Differential operators on Jacobi forms and special values of certain Dirichlet series

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Abstract. We construct Jacobi cusp forms by computing the adjoint of a certain linear map constructed using bilinear holomorphic differential operators with respect to the Petersson scalar product. The Fourier coefficients of the Jacobi cusp forms constructed involve special values of the shifted convolution of Dirichlet series of Rankin-Selberg type.

1. Introduction

The derivative of a modular form is not in general a modular form, but one can construct modular forms by using certain combinations of derivatives of modular forms. Rankin [13, 14] gave a general description of differential operators which map modular forms to modular forms. For every non-negative integer ν, Cohen [7] explicitly constructed certain bilinear operators from $M_k \times M_l$ to $M_{k+l+2\nu}$, where M_i denotes the space of holomorphic modular forms of weight i for the group $SL_2(\mathbb{Z})$. Zagier [16] studied algebraic properties of these bilinear operators and called them Rankin-Cohen brackets. For example, the first bracket $[\cdot,\cdot]_1$ satisfies the Jacobi identity

$$[[f,g]_1,h]_1 + [[g,h]_1,f]_1 + [[h,f]_1,g]_1 = 0, \ (f \in M_k, g \in M_l, h \in M_m)$$

giving M_* the structure of a graded Lie algebra. Kohnen [12] constructed certain elliptic cusp forms whose Fourier coefficients involve special values of certain Dirichlet series of Rankin-Selberg type by computing the adjoint of the product map (i.e. the map $f \mapsto fg$, for a fixed modular form g) with respect to the Petersson scalar product. Recently, Herrero [10] generalized the work of Kohnen by computing the adjoint of certain linear maps constructed using Rankin-Cohen brackets with respect to the Petersson scalar product. Herrero constructed cusp forms whose Fourier coefficients involve special values of Dirichlet series similar to those which appeared in the work of Kohnen, with additional factors arising due to the binomial coefficients appearing in the Rankin-Cohen brackets.

The work of Kohnen [12] has been generalized by Choie, Kim and Knopp [5] and Sakata [15] to the case of Jacobi forms. Choie [2, 3] studied the Rankin-Cohen brackets for Jacobi forms by using the heat operator acting on Jacobi forms. Recently in [11], we generalized the work of Herrero to the case of Jacobi forms. We

2000 *Mathematics Subject Classification.* Primary: 11F50; Secondary: 11F25, 11F66.

Key words and phrases. Jacobi Forms, Differential operators, Adjoint map.

explicitly computed the adjoint of certain linear maps constructed using Rankin-Cohen brackets with respect to the Petersson scalar product. Böecherer [1] studied the general bilinear holomorphic differential operators on the Jacobi group by using the Maass operator and proved that the space of bilinear holomorphic differential operators raising the weight by ν is in general of dimension equal to $\lfloor \frac{\nu}{2} \rfloor + 1$. Choie and Eholzer [4] explicitly constructed a family of dimension $\lfloor \frac{\nu}{2} \rfloor + 1$ of Rankin-Cohen type operators defined on the space of Jacobi forms raising the weight by $\nu \in \mathbb{N}$.

In this article, we consider certain linear maps defined on the space of Jacobi forms which are constructed using Rankin-Cohen type operators (studied in [4]). We explicitly compute their adjoints with respect to the Petersson scalar product. The Fourier coefficients of the image of a Jacobi cusp forms under the adjoint maps involve special values of shifted convolutions of Dirichlet series of Rankin-Selberg type.

2. Preliminaries on Jacobi forms of scalar index

Let $\mathbb{C}$ and $\mathcal{H}$ be the complex plane and the complex upper half-plane, respectively. The Jacobi group $\Gamma^J := SL_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$ acts on $\mathcal{H} \times \mathbb{C}$ in the usual way by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

Let k and m be fixed positive integers. If $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in \Gamma^J$ and ϕ is a complex-valued function defined on $\mathcal{H} \times \mathbb{C}$, then define

$$\phi|_{k,m}\gamma := (c\tau + d)^{-k} e^{2\pi i m(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z)} \ \phi(\gamma \cdot (\tau, z)).$$

