma_5(n),\\
\text{(vi)} \quad \sum_{m=1}^{n-1} m\sigma(m)\sigma_5(n-m) &= \frac{1}{504}n\sigma(n) - \frac{1}{84}n^2\sigma_5(n) + \frac{5}{504}n\sigma_7(n),\\
\text{(vii)} \quad \sum_{m=1}^{n-1} \sigma(m)\sigma_5(n-m) &= \frac{1}{504}\sigma(n) - \frac{1}{12}n\sigma_5(n) + \frac{1}{24}\sigma_5(n) + \frac{5}{126}\sigma_7(n),\\
\text{(viii)} \quad \sum_{m=1}^{n-1} \sigma(m)\sigma_7(n-m) &= -\frac{1}{480}\sigma(n) + \frac{1}{24}\sigma_7(n) + \frac{11}{480}\sigma_9(n) - \frac{1}{16}n\sigma_7(n).
\end{aligned}$$

Corollary 2.10.

$$\sum_{m=1}^{n-1}(2m^3 - 3m^2n + mn^2)\sigma(m)\sigma(n-m) = 0. \tag{3}$$

Using Theorem 2.8 (i)–(iii) and Theorem 2.5 (i), we get another formula for $\tau(n)$, given in the following corollary.

Corollary 2.11.

$$\tau(n) = 50n^4\sigma_3(n) - 7n^4(12n-5)\sigma(n) - 840\sum_{m=1}^{n-1} m^4\sigma(m)\sigma(n-m). \tag{4}$$

We end by stating some congruence relations among the divisor functions. These congruences follow as a consequence of the above convolution identities and the congruences of $\tau(n)$ (Corollary 2.8).

Corollary 2.12.

$$\begin{aligned}
\text{(i)} \quad (6n-5)\sigma(n) &\equiv \sigma_3(n) \pmod{24},\ \gcd(n,6)=1,\\
\text{(ii)} \quad \sigma(n) + 2n\sigma_3(n) &\equiv 3\sigma_5(n) \pmod{16},\ 2 \nmid n,\\
\text{(iii)} \quad n\sigma(n) + 5n\sigma_7(n) &\equiv 6n^2\sigma_5(n) \pmod{2^3\cdot 3^2\cdot 7},\ \gcd(n,42)=1,\\
\text{(iv)} \quad 20\sigma_7(n) + 11\sigma_9(n) &\equiv \sigma(n) + 30n\sigma_7(n) \pmod{2^5\cdot 3\cdot 5},\\
\text{(v)} \quad 5\sigma(n) + 6n\sigma_7(n) &\equiv 11\sigma_9(n) \pmod{2^5},\ 2 \nmid n,\\
\text{(vi)} \quad \sigma(n) + 2n\sigma_3(n) &\equiv 3\sigma_5(n) \pmod{2^4\cdot 5},\ \gcd(n,10)=1,\\
\text{(vii)} \quad \sigma(n) + 10(3n-2)\sigma_7(n) &\equiv 11\sigma_9(n) \pmod{2^3\cdot 3\cdot 5},\ \gcd(n,30)=1.
\end{aligned}$$

3. Preliminaries

In this section we shall provide some well-known facts about modular forms and give the definitions of Rankin-Cohen bracket and quasimodular forms, which are essential in proving our theorems. For basic details of the theory of modular forms and quasimodular forms, we refer to [13, 2, 7, 12].

For an even integer $k \geq 4$, let $M_k(1)$ (resp. $S_k(1)$) denote the vector space of modular forms (resp. cusp forms) of weight k for the full modular group $SL_2(\mathbb{Z})$. Let E_k be the normalized Eisenstein series of weight k in $M_k(1)$, given by

$$E_k(z) = 1 - \frac{4k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n,$$

where B_k is the k-th Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty}\frac{B_m}{m!}x^m.$$

The first few Eisenstein series are as follows.

$$\begin{aligned}
E_4 &= 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n,\\
E_6 &= 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n,\\
E_8 &= 1 + 480\sum_{n\geq 1}\sigma_7(n)q^n,\\
E_{10} &= 1 - 264\sum_{n\geq 1}\sigma_9(n)q^n,\\
E_{12} &= 1 + \frac{65520}{691}\sum_{n\geq 1}\sigma_{11}(n)q^n.
\end{aligned} \tag{5}$$

Since the space $M_k(1)$ is one-dimensional for $4 \leq k \leq 10$ and for $k = 14$ and $S_{12}(1)$ is one-dimensional, we have the following well-known identities:

Theorem 3.1.

$$\begin{aligned}
&\text{(i)} & E_8(z) &= E_4^2(z),\\
&\text{(ii)} & E_{10}(z) &= E_4(z)E_6(z),\\
&\text{(iii)} & E_{12}(z) - E_8(z)E_4(z) &= \left(\tfrac{65520}{691} - 720\right)\Delta(z),\\
&\text{(iv)} & E_{12}(z) - E_6^2(z) &= \left(\tfrac{65520}{691} + 1008\right)\Delta(z),
\end{aligned}$$

Rankin-Cohen Brackets: The derivative of a modular form is not a modular form. However, the works of Rankin and Cohen [11, 1] lead to the concept of Rankin-Cohen brackets which is defined in the following.

Definition: Let $f \in M_k(1)$ and $g \in M_l(1)$ be modular forms of weights k and l respectively. For each $\nu \geq 0$, define the ν-th *Rankin-Cohen bracket* (in short RC bracket) of f and g by

$$[f,g]_\nu := \sum_{r=0}^{\nu} (-1)^r \binom{\nu+k-1}{\nu-r}\binom{\nu+l-1}{r} D^{(r)}f\, D^{(\nu-r)}g, \tag{6}$$

where we have set $D := \frac{1}{2\pi i}\frac{d}{dz}$. When $\nu = 1$, we write $[f,g]$ instead of $[f,g]_1$.

Theorem 3.2. *([14, pp. 58–61]) Let $f \in M_k(1)$ and $g \in M_l(1)$. Then $[f,g]_\nu$ is a modular form of weight $k+l+2\nu$ for $SL_2(\mathbb{Z})$. It is a cusp form if $\nu \geq 1$.*

Quasimodular forms: We now present some basics of quasimodular forms. Another important Eisenstein series is the weight 2 Eisenstein series E_2 given by

$$E_2(z) = 1 - 24\sum_{n\geq 1} \sigma(n) q^n. \tag{7}$$

It is not a modular form because it doesn't satisfy the required transformation property under the action of $SL_2(\mathbb{Z})$. However, it plays a fundamental role in defining the concept of quasimodular forms, which was formally introduced by M. Kaneko and D. Zagier [2].

Definition: Let $k \geq 1, s \geq 0$ be natural numbers. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is defined to be a quasimodular form of weight k, depth s on $SL_2(\mathbb{Z})$, if there exist holomorphic functions $f_0, f_1, \ldots, f_s$ on $\mathcal{H}$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{i=0}^{s} f_i(z)\left(\frac{c}{cz+d}\right)^i, \tag{8}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ and such that f_s is holomorphic at infinity and not identically vanishing.

Remark 3.1. It is a fact that if f is a quasimodular form of weight k and depth s, not identically zero, then k is even and $s \leq k/2$.

The space of all quasimodular forms of weight k, depth s on $SL_2(\mathbb{Z})$ is denoted by $\widetilde{M}_k^{\leq s}(1)$. Note that E_2 is a quasimodular form of weight 2 and depth 1, and so $E_2 \in \widetilde{M}_2^{\leq 1}(1)$.

We need the following lemma (see [12] for details).

Lemma 3.3. *(*[12, Lemma 1.17]*) Let $k \geq 2$ be even. Then*

$$\widetilde{M}_k^{\leq k/2}(1) = \bigoplus_{i=0}^{k/2-1} D^i M_{k-2i}(1) \oplus \mathbb{C}D^{k/2-1}E_2.$$

Definition: (Rankin-Cohen bracket of quasimodular forms) Let $f \in \widetilde{M}_k^{\leq s}(1)$ and $g \in \widetilde{M}_l^{\leq t}(1)$ be quasimodular forms of weights k, l and depths s, t respectively. For each $n \geq 0$, define the n-th *Rankin-Cohen Bracket* (in short RC bracket) of f and g by

$$\Phi_{n;k,s;l,t}(f,g) := \sum_{r=0}^{\nu}(-1)^r \binom{k-s+n-1}{n-r}\binom{l-t+n-1}{r} D^{(r)}f\ D^{(n-r)}g. \tag{9}$$

Theorem 3.4. *(*[8, Theorem 1]*) Let $k, l \geq 2$ and $s, t \geq 0$ with $s \leq k/2$ and $t \leq l/2$. Then for $n \geq 0$, the RC bracket $\Phi_{n;k,s;l,t}(f,g)$ is a quasimodular form of weight $k+l+2n$ and depth $s+t$ for $SL_2(\mathbb{Z})$.*

