RANKIN-COHEN BRACKETS ON HILBERT MODULAR FORMS AND SPECIAL VALUES OF CERTAIN DIRICHLET SERIES

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Abstract. Given a fixed Hilbert modular form, we consider a family of linear maps between the spaces of Hilbert cusp forms by using the Rankin-Cohen brackets and then we compute the adjoint maps of these linear maps with respect to the Petersson scalar product. The Fourier coefficients of the Hilbert cusp forms constructed using this method involve special values of certain Dirichlet series of Rankin-Selberg type associated to Hilbert cusp forms.

1. Introduction

W. Kohnen [16] constructed certain linear maps between spaces of modular forms with the property that the Fourier coefficients of image of a modular form involve special values of certain Dirichlet series attached to these forms using the existence of adjoint linear maps and properties of Poincaré series. In fact, Kohnen constructed the adjoint map with respect to the usual Petersson scalar product of the product map by a fixed cusp form. This result has been generalized by several authors to other automorphic forms (see the list [17, 19, 4, 22]). In particular, M. H. Lee [18], X. Wang and D. Pei [23] and Wang [24] have analogous results for Hilbert modular forms.

There are many interesting connections between differential operators and modular forms and many interesting results have been found. In [20, 21], R. A. Rankin gave a general description of the differential operators which send modular forms to modular forms. In [7], H. Cohen constructed bilinear operators and obtained elliptic modular forms with interesting Fourier coefficients. In [25, 26], D. Zagier studied the algebraic properties of these bilinear operators and called them Rankin–Cohen brackets.

Recently the work of Kohnen in [16] has been generalized by S. D. Herrero in [15], where the author constructed the adjoint map using the Rankin-Cohen brackets by a fixed cusp form instead of product map. Rankin–Cohen brackets for Jacobi forms were studied by Y. Choie [1, 2] by using the heat operator. The Rankin-Cohen type differential operators for Siegel modular forms of genus two were studied by Choie and W. Eholzer [3] explicitly and the existence of such operators for general genus were established by W. Eholzer and T. Ibukiyama [8]. A. K. Jha and second author generalized the work of Herrero to the case of Jacobi forms in [12, 14] and to Siegel

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modular forms of degree two in [13]. We generalize the work of Herrero to the case of Hilbert modular forms in this article. As an application one can give a different proof of a result of Y. Choie, H. Kim and O. K. Richter ([5], Theorem 3) using our method.

2. Preliminaries

Let K be a totally real number field over $\mathbb{Q}$ with degree n and $\mathcal{O}_K$ be the ring of its algebraic integers. Let

$$\Gamma_K = SL_2(\mathcal{O}_K) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathcal{O}_K, ad - bc = 1 \right\}.$$

Let $\mathbb{H}$ be the upper half plane. For $\gamma = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right) \in SL_2(\mathbb{R})^n$ and $z = (z_1, \cdots, z_n) \in \mathbb{H}^n$ define the action,

$$\gamma \circ z = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \cdots, \frac{a_n z_n + b_n}{c_n z_n + d_n} \right).$$

Let $\sigma_1, \sigma_2, ..., \sigma_n$ be all the embedding of K into $\mathbb{R}$, then Γ_K can be embedded into $SL_2(\mathbb{R})^n$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \left(\begin{pmatrix} \sigma_1(a) & \sigma_1(b) \\ \sigma_1(c) & \sigma_1(d) \end{pmatrix}, \cdots, \begin{pmatrix} \sigma_n(a) & \sigma_n(b) \\ \sigma_n(c) & \sigma_n(d) \end{pmatrix} \right).$$

We write $\alpha_i = \sigma_i(\alpha)$ for $\alpha \in K$ and $1 \leqslant i \leqslant n$. The trace and norm of $\alpha \in K$ are defined by $tr(\alpha) = \sum_{i=1}^{n} \alpha_i$ and $N(\alpha) = \prod_{i=1}^{n} \alpha_i$. The trace and norm of an element $\alpha \in \mathbb{C}^n$ are given by the sum and by the product of its components, respectively. More generally, if $c = (c_1, \cdots, c_n), d = (d_1, \cdots, d_n), k = (k_1, \cdots, k_n)$ and $m = (m_1, m_2, \cdots, m_n) \in \mathbb{C}^n$, then the trace and norm are defined by

$$tr(mz) := \sum_{i=1}^{n} m_i z_i$$

and

$$(cz + d)^k := \prod_{i=1}^{n} (c_i z_i + d_i)^{k_i}.$$

Let $k = (k_1, \cdots, k_n) \in \mathbb{N}_0^n$. For $\gamma = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right) \in SL_2(\mathbb{R})^n$ and a function $f : \mathbb{H}^n \to \mathbb{C}$ define the slash operator

$$(f \mid_k \gamma)(z) = j(\gamma, z)^{-k} f(\gamma \circ z), \text{ where } j(\gamma, z) = (cz + d).$$