Let $J_{k,m}$ be the space of Jacobi forms of weight k and index m on Γ^J, i.e. the space of holomorphic functions $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying $\phi|_{k,m}\gamma = \phi$, for all $\gamma \in \Gamma^J$ and having a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, \\ 4nm - r^2 \geq 0}} c(n, r) q^n \zeta^r \qquad (q = e^{2\pi i \tau}, \zeta = e^{2\pi i z}).$$

Furthermore, we say ϕ is a cusp form if and only if $c(n, r) \neq 0 \implies n > r^2/4m$. We denote the space of all Jacobi cusp forms by $J^{cusp}_{k,m}$. We define the Petersson scalar product on $J^{cusp}_{k,m}$ as

$$\langle \phi, \psi \rangle = \int_{\Gamma^J \backslash \mathcal{H} \times \mathcal{C}} \phi(\tau, z) \overline{\psi(\tau, z)} v^k e^{\frac{-4\pi m y^2}{v}} dV_J,$$

where $\tau = u + iv, z = x + iy$ and $dV_J = \dfrac{dudvdxdy}{v^3}$ is an invariant measure under the action of Γ^J on $\mathcal{H} \times \mathbb{C}$. The space $(J^{cusp}_{k,m}, \langle . \, , \, . \rangle)$ is a finite dimensional Hilbert space. For more details on the theory of Jacobi forms, we refer the reader to [8]. The following lemma describes the growth of the Fourier coefficients of a Jacobi form.

LEMMA 2.1. *If $k > 3$ and $\phi \in J_{k,m}$ has Fourier coefficients $c(n,r)$, then*

$$c(n,r) \ll |r^2 - 4nm|^{k-\frac{3}{2}},$$

and, moreover if ϕ is a cusp form, then

$$c(n,r) \ll |r^2 - 4nm|^{\frac{k}{2}-\frac{1}{2}}.$$

For a proof, we refer to [6].

2.1. Poincaré series. Let m, n and r be fixed integers such that $r^2 < 4mn$. Let

$$\Gamma_\infty^J := \left\{ \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) : t, \mu \in \mathbb{Z} \right\}$$

be the stabilizer of $q^n \zeta^r$ in Γ^J. Let

$$P_{k,m;(n,r)}(\tau, z) := \sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma^J} e^{2\pi i(n\tau + rz)}|_{k,m}\gamma \tag{2.1}$$

be the (n,r)-th Poincaŕe series of weight k and index m. It is well-known that $P_{k,m;(n,r)} \in J_{k,m}^{cusp}$ for $k > 2$ [9]. This series has the following property:

LEMMA 2.2. *Let $\phi \in J_{k,m}^{cusp}$ has the Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, \\ 4nm - r^2 > 0}} c(n,r) q^n \zeta^r.$$

Then

$$\langle \phi, P_{k,m;(n,r)} \rangle = \frac{m^{k-2}\Gamma(k - \frac{3}{2})(4mn - r^2)^{\frac{3}{2}-k}}{2\pi^{k-\frac{3}{2}}} \; c(n,r), \tag{2.2}$$

where $\Gamma(\cdot)$ denotes the usual gamma function.

One can get explicit Fourier expansion of $P_{k,m;(n,r)}$, for details and a proof of the Lemma 2.2, we refer to [9].

2.2. Differential operators on Jacobi forms. For an integer m, we define the heat operator

$$L_m := \frac{1}{(2\pi i)^2} \; (8\pi i m \; \partial_\tau - \partial_z^2),$$

where ∂_τ and ∂_z are the derivative with respect to τ and z, respectively. Note that

$$L_m(e^{(2\pi i(n\tau + rz))}) = (4mn - r^2)e^{(2\pi i(n\tau + rz))}.$$

Let k_1, k_2, m_1 and m_2 be positive integers, and let ϕ and ψ be two complex-valued holomorphic functions defined on $\mathcal{H} \times \mathbb{C}$. Then, for any complex number X and non-negative integer ν, define the 2ν-th and $(2\nu+1)$-th Rankin-Cohen type brackets of ϕ and ψ as