4. PROOFS

4.1. **Proof of Theorem 2.1:** Differentiating (i) of Theorem 3.1 twice, we get

$$D^2E_8 = 2(DE_4)^2 + 2E_4D^2E_4. \tag{10}$$

Now consider the RC bracket $[E_4, E_4]_2 = 20D^2E_4E_4 - 25(DE_4)^2 \in S_{12}(1) = \mathbb{C}\Delta$. Substituting for $E_4D^2E_4$ from (10), we get

$$2D^2E_8 - 9(DE_4)^2 = 960\Delta. \tag{11}$$

Comparing the n-th Fourier coefficients we obtain (i).

On the other hand, in (11) substituting for $(DE_4)^2$ from (10), we get

$$-5D^2E_8 + 18E_4D^2E_4 = 1920\Delta,$$

from which we obtain the identity (ii). To obtain (iii) we differentiate (11) and compare the coefficients. Finally to prove (iv), we differentiate (10) and use it in (11) to get

$$960D\Delta = -D^3E_8 + 6E_4D^3E_4.$$

Now comparing the n-th Fourier coefficients we get (iv). This completes the proof of Theorem 2.1.

4.2. **Proof of Theorem 2.2:** We now take differentiation of Theorem 3.1 (ii) to get

$$DE_{10} = E_6DE_4 + E_4DE_6. \tag{12}$$

Now consider the RC bracket $[E_4, E_6]$ which belongs to $S_{12}(1)$ and so is a constant multiple of Δ. Thus, we have

$$4E_4DE_6 - 6E_6DE_4 = -3456\ \Delta. \tag{13}$$

Substituting for E_4DE_6 from (12) in (13) and comparing the n-th Fourier coefficients, we get the identity (i). Instead, if we substitute for E_6DE_4 from (12) in (13), we get the identity (ii). Taking derivative of (12) we get

$$D^2E_{10} = E_6D^2E_4 + 2(DE_4)(DE_6) + E_4D^2E_6. \tag{14}$$

Differentiating (13) we get

$$-2(DE_4)(DE_6) + 4E_4D^2E_6 - 6E_6D^2E_4 = -3456\ D\Delta. \tag{15}$$

Eliminating $E_4D^2E_6$ from (14) and (15) we obtain

$$2D^2E_{10} - 5(E_6D^2E_4 + (DE_4)(DE_6)) = -1728\ D\Delta. \tag{16}$$

Now, consider the RC bracket $[E_4, E_6]_2$. Since it belongs to $S_{14}(1)$, it must be zero. So, we get the following

$$10E_4D^2E_6 - 35(DE_4)(DE_6) + 21E_6D^2E_4 = 0. \tag{17}$$

Eliminating $(DE_4)(DE_6)$ from (14) and (16), we get

$$-D^2E_{10} - 5E_6D^2E_4 + 5E_4D^2E_6 + 3456\ D\Delta = 0, \tag{18}$$

and from (14) and (17), we get

$$-35D^2E_{10} + 77E_6D^2E_4 + 55E_4D^2E_6 = 0. \tag{19}$$

Now, eliminating $E_4D^2E_6$ from (18) and (19), we obtain

$$24D^2E_{10} - 132E_6D^2E_4 + 38016\ D\Delta = 0,$$

whose n-th Fourier coefficient give the identity (iii). In the last step if we eliminate $E_6D^2E_4$ instead of $E_4D^2E_6$, we obtain the identity (v). For the proof of identity (iv), we first eliminate $E_6D^2E_4$ from (14) & (16) and (14) & (17) to get

$$\begin{aligned} -3D^2E_{10} + 5(DE_4)(DE_6) + 5E_4D^2E_6 + 1728\ D\Delta &= 0, \\ -21D^2E_{10} + 77(DE_4)(DE_6) + 11E_4D^2E_6 &= 0. \end{aligned}$$

The required identity (iv) follows by eliminating $E_4D^2E_6$ from the above two equations. This completes the proof of Theorem 2.2.

4.3. **Proof of Theorem 2.3:** This identity follows easily by differentiating (iv) of Theorem 3.1 and comparing the n-th Fourier coefficients.