A Hilbert modular form of weight $k \in \mathbb{N}_0^n$ for the group Γ_K is a holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ such that $f \mid_k \gamma = f$, for all $\gamma \in \Gamma_K$. In addition, f is called a cusp form if f vanishes at all cusps of Γ_K. Let $M_k(\Gamma_K)$ denotes the space of Hilbert modular forms of weight $k \in \mathbb{N}_0^n$ for the group Γ_K and $S_k(\Gamma_K)$ be the subspace of cusp forms.

These are finite dimensional complex vector spaces and $S_k(\Gamma_K)$ is a Hilbert space with respect to the Petersson inner product

$$\langle f, g\rangle := \int_{\Gamma_K\backslash\mathbb{H}^n} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}, \tag{1}$$

where $z = x + iy,\ dx = dx_1 \cdots dx_n$ and $dy = dy_1 \cdots dy_n$.

For $\alpha \in \mathcal{O}_K$, by $\alpha \succeq 0$ we mean either $\alpha = 0$ or α is totally positive (all the conjugates of α are positive) and by $\alpha \gg 0$ we mean α is totally positive. By Koecher principle, $f \in M_k(\Gamma_K)$ has a Fourier expansion at the cusp ∞ of the form

$$f(z) = \sum_{\substack{m\in\mathcal{O}_K^*\\ m\succeq 0}} a_m e^{2\pi i tr(mz)},$$

where $\mathcal{O}_K^* = \{\mu \in K \mid tr(\mu\lambda) \in \mathbb{Z} \text{ for all } \lambda \in \mathcal{O}_K\}$. For an integer $x \in \mathbb{N}_0$, we denote $\overrightarrow{x} := (x, \cdots , x) \in \mathbb{N}_0^n$. For $\nu = (\nu_1, \cdots , \nu_n) \in \mathbb{N}_0^n$ and $z = (z_1, \cdots , z_n) \in \mathbb{C}^n$, we denote

$$|\nu| = \sum_{i=1}^n \nu_i,\ \ \nu! = \prod_{i=1}^n \nu_i! \text{ and } z^\nu = \prod_{i=1}^n z_i^{\nu_i}.$$

One has the following growth condition on the Fourier coefficients of a Hilbert modular form.

Proposition 2.1. *(Hecke) Let* $f(z) = \sum_{\substack{m\in\mathcal{O}_K^*\\ m\succeq 0}} a_m e^{2\pi i tr(mz)} \in M_k(\Gamma_K)$, *then*

$$a_m \ll m^{k-\overrightarrow{1}}, \tag{2}$$

If f is a cusp form, then

$$a_m \ll m^{\frac{k}{2}}. \tag{3}$$

For a proof, we refer to [10].

2.1. **Eisenstein series.** Let $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathcal{O}_K \right\}$ and let $\overrightarrow{k} = (k, \cdots , k) \in \mathbb{N}_0^n$. Define

$$E_k(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} (1|_{\overrightarrow{k}}\gamma)(z), \tag{4}$$

the Hilbert Eisenstein series. It is well known that (see [10]) E_k is a Hilbert modular form of weight $\overrightarrow{k}$ on Γ_K for $k > 2$.

2.2. **Poincaré series.** For $\mu \in \mathcal{O}_K, \mu \gg 0$ and $k = (k_1, \cdots , k_n) \in \mathbb{N}_0^n$, define

$$\mathcal{P}_{k,\mu}(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} (e^{2\pi i tr(\mu z)}|_k\gamma)(z). \tag{5}$$

It is well known that $\mathcal{P}_{k,\mu} \in S_k(\Gamma_K)$ if $\mu \gg 0$ and $k_j > 2$ for all $1 \leqslant j \leqslant n$.
One has the following characterization:

Theorem 2.2. [10] *If* $f(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} a_m e^{2\pi i tr(mz)} \in S_k(\Gamma_K)$, *then*

$$\langle f, \mathcal{P}_{k,\mu} \rangle = vol(\mathcal{O}_K/\mathbb{R}^n) \frac{(k - \overrightarrow{2})!}{(4\pi\mu)^{k-\overrightarrow{1}}} \ a_\mu. \tag{6}$$

For more details on the theory of Hilbert modular forms, we refer to [10].