$$[\phi, \psi]_{X,\ 2\nu}^{k_1,k_2,m_1,m_2} := \sum_{\substack{\alpha,\beta,\gamma \in \mathbb{N}\cup\{0\} \\ \alpha+\beta+\gamma = \nu}} C_{\alpha,\beta,\gamma}(k_1,k_2)(1 + m_1 X)^\beta (2 - m_2 X)^\alpha L_{m_1+m_2}^\gamma \left(L_{m_1}^\alpha(\phi) L_{m_2}^\beta(\psi) \right),$$

$$[\phi,\ \psi]_{X,\ 2\nu+1}^{k_1,k_2,m_1,m_2} := m_1[\phi, \partial_z \psi]_{X,\ 2\nu}^{k_1,k_2,m_1,m_2} - m_2[\partial_z \phi,\ \psi]_{X,\ 2\nu}^{k_1,k_2,m_1,m_2},$$

where the coefficients $C_{\alpha,\beta,\gamma}(k_1,k_2)$ are given by

$$C_{\alpha,\beta,\gamma}(k_1,k_2)=\frac{(k_1+\nu-3/2)_{\beta+\gamma}}{\alpha!}\frac{(k_2+\nu-3/2)_{\alpha+\gamma}}{\beta!}\frac{(-(k_1+k_2+\nu-3/2))_{\alpha+\beta}}{\gamma!},$$

and

$$(x)_m=\prod_{0\leqslant i\leqslant m-1}(x-i),$$

with $x!=\Gamma(x+1)$. Using the action of the heat operator, one can verify that

$$[\phi|_{k_1,m_1}\gamma,\psi|_{k_2,m_2}\gamma]_{X,\nu}^{k_1,k_2,m_1,m_2}=[\phi,\psi]_{X,\nu}^{k_1,k_2,m_1,m_2}|_{k_1+k_2+\nu,m_1+m_2}\gamma,\ \forall\gamma\in\Gamma^J. \tag{2.3}$$

This implies the following result, which was proved in [4].

THEOREM 2.3. *Let $\phi\in J_{k_1,\ m_1}$ and $\psi\in J_{k_2,\ m_2}$, where k_1,k_2,m_1 and m_2 are positive integers. Then, for any $X\in\mathbb{C}$ and any non-negative integer ν, the function $[\phi,\psi]_{X,\nu}^{k_1,k_2,m_1,m_2}$ is a Jacobi form of weight $k_1+k_2+\nu$ and index m_1+m_2. Moreover, $[\phi,\psi]_{X,\nu}^{k_1,k_2,m_1,m_2}$ is a Jacobi cusp form for $\nu>1$. In fact, $[\ ,\ ]_{X,\nu}^{k_1,k_2,m_1,m_2}$ is a bilinear map from $J_{k_1,\ m_1}\times J_{k_2,\ m_2}$ to $J_{k_1+k_2+\nu,\ m_1+m_2}$.*

REMARK 2.1. For $\nu\in 2\mathbb{N}$, the operator $\left(\dfrac{d}{dX}\right)^{\nu/2}[\phi,\psi]_{X,\nu}^{k_1,k_2,m_1,m_2}$ is, up to a scalar multiple of the Rankin-Cohen bracket, studied in [2]. If we take $\nu=1$ in Theorem 2.3, we obtain Theorem 9.5 [8]. These are some applications of the Rankin-Cohen brackets.

3. Statement of the Theorems

Let $\psi\in J_{k_2,\ m_2}^{cusp}$ and ν be a non-negative integer. For a complex number X, we define the map

$$T_{\psi,X,\nu}:J_{k_1,\ m_1}^{cusp}\to J_{k_1+k_2+\nu,\ m_1+m_2}^{cusp}$$

by

$$T_{\psi,X,\nu}(\phi)=[\phi,\psi]_{X,\nu}^{k_1,k_2,m_1,m_2}.$$

This is a $\mathbb{C}$-linear map of finite dimensional Hilbert spaces and therefore there exists an adjoint map

$$T_{\psi,X,\nu}^{*}:J_{k_1+k_2+\nu,\ m_1+m_2}^{cusp}\to J_{k_1,\ m_1}^{cusp},$$

such that

$$\langle\phi,\ T_{\psi,X,\nu}(\omega)\rangle=\langle T_{\psi,\ X,\ \nu}^{*}(\phi),\ \omega\rangle,$$

for all $\phi\in J_{k_1+k_2+\nu,\ m_1+m_2}^{cusp}$ and $\omega\in J_{k_1,\ m_1}^{cusp}$.