4.4. **Proof of Theorem 2.4:** Differentiating (iii) of Theorem 3.1 gives

$$DE_{12} - E_4DE_8 - E_8DE_4 = \left(\frac{65520}{691} - 720\right) D\Delta. \tag{20}$$

Next, using the fact that the RC bracket $[E_4, E_8]$ is zero (because it belongs to the space $S_{14}(1)$) we get

$$E_4DE_8 = 2E_8DE_4. \tag{21}$$

Substituting for E_4DE_8 from (21) in (20) and comparing the n-th Fourier coefficients yield identity (i). On the other hand, substituting for E_8DE_4 from (21) in (20) gives

$$DE_{12} - \frac{3}{2}E_4DE_8 = \left(\frac{65520}{691} - 720\right) D\Delta, \tag{22}$$

from which (ii) follows.

4.5. **Proof of Theorem 2.5:** As mentioned before, we make use of the theory of quasimodular forms. Recall that the Eisenstein series E_2 is a quasimodular form of weight 2 and depth 1 on $SL_2(\mathbb{Z})$. So, $D^iE_2 \in \widetilde{M}_{2i+2}^{\leq i+1}(1)$. Consider the following 6 quasimodular forms of weight 12 on $SL_2(\mathbb{Z})$ which are RC brackets of functions involving D^iE_2.

(i) $f_1(z) = \Phi_{1;8,4;2,1}(D^3E_2, E_2) = 4D^3E_2DE_2 - E_2D^4E_2,$

(ii) $f_2(z) = \Phi_{1;6,3;4,2}(D^2E_2, DE_2) = 3(D^2E_2)^2 - 2D^3E_2DE_2,$

(iii) $f_3(z) = \Phi_{2;6,3;2,1}(D^2E_2, E_2) = 6(D^2E_2)^2 - 8D^3E_2DE_2 + E_2D^4E_2,$

(iv) $f_4(z) = \Phi_{2;4,2;4,2}(DE_2, DE_2) = 6D^3E_2DE_2 - 9(D^2E_2)^2,$

(v) $f_5(z) = \Phi_{3;4,2;2,1}(DE_2, E_2) = 16D^3E_2DE_2 - 18(D^2E_2)^2 - E_2D^4E_2,$

(vi) $f_6(z) = \Phi_{4;2,1;2,1}(E_2, E_2) = -32D^3E_2DE_2 + 36(D^2E_2)^2 + 2E_2D^4E_2.$

First note that $f_6(z) = -2f_5(z)$ and $f_4(z) = -3f_2(z)$. Using the definition of the RC brackets one finds that $f_1, f_2 \in \widetilde{M}_{12}^{\leq 5}(1)$, $f_3, f_4 \in \widetilde{M}_{12}^{\leq 4}(1)$, $f_5 \in \widetilde{M}_{12}^{\leq 3}(1)$ and $f_6 \in \widetilde{M}_{12}^{\leq 2}(1)$. Considering all the functions f_i, $1 \leq i \leq 6$ in the space $\widetilde{M}_{12}^{\leq 6}(1)$ and using the decomposition stated in Lemma 3.3, we have the following expressions for the f_i's.

$$\begin{aligned} f_1(z) &= \frac{24}{7}\Delta(z) + \frac{3}{35}D^4E_4,\\ f_2(z) &= -\frac{24}{7}\Delta(z) + \frac{1}{70}D^4E_4,\\ f_3(z) &= -\frac{72}{7}\Delta(z) - \frac{2}{35}D^4E_4,\\ f_5(z) &= 24\Delta(z), \end{aligned} \tag{23}$$

Using the definitions of f_i and (23), we express the RC brackets appearing in the definitions of f_i in terms of $\Delta(z)$ and $D^4E_4(z)$. This way we get four expressions corresponding to the functions f_1, f_2, f_3 and f_5. Comparing the Fourier expansions corresponding to f_5 gives the required Niebur's identity (i). It also follows by eliminating D^4E_4 between any two expressions corresponding to f_i's, $1 \leq i \leq 3$. The rest of the identities ((ii) to (iv)) are obtained directly by comparing the Fourier coefficients of the expressions corresponding to f_i, $1 \leq i \leq 3$, respectively.

