RANKIN-COHEN TYPE OPERATORS FOR HILBERT-JACOBI FORMS

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Abstract. We construct Rankin-Cohen type differential operators on the space of Hilbert-Jacobi forms. This generalizes a result of Choie and Eholzer (J. Number Theory, 68, 160–177 [1998]) in the case of Jacobi forms to Hilbert-Jacobi forms.

1. Introduction

There are many interesting connections between differential operators and modular forms and many interesting results have been studied. In particular, Rankin [7, 8] gave a general description of the differential operators which send modular forms to modular forms. Cohen [5] constructed certain covariant bilinear operators and obtained modular forms with interesting Fourier coefficients. Zagier [10, 11] called these covariant bilinear operators as Rankin–Cohen operators and studied their algebraic properties.

Rankin-Cohen type operators for Jacobi forms on $\mathbb{H} \times \mathbb{C}$ have been studied using heat operators in [2, 3]. Using Maass operator Böcherer [1] showed that the space of bilinear