2.3. **Rankin-Cohen Brackets.** For $t = (t_1, \cdots, t_n) \in \mathbb{N}_0^n$, let $f^{(t)} := \frac{\partial^{|t|}}{\partial z_1^{t_1} \partial z_2^{t_2} \ldots \partial z_n^{t_n}} f(z)$. Let $f_i : \mathbb{H}^n \to \mathbb{C}$ be holomorphic for $i = 1, 2$ and $k = (k_1, \cdots, k_n), l = (l_1, \cdots, l_n) \in \mathbb{N}_0^n$. For all $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}_0^n$, define the ν-th Rankin-Cohen bracket by

$$[f_1, f_2]_\nu := \sum_{\substack{t \in \mathbb{N}_0^n \\ 0 \leqslant t_i \leqslant \nu_i}} (-1)^{|t|} \binom{k + \nu - \overrightarrow{1}}{\nu - t} \binom{l + \nu - \overrightarrow{1}}{t} f_1^{(t)}(z) f_2^{(\nu - t)}(z). \tag{7}$$

Theorem 2.3. [5] *For all* $M \in SL_2(\mathbb{R})^n$,

$$[f_1|_k M,\ f_2|_l M]_\nu = [f_1, f_2]|_{k+l+2\nu} M. \tag{8}$$

In particular, if $f_1 \in M_k(\Gamma_K)$ *and* $f_2 \in M_l(\Gamma_K)$ *then*

$$[f_1, f_2]_\nu \in M_{k+l+2\nu}(\Gamma_K),$$

and if $\nu \neq 0$, *then*

$$[f_1, f_2]_\nu \in S_{k+l+2\nu}(\Gamma_K).$$

Remark 2.1. For each $\nu \in \mathbb{N}_0^n$, $[\ ,\ ]_\nu$ is a bilinear operator on the space of Hilbert modular forms.

Remark 2.2. Let $s = (s_1, \cdots, s_n) \in \mathbb{C}^n$. The series

$$\sum_{\substack{m \in O_K^* \\ m \gg 0}} \frac{1}{m^s}$$

converges absolutely if $Re(s_i) > n$ for some $i,\ 1 \leq i \leq n$.

Remark 2.3. Let $s = (s_1, \cdots, s_n) \in \mathbb{C}^n$. Then the series

$$\sum_{\substack{m,n \in \mathcal{O}_K^* \\ n > 0, m \gg 0}} \frac{1}{(m+n)^s},$$

converges absolutely if $Re(s_i) > 2n$ for some $i,\ 1 \leq i \leq n$.

3. Statement of the theorem

For a fixed $g \in M_l(\Gamma_K)$ and $\nu \in \mathbb{N}_0^n$, consider the linear map,

$$T_{g,\nu} : S_k(\Gamma_K) \longrightarrow S_{k+l+2\nu}(\Gamma_K)$$

defined by

$$f \longmapsto [f, g]_\nu. \tag{9}$$

Since $S_k(\Gamma_K)$ is a finite dimensional Hilbert space, there exists the adjoint map

$$T^*_{g,\nu} : S_{k+l+2\nu}(\Gamma_K) \longrightarrow S_k(\Gamma_K) \tag{10}$$

satisfying

$$\langle T^*_{g,\nu}\ f, h\rangle = \langle f, T_{g,\nu}h\rangle,\ \forall\ f \in S_{k+l+2\nu}(\Gamma_K) \text{ and } h \in S_k(\Gamma_K).$$

We compute the Fourier coefficients of $T^*_{g,\nu}(f)$ explicitly which involve certain Dirichlet series associated to the Fourier coefficients of f and g.