As an application one can obtain certain arithmetic information of Fourier coefficients of Jacobi forms using $T_{\psi,X,\nu}^{*}$ (refer to [11]) and analogous results in the case modular forms (refer to [12, 10]). In the following theorems, we exhibit the Fourier coefficients of $T_{\psi,X,\nu}^{*}(\phi)$ for $\phi\in J_{k_1+k_2+\nu,\ m_1+m_2}^{cusp}$ for ν even and odd. These Fourier coefficients involve special values of certain shifted convolution of Dirichlet series of Rankin-Selberg type associated to ϕ and ψ.

THEOREM 3.1. *Let k_1,k_2,m_1,m_2 be positive integers, such that $k_1>4,k_2>3$ and ν be a non-negative integer. Let $\psi\in J_{k_2,\ m_2}^{cusp}$ has Fourier expansion*

$$\psi(\tau,z)=\sum_{\substack{n_1,r_1\in\mathbb{Z},\\4m_2n_1-r_1^2>0}}a(n_1,r_1)q^{n_1}\zeta^{r_1}.$$

Then the image of any cusp form $\phi \in J^{cusp}_{k_1+k_2+2\nu,\ m_1+m_2}$ *with Fourier expansion*

$$\phi(\tau,z) = \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}} b(n_2,r_2)q^{n_2}\zeta^{r_2}$$

under $T^*_{\psi,X,2\nu}$ *is given by*

$$T^*_{\psi,X,2\nu}(\phi)(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z},\\ 4m_1n-r^2>0}} c_{\nu,X}(n,r)q^n\zeta^r,$$

where

$$\begin{aligned} c_{\nu,X}(n,r) &= \frac{(4m_1n-r^2)^{k_1-3/2}\ (m_1+m_2)^{k_1+k_2+2\nu-2}\ \Gamma(k_1+k_2+2\nu-\frac{3}{2})}{\pi^{k_2+2\nu}\ m_1^{k_1-2}\ \Gamma(k_1-\frac{3}{2})} \\ &\times \sum_{\alpha+\beta+\gamma=\nu} C_{\alpha,\beta,\gamma}(k_1,k_2)(1+m_1X)^{\beta}(2-m_2X)^{\alpha}\ (4m_1n-r^2)^{\alpha} \\ &\times \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0\\ 4(m_1+m_2)(n+n_1)-(r+r_1)^2>0}} \frac{(4m_2n_1-r_1^2)^{\beta}\ \overline{a(n_1,r_1)}\ b(n+n_1,r+r_1)}{(4(m_1+m_2)(n+n_1)-(r+r_1)^2)^{k_1+k_2+2\nu-(\gamma+\frac{3}{2})}}. \end{aligned}$$

Theorem 3.2. *Let* k_1, k_2, m_1 *and* m_2 *be positive integers such that* $k_1 > 4, k_2 > 3$ *and* ν *be a non-negative integer. Let* $\psi \in J^{cusp}_{k_2,\ m_2}$ *has Fourier expansion*

$$\psi(\tau,z) = \sum_{\substack{n_1,r_1\in\mathbb{Z},\\ 4m_2n_1-r_1^2>0}} a(n_1,r_1)q^{n_1}\zeta^{r_1}.$$