4.6. **Proof of Theorem 2.6:** The identities (i) and (ii) follow directly by comparing the Fourier coefficients of the RC brackets $[E_4, E_6]_1 = 3456\Delta$ and $[E_4, E_4]_2 = 960\Delta$ respectively. Now we prove (iii). Differentiating twice the expression (iv) of Theorem 3.1 we get

$$2(DE_6)^2 + 2E_6D^2E_6 = D^2E_{12} - \alpha D^2\Delta, \tag{24}$$

where $\alpha = \left(\frac{65520}{691} + 1008\right)$. Since $\dim S_{16}(1) = 1$, we get $[E_6, E_6]_2 = 42D^2E_6 - 49(DE_6)^2 = -\frac{49}{48}[E_6, E_8] = -\frac{49}{48}(8E_8DE_6 - 6E_6DE_8)$. Substituting for $E_6D^2E_6$ from equation (24) and comparing the Fourier coefficients both the sides give the identity (iii). In the above argument, replacing $[E_6, E_8]$ by $[E_4, E_8]_2$ one gets the identity (iv) and replacing by $[E_4, E_{10}]$ one gets (v).

4.7. **Proof of Theorem 2.7:** All the identities are straight forward comparison of the Fourier coefficients of the following quasimodular forms of weight 12 (using the decomposition given in Lemma 3.3) respectively: E_2E_{10}, DE_2E_8, $D^2E_2E_6$, $D^3E_2E_4$, $\Phi_{1;2,1;8,0}(E_2, E_8)$, $\Phi_{1;4,2;6,0}(DE_2, E_6)$, $\Phi_{1;6,3;4,0}(D^2E_2, E_4)$, $\Phi_{2;2,1;6,0}(E_2, E_6)$ and $\Phi_{2;4,2;4,0}(DE_2, E_4)$.

4.8. **Proof of Theorem 2.9:** Using any two the f_i's, $1 \leq i \leq 3$, in (23) and eliminating $\Delta(z)$, we get the following expression among the divisor functions.

$$24\sum_{m=1}^{n-1}(4m^3 - 3m^2n)\sigma(m)\sigma(n-m) = n^3\sigma(n) - n^3\sigma_3(n). \tag{25}$$

From Niebur's identity, we have

$$\Delta(z) = E_2D^4E_2(z) - 16DE_2\ D^3E_2 + 18(D^2E_2)^2. \tag{26}$$

It is well known that E_2 satisfies the following transformation property with respect to $SL_2(\mathbb{Z})$. For $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$E_2(\gamma(z)) = (cz+d)^2E_2(z) + \frac{12c(cz+d)}{2\pi i}. \tag{27}$$

Using this transformation property and its successive derivatives, we see that the right-hand side expression in (26) is invariant under the stroke operation with respect to $SL_2(\mathbb{Z})$ with weight 12, provided the following identities are true.

$$3(DE_2)^2 - 2E_2D^2E_2 + 2D^3E_2 = 0, \tag{28}$$

$$D^4E_2 - E_2D^3E_2 + 2DE_2D^2E_2 = 0. \tag{29}$$

It is easy to see that the derivative of (28) gives (29) and so it is enough to show the identity (28). Note that the forms $(DE_2)^2$ and $E_2D^2E_2$ are quasimodular forms of weight 8 and depth 4. So, by using Lemma 3.3, we have

$$(DE_2)^2 = \frac{1}{5}DE_6 + 2D^3E_2,$$
$$E_2D^2E_2 = \frac{3}{10}DE_6 + 4D^3E_2.$$

Eliminating DE_6 from the above two equations, we get (28). Thus we have demonstrated another proof of the Niebur's identity. Now (28) and (29) give rise to the following identities:

$$12\sum_{m=1}^{n-1}(5m^2 - 3mn)\sigma(m)\sigma(n-m) = n^2\sigma(n) - n^3\sigma(n), \tag{30}$$

$$24\sum_{m=1}^{n-1}(3m^3 - 2m^2n)\sigma(m)\sigma(n-m) = n^3\sigma(n) - n^4\sigma(n). \tag{31}$$

By simple manipulations of the equations (25), (30) and (31), we get the identities (i) – (iii). Identity (iv) is obtained by taking the difference of the identities (iii) and (iv) of Theorem 2.7. Next we prove (v). Eliminating the expression $\sum m^3\sigma(m)\sigma_3(n-m)$ from (iv) and (ix) of Theorem 2.7 and substituting for the expression $\sum m^2\sigma(m)\sigma_3(n-m)$ from (iv), we get (v). Eliminating $\tau(n)$ from (iii) and (iv) of Theorem 2.7 gives the identity (vi). To prove (vii) we first eliminate $\tau(n)$ between (vi) and (viii) of Theorem 2.7 and then use the identity (vi) of Theorem 2.9. Finally eliminating $\tau(n)$ between (ii) and (v) of Theorem 2.7 gives the identity (viii). This completes the proof.

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