Theorem 3.1. *Suppose $k, l, \nu \in \mathbb{N}_0^n$ with $k_i \geqslant 4n + 2$ for some i. Let $g \in M_l(\Gamma_K)$ with Fourier expansion*

$$g(z) = \sum_{\substack{m\in\mathcal{O}_K^* \\ m\succeq 0}} b_m e^{2\pi i tr(mz)}.$$

Suppose that either (a) g is a cusp form or (b) g is not cusp form and $k_i - l_i > 4n$ for some i. Then the image of any cusp form $f(z) \in S_{k+l+2\nu}$ with Fourier expansion

$$f(z) = \sum_{\substack{m\in\mathcal{O}_K^* \\ m\gg 0}} a_m e^{2\pi i tr(mz)},$$

under $T^*_{g,\nu}$ is given by

$$T^*_{g,\nu}(f)(z) = \sum_{\substack{\mu\in\mathcal{O}_K^* \\ \mu\gg 0}} c_\mu e^{2\pi i tr(\mu z)},$$

where

$$c_\mu = \frac{\Gamma(k + l + 2\nu - \overrightarrow{1})}{(4\pi)^{l+2\nu}\Gamma(k - \overrightarrow{1})}\mu^{k-\overrightarrow{1}} \sum_{\substack{m\in\mathcal{O}_K^* \\ m\gg 0}} \frac{a_{m+\mu}\bar{b}_m}{(m + \mu)^{k+l+2\nu-\overrightarrow{1}}}\varepsilon^{k,l,\nu}_{\mu,m} \tag{11}$$

and

$$\varepsilon^{k,l,\nu}_{\mu,m} = \sum_{\substack{t\in\mathbb{N}_0^n \\ 0\leqslant t_i\leqslant\nu_i}} (-1)^{|t|}\binom{k + \nu - \overrightarrow{1}}{\nu - t}\binom{l + \nu - \overrightarrow{1}}{t}\mu^t m^{\nu-t}. \tag{12}$$

Remark 3.1. Using the estimates in Proposition 2.1, we observe that the series in (11) converges absolutely.

Remark 3.2. The above result generalises the work of Wang and Pei [23] and Wang [24] where the authors computed the adjoint map for $\nu = 0$.

We need the following Lemma.

Lemma 3.2. *Let f and g be Hilbert modular forms with Fourier coefficients a_m and b_m respectively as in Theorem 3.1. Then the series*

$$\sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg 0, m\succeq 0}} \frac{|a_n b_m m^{\nu}|}{(n+m+\mu)^{k+l+2\nu-\overrightarrow{1}}} \tag{13}$$

converges.

Proof. Using Proposition 2.1, we have $a_n \ll n^{\frac{k+l+2\nu}{2}}$ and $b_m \ll m^{\frac{l}{2}}$ (if g is a Hilbert cusp form). Hence the series (13) satisfies

$$\ll \sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg 0, m\succeq 0}} \frac{1}{(n+m+\mu)^{k/2-\overrightarrow{1}}},$$

which converges absolutely using Remark 2.3 as $k_i \geq 4n+2$ for some i. If g is not a cusp form, then $b_m \ll m^{l-1}$ and the series (13) satisfies

$$\ll \sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg 0, m\succeq 0}} \frac{1}{(n+m+\mu)^{k/2-l/2}},$$

which converges absolutely using Remark 2.3 as $k_i - l_i > 4n$ for some i. □

Proposition 3.3. *Let f and g be Hilbert modular forms as in Theorem 3.1 and $\mu(\gg 0) \in \mathcal{O}_K^*$. Then the series*

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} \int_{\Gamma_K\backslash\mathbb{H}^n} |f(z)\overline{[e^{2\pi i tr(\mu z)}|_k\gamma, g]_\nu}(z)\ y^{k+l+2\nu}|\frac{dxdy}{y^2} \tag{14}$$

converges.