Then the image of any cusp form $\phi \in J^{cusp}_{k_1+k_2+2\nu+1,\ m_1+m_2}$ *with Fourier expansion*

$$\phi(\tau,z) = \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}} b(n_2,r_2)q^{n_2}\zeta^{r_2}$$

under $T^*_{\psi,X,2\nu+1}$ *is given by*

$$T^*_{\psi,X,2\nu+1}(\phi)(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z},\\ 4m_1n-r^2>0}} c_{\nu,X}(n,r)q^n\zeta^r,$$

where

$$\begin{aligned} c_{\nu,X}(n,r) &= \frac{2i(4m_1n-r^2)^{k_1-3/2}\ (m_1+m_2)^{k_1+k_2+2\nu-1}\ \Gamma(k_1+k_2+2\nu-\frac{1}{2})}{\pi^{k_2+2\nu}\ m_1^{k_1-2}\ \Gamma(k_1-\frac{3}{2})} \\ &\times \sum_{\alpha+\beta+\gamma=\nu} C_{\alpha,\beta,\gamma}(k_1,k_2)(1+m_1X)^{\beta}(2-m_2X)^{\alpha}\ (4m_1n-r^2)^{\alpha} \\ &\times \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0\\ 4(m_1+m_2)(n+n_1)-(r+r_1)^2>0}} \frac{(4m_2n_1-r_1^2)^{\beta}\ (m_1r_1-m_2r)\overline{a(n_1,r_1)}\ b(n+n_1,r+r_1)}{(4(m_1+m_2)(n+n_1)-(r+r_1)^2)^{k_1+k_2+2\nu-(\gamma+\frac{1}{2})}}. \end{aligned}$$

4. Proofs

We follow the same exposition as in the proof of Theorem 3.1 in [11]. The following lemma is used to prove the above theorems.

Lemma 4.1. *Using the same notation as in Theorem 3.1, we have*

$$\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J}\int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}} \mid \phi(\tau,z)\overline{\left[e^{2\pi i(n\tau+rz)}\mid_{k_1,m_1}\gamma,\ \psi(\tau,z)\right]^{k_1,k_2,m_1,m_2}_{X,2\nu}}\Im(\tau)^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)(\Im(z))^2}{\Im(\tau)}}\mid dV_J$$

converges.

Proof. The proof is analogous to that of Lemma 4.1 in [11]. □

We now give a proof of Theorem 3.1. Write

$$T^*_{\psi,X,2\nu}(\phi)(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z},\\ 4m_1n-r^2>0}} c_{\nu,X}(n,r)q^n\zeta^r.$$

Consider the (n,r)-th Poincaŕe series of weight k_1 and index m_1, as given in (2.1). Using Lemma 2.2, we have

$$\langle T^*_{\psi,\ X,\ 2\nu}\phi,P_{k_1,m_1;(n,r)}\rangle=\frac{m_1^{k_1-2}\Gamma(k_1-\frac{3}{2})(4m_1n-r^2)^{\frac{3}{2}-k_1}}{2\pi^{k_1-\frac{3}{2}}}\ c_{\nu,X}(n,r).$$

On the other hand, by the definition of the adjoint map, we have

$$\langle T^*_{\psi,\ X,\ 2\nu}\ \phi,P_{k_1,m_1;(n,r)}\rangle=\langle\phi,T_{\psi,\ X,\ 2\nu}\ (P_{k_1,m_1;(n,r)})\rangle=\langle\phi,[P_{k_1,m_1;(n,r)},\psi]^{k_1,k_2,m_1,m_2}_{X,2\nu}\rangle.$$

Hence, we obtain

$$c_{\nu,X}(n,r)=\frac{2\pi^{k_1-\frac{3}{2}}(4m_1n-r^2)^{k_1-\frac{3}{2}}}{m_1^{k_1-2}\Gamma(k_1-\frac{3}{2})}\langle\phi,[P_{k_1,m_1;(n,r)},\psi]^{k_1,k_2,m_1,m_2}_{X,2\nu}\rangle. \tag{4.1}$$

By definition,

$$\begin{aligned}&\langle\phi,[P_{k_1,m_1;(n,r)},\psi]^{k_1,k_2,m_1,m_2}_{X,\ 2\nu}\rangle=\\ &\int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}}\phi(\tau,z)\overline{\left[P_{k_1,m_1;(n,r)}(\tau,z),\psi(\tau,z)\right]^{k_1,k_2,m_1,m_2}_{X,\ 2\nu}}\Im(\tau)^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)(\Im(z))^2}{\Im(\tau)}}dV_J\\ &=\int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}}\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J}\phi(\tau,z)\overline{\left[e^{2\pi i(n\tau+rz)}\mid_{k_1,m_1}\gamma,\ \psi(\tau,z)\right]^{k_1,k_2,m_1,m_2}_{X,\ 2\nu}}\ \Im(\tau)^{k_1+k_2+2\nu}e^{\frac{-4\pi(m_1+m_2)(\Im(z))^2}{\Im(\tau)}}dV_J.\end{aligned}$$