Proof. For any $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_K$, changing the variable z to $\gamma^{-1}\circ z$ for each integral, the sum (14) equals to

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} \int_{\gamma(\Gamma_K\backslash\mathbb{H}^n)} |f(\gamma^{-1}\circ z)\overline{[e^{2\pi i tr(\mu z)}|_k\gamma, g]_\nu}(\gamma^{-1}\circ z)|\frac{y^{k+l+2\nu}}{|j(\gamma^{-1},z)|^{2(k+l+2\nu)}}\frac{dxdy}{y^2}$$

By (8), the sum is equal to

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K} \int_{\gamma(\Gamma_K\backslash\mathbb{H}^n)} |f(z)\overline{[e^{2\pi i tr(\mu z)}, g]_\nu(z)}|y^{k+l+2\nu}\frac{dxdy}{y^2}.$$

Now using the Rankin-Selberg unfolding argument, the above sum is equal to

$$\int_{\Gamma_\infty\backslash\mathbb{H}^n} |f(z)\overline{[e^{2\pi i tr(\mu z)}, g]_\nu(z)}|y^{k+l+2\nu}\frac{dxdy}{y^2}.$$

Replacing $f(z)$ and $g(z)$ with their Fourier expansions and using the definition of Rankin-Cohen brackets, the last integral is majorized by

$$\sum_{\substack{t\in\mathbb{N}_0^n\\ 0\leqslant t_i\leqslant\nu_i}} \alpha_{\nu,\mu}(t)\int_{\Gamma_\infty\backslash\mathbb{H}^n}\sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg0\\ m\succeq0}}|a_n\overline{b_m}m^{\nu-t}e^{2\pi i tr(nz)}\overline{e^{2\pi i tr((m+\mu)z)}}|y^{k+l+2\nu}\frac{dxdy}{y^2}.$$

where

$$\alpha_{\nu,\mu}(t)=|(-1)^{|t|}\binom{k+\nu-\overrightarrow{1}}{\nu-t}\binom{l+\nu-\overrightarrow{1}}{t}(2\pi i\mu)^t|.$$

The above sum is a finite sum and now it suffices to show that the integral

$$\mathcal{I}_t=\int_{\Gamma_\infty\backslash\mathbb{H}^n}\sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg0\\ m\succeq0}}|a_n\overline{b_m}m^{\nu-t}e^{2\pi i tr(nz)}\overline{e^{2\pi i tr((m+\mu)z)}}|y^{k+l+2\nu}\frac{dxdy}{y^2}$$

is finite for each t. We choose $\mathbb{R}^n\setminus\mathcal{O}_K^*\times(0,\infty)^n$ as a fundamental domain for the action of Γ_∞ on $\mathbb{H}^n$ and integrating over it, we have

$$\ll\sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg0,m\succeq0}}\frac{|a_nb_mm^\nu|}{(n+m+\mu)^{k+l+2\nu-\overrightarrow{1}}}.$$

Using Lemma 3.2, the above series converges.

□

Now we give a proof of the Theorem 3.1.

Proof. Let $T^*_{g,\nu}(f)(z)=\sum_{\substack{\mu\in\mathcal{O}_K^*\\ \mu\gg0}}c_\mu e^{2\pi i tr(\mu z)}$. Using Theorem 2.2 we have

$$\begin{aligned}
vol(\mathcal{O}_K/\mathbb{R}^n)(4\pi\mu)^{\overrightarrow{1}-k}(k-\overrightarrow{2})!\ c_\mu &= \langle T^*_{g,\nu}f\ ,\mathcal{P}_{k,\mu}\rangle\\
&=\langle f\ ,T_{g,\nu}(\mathcal{P}_{k,\mu})\rangle\\
&=\langle f\ ,[\mathcal{P}_{k,\mu},\ g]_\nu\rangle\\
&=\int_{\Gamma_K\backslash\mathbb{H}^n}f(z)\overline{[\mathcal{P}_{k,\mu},\ g]_\nu(z)}y^{k+l+2\nu}\frac{dxdy}{y^2}\\
&=\int_{\Gamma_K\backslash\mathbb{H}^n}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K}f(z)\overline{[e^{2\pi i tr(\mu z)}|_k\gamma,\ g]_\nu(z)}\ y^{k+l+2\nu}\frac{dxdy}{y^2}.
\end{aligned}$$

By Proposition 3.3, the above expression is absolutely convergent, hence one can interchange the summation and the integration. The change of variable z to $\gamma^{-1}\circ z$ for each integral gives

$$vol(\mathcal{O}_K/\mathbb{R}^n)(4\pi\mu)^{\overrightarrow{1}-k}(k-\overrightarrow{2})!c_\mu = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_K}\int_{\gamma(\Gamma_K\backslash\mathbb{H}^n)} f(z)\overline{[e^{2\pi itr(\mu z)}, g]_\nu}(z)\ y^{k+l+2\nu}\frac{dxdy}{y^2}. \tag{15}$$

Using the Rankin-Selberg unfolding argument, the right hand side of (15) is equal to

$$\int_{\Gamma_\infty\backslash\mathbb{H}^n} f(z)\overline{[e^{2\pi itr(\mu z)},\ g]_\nu}(z)\ y^{k+l+2\nu}\frac{dxdy}{y^2}. \tag{16}$$

Using the definition of Rankin-Cohen bracket (7), the above integral is equal to

$$\sum_{\substack{t\in\mathbb{N}_0^n\\ 0\leqslant t_i\leqslant\nu_i}}(-1)^{|t|}\binom{k+\nu-\overrightarrow{1}}{\nu-t}\binom{l+\nu-\overrightarrow{1}}{t}\int_{\Gamma_\infty\backslash\mathbb{H}^n} f(z)\overline{e^{2\pi itr(\mu z)^{(t)}}}\,\overline{g^{(\nu-t)}}(z)\ y^{k+l+2\nu}\frac{dxdy}{y^2}.$$

Substituting $f(z)$ and $g(z)$ by their Fourier expansions and observing the repeated action of differential operators,

$$\begin{aligned} e^{2\pi itr(\mu z)^{(t)}} &= (2\pi i\mu)^t e^{2\pi itr(\mu z)}\\ g^{(\nu-t)}(z) &= \sum_{\substack{m\in\mathcal{O}_K^*\\ m\succeq 0}}(2\pi im)^{\nu-t}b_m e^{2\pi itr(mz)}, \end{aligned}$$

the integral (16) equals

$$\sum_{\substack{t\in\mathbb{N}_0^n\\ 0\leqslant t_i\leqslant\nu_i}}(-1)^{|t|}\binom{k+\nu-\overrightarrow{1}}{\nu-t}\binom{l+\nu-\overrightarrow{1}}{t}(2\pi i\mu)^t$$
$$\times\int_{\mathbb{R}^n\backslash\mathcal{O}_K^*\times(0,\infty)^n}\sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg 0\\ m\succeq 0}} a_n\overline{b_m}(2i\pi m)^{\nu-t}e^{2i\pi tr(nz)}\overline{e^{2\pi itr((m+\mu)z)}}y^{k+l+2\nu}\frac{dxdy}{y^2}.$$

Writing $z = x+iy$ and choosing $\mathbb{R}^n\setminus\mathcal{O}_K^*\times(0,\infty)^n$ as a fundamental domain for $\Gamma_\infty\setminus\mathbb{H}^n$ (see [10]) the above expression equals

$$\sum_{\substack{t\in\mathbb{N}_0^n\\ 0\leqslant t_i\leqslant\nu_i}}(-1)^{|t|}\binom{k+\nu-\overrightarrow{1}}{\nu-t}\binom{l+\nu-\overrightarrow{1}}{t}(2\pi i\mu)^t$$
$$\times\int_{\mathbb{R}^n\backslash\mathcal{O}_K^*\times(0,\infty)^n}\sum_{\substack{n,m\in\mathcal{O}_K^*\\ n\gg 0\\ m\succeq 0}} a_n\overline{b_m}(2i\pi m)^{\nu-t}e^{2i\pi tr((n-(m+\mu))x)}e^{-2\pi tr((n+(m+\mu))y)}y^{k+l+2\nu}\frac{dxdy}{y^2}.$$

Integrating over x first, we have ([10])

$$\int_{\mathbb{R}^n \setminus O_K^*} e^{2i\pi tr((n-(m+\mu))x)} dx = vol(\mathbb{R}^n \setminus O_K^*) \tag{17}$$

if $n = m + \mu$ and zero, otherwise. Using (17) in the previous integral, the integral (16) equals

$$= vol(\mathbb{R}^n \setminus O_K^*) \sum_{\substack{t \in \mathbb{N}_0^n \\ 0 \leqslant t_i \leqslant \nu_i}} (-1)^{|t|} \binom{k+\nu-\overrightarrow{1}}{\nu - t} \binom{l+\nu-\overrightarrow{1}}{t} (2\pi i \mu)^t$$
$$\times \int_{(0,\infty)^n} \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} a_{(m+\mu)} \overline{b_m} (\overline{2\pi i m})^{\nu - t} e^{-4\pi tr((m+\mu)y)}\ y^{k+l+2\nu} \frac{dy}{y^2}.$$