By Lemma 4.1, we can interchange the order of summation and integration in the above equation. Hence, $\langle\phi,[P_{k_1,m_1;(n,r)},\psi]^{k_1,k_2,m_1,m_2}_{X,\ 2\nu}\rangle$ equals

$$\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J}\int_{\Gamma^J\backslash\mathcal{H}\times\mathbb{C}}\phi(\tau,z)\overline{\left[e^{2\pi i(n\tau+rz)}\mid_{k_1,m_1}\gamma,\psi(\tau,z)\right]^{k_1,k_2,m_1,m_2}_{X,\ 2\nu}}\ \Im(\tau)^{k_1+k_2+2\nu}\ e^{\frac{-4\pi(m_1+m_2)(\Im(z))^2}{\Im(\tau)}}\ dV_J.$$

Using equation (2.3), the fact that

$$\phi(\tau,z)\overline{\psi\mid_{k,m}\gamma(\tau,z)}\ \Im(\tau)^k\ e^{\frac{-4\pi m\Im(z)^2}{\Im(\tau)}}=\phi(\gamma(\tau,z))\overline{\psi\mid_{k,m}\gamma(\tau,z)}\ \Im(\gamma\tau)^k\ e^{\frac{-4\pi m\Im(z)^2}{\Im(\gamma\tau)}}$$

and the change of variable $(\tau,z)\mapsto\gamma^{-1}\cdot(\tau,z)$ and equation (2.3), the above equals

$$\sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J}\int_{\gamma\cdot\Gamma^J\backslash\mathcal{H}\times\mathbb{C}}\phi(\tau,z)\overline{\left[e^{2\pi i(n\tau+rz)},\ \psi(\tau,z)\right]^{k_1,k_2,m_1,m_2}_{X,\ 2\nu}}\ \Im(\tau)^{k_1+k_2+2\nu}\ e^{\frac{-4\pi(m_1+m_2)(\Im(z))^2}{\Im(\tau)}}\ dV_J.$$

Using Rankin's unfolding argument, $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_{X,2\nu}^{k_1,k_2,m_1,m_2}\rangle$ equals

$$\int\limits_{\Gamma^J_\infty\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z)\overline{\left[e^{2\pi i(n\tau+rz)},\psi(\tau,z)\right]_{X,\ 2\nu}^{k_1,k_2,m_1,m_2}}\ \Im(\tau)^{k_1+k_2+2\nu}\ e^{\frac{-4\pi(m_1+m_2)(\Im(z))^2}{\Im(\tau)}}\ dV_J$$

$$= \sum_{\substack{\alpha,\beta,\gamma\in\mathbb{N}\cup\{0\}\\ \alpha+\beta+\gamma=\nu}} C_{\alpha,\beta,\gamma}(k_1,k_2)\ (1+m_1X)^\beta\ (2-m_2X)^\alpha$$

$$\times \int\limits_{\Gamma^J_\infty\backslash\mathcal{H}\times\mathbb{C}} \phi(\tau,z)\overline{L^\gamma_{m_1+m_2}\left(L^\alpha_{m_1}(e^{2\pi i(n\tau+rz)})L^\beta_{m_2}(\psi(\tau,z))\right)}\ \Im(\tau)^{k_1+k_2+2\nu}\ e^{\frac{-4\pi(m_1+m_2)(\Im(z))^2}{\Im(\tau)}}\ dV_J.$$