Integrating over y, we have

$$\int_{(0,\infty)^n} e^{-4\pi tr((m+\mu)y)}\ y^{k+l+2\nu} \frac{dy}{y^2} = \frac{\Gamma(k+l+2\nu-\overrightarrow{1})}{(4\pi)^{k+l+2\nu-\overrightarrow{1}}} \frac{1}{(m+\mu)^{k+l+2\nu-\overrightarrow{1}}}, \tag{18}$$

where $\Gamma(k+l+2\nu-\overrightarrow{1}) = \prod_{i=1}^n \Gamma(k_i + l_i + 2\nu_i - 1)$. Finally, substituting (18) in the previous integral, the integral (16) is equal to

$$\sum_{\substack{t \in \mathbb{N}_0^n \\ 0 \leqslant t_i \leqslant \nu_i}} (-1)^{|t|} \binom{k+\nu-\overrightarrow{1}}{\nu - t} \binom{l+\nu-\overrightarrow{1}}{t} (2\pi i \mu)^t vol(\mathcal{O}_K / \mathbb{R}^n)$$
$$\times \frac{\Gamma(k+l+2\nu-\overrightarrow{1})}{(4\pi)^{k+l+2\nu-\overrightarrow{1}}} \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} \frac{a_{m+\mu} \overline{b_m} (\overline{2\pi i m})^{\nu - t}}{(m+\mu)^{k+l+2\nu-\overrightarrow{1}}}.$$

Hence,

$$c_\mu = \frac{(2\pi i)^{|\nu|} \Gamma(k+l+2\nu-\overrightarrow{1})}{(4\pi)^{l+2\nu} \Gamma(k-\overrightarrow{1})} \mu^{k-\overrightarrow{1}} \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} \frac{a_{m+\mu} \overline{b}_m}{(m+\mu)^{k+l+2\nu-\overrightarrow{1}}} \varepsilon_{\mu,m}^{k,l,\nu}, \tag{19}$$

where $\varepsilon_{\mu,m}^{k,l,\nu}$ is given by (12). This completes the proof. □

As an application one can give a different proof of Theorem 3, [5]. Choie, Kim and Richter [5] computed the Petersson scalar product $\langle f, [E_k, g]_\nu \rangle$ in terms of special values of a certain Rankin-Selberg convolution of Hilbert modular forms f, g which generalises the work of Zagier for the case of modular forms [25].

Theorem 3.4. [5] *Let $k > 2$ be a natural number and $l, \nu \in N_0^n$ with $k - l_i > 2n$ for some $i, 1 \leq i \leq n$. Suppose that $f \in S_{k+l+2\nu}(\Gamma_K)$ with Fourier expansion*

$$f(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \gg 0}} a_m e^{2\pi i tr(mz)},$$

and $g \in M_l(\Gamma_K)$ with Fourier expansion

$$g(z) = \sum_{\substack{m \in \mathcal{O}_K^* \\ m \succeq 0}} b_n e^{2\pi i tr(mz)}.$$

Then

$$\langle f, [E_k, g]_\nu \rangle = vol(\mathcal{O}_K/\mathbb{R}^n)(2i\pi)^{|\nu|} \frac{(\overrightarrow{k} + l + 2\nu - \overrightarrow{2})!(\overrightarrow{k} + \nu - \overrightarrow{1})!}{(4\pi)^{|\overrightarrow{k}+l+2\nu-\overrightarrow{1}|}(\overrightarrow{k} - \overrightarrow{1})!\nu!} \sum_{\substack{n \in \mathcal{O}_K^* \\ n \gg 0}} \frac{a_n \overline{b_n}}{n^{k+l+\nu-\overrightarrow{1}}}.$$

Following the method of Zagier [25], the authors [5] expressed $[E_k, g]_\nu$ as a linear combination of Hilbert-Poincaré series and then used the characterization property of Hilbert Poincaré series given in Theorem 2.2 to compute the inner product. Following the method of proof of Theorem 3.1, one can give a different proof of Theorem 3.4 by evaluating the integral

$$\int_{\Gamma_K \backslash \mathbb{H}^n} f(z) \overline{[E_k,\ g]_\nu} y^{k+l+2\nu} \frac{dxdy}{y^2}$$

using the Rankin-Selberg unfolding argument.

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