Inserting the Fourier expansions of ϕ and ψ and using the iterated action of the heat operators $L_{m_1},\ L_{m_2}$:

$$\begin{aligned} L^\alpha_{m_1}(e^{(2\pi i(n\tau+rz))}) &= (4m_1n-r^2)^\alpha e^{(2\pi i(n\tau+rz))},\\ L^\beta_{m_2}\psi(\tau,z) &= \sum_{\substack{n_1,r_1\in\mathbb{Z},\\ 4m_2n_1-r_1^2>0}} (4m_2n_1-r_1^2)^\beta a(n_1,r_1)e^{(2\pi i(n_1\tau+r_1z))}, \end{aligned}$$

and $L_{m_1+m_2}$ using (2.2), $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_{X,\ 2\nu}^{k_1,k_2,m_1,m_2}\rangle$ equals

$$\sum_{\substack{\alpha,\beta,\gamma\in\mathbb{N}\cup\{0\}\\ \alpha+\beta+\gamma=\nu}} C_{\alpha,\beta,\gamma}(k_1,k_2)\ (1+m_1X)^\beta\ (2-m_2X)^\alpha\ (4m_1n-r^2)^\alpha$$

$$\times \sum_{\substack{n_2,r_2\in\mathbb{Z},\\ 4(m_1+m_2)n_2-r_2^2>0}} b(n_2,r_2)\sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0}} (4m_2n_1-r_1^2)^\beta(4(m_1+m_2)(n+n_1)-(r+r_1)^2)^\gamma\ \overline{a(n_1,r_1)}$$

$$\times \int\limits_{\Gamma^J_\infty\backslash\mathcal{H}\times\mathbb{C}} e^{2\pi i(n_2\tau+r_2z)}\ \overline{e^{2\pi i((n+n_1)\tau+(r+r_1)z)}}\ \Im(\tau)^{k_1+k_2+2\nu}\ e^{\frac{-4\pi(m_1+m_2)(\Im(z))^2}{\Im(\tau)}}\ dV_J.$$

A fundamental domain for the action of Γ^J_∞ on $\mathcal{H}\times\mathbb{C}$ is given by the set

$$\{(u,v,x,y) : u\in[0,1], v\in[0,\infty], x\in[0,1], y\in\mathbb{R}\}.$$

Integrating in this region, $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_{X,\ 2\nu}^{k_1,k_2,m_1,m_2}\rangle$ equals

$$\frac{(m_1+m_2)^{k_1+k_2+2\nu-2}\Gamma(k_1+k_2+2\nu-\frac{3}{2})}{2\pi^{k_1+k_2+2\nu-\frac{3}{2}}}\sum_{\substack{\alpha,\beta,\gamma\in\mathbb{N}\cup\{0\}\\ \alpha+\beta+\gamma=\nu}} C_{\alpha,\beta,\gamma}(k_1,k_2)(1+m_1X)^\beta(2-m_2X)^\alpha$$

$$\times\ (4m_1n-r^2)^\alpha \sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0\\ 4(m_1+m_2)(n+n_1)-(r+r_1)^2>0}} \frac{(4m_2n_1-r_1^2)^\beta\ \overline{a(n_1,r_1)}\ b(n+n_1,r+r_1)}{(4(m_1+m_2)(n+n_1)-(r+r_1)^2)^{k_1+k_2+2\nu-(\gamma+\frac{3}{2})}}.$$

In the above, we use the orthogonality relations for the exponential function to compute the integrals in x and u, the Gaussian integral to compute the integral in y and the integral representation of the Gamma function to compute the integral in v. Inserting the expression for $\langle \phi, [P_{k_1,m_1;(n,r)}, \psi]_{X,\ 2\nu}^{k_1,k_2,m_1,m_2}\rangle$ in (4.1), we obtain the required expression for $c_{\nu,X}(n,r)$ given in Theorem 3.1.

The proof of Theorem 3.2 is analogous to that of Theorem 3.1.

Acknowledgements. We thank the organizers for giving us the opportunity to participate in "Building Bridges: 3rd EU/US Summer School and Workshop on Automorphic Forms and Related Topics" during July 11-22, 2016 at the University of Sarajevo, Bosnia and Herzegovina. The first author would like to thank the National Board of Higher Mathematics (NBHM), India for providing travel support to attend the workshop. The first author is financially supported by the Council of Scientific and Industrial Research (CSIR), India. The second author is partially funded by the SERB grant SR/FTP/MS-053/2012. Finally, the authors thank the referee for careful reading of the paper and many helpful suggestions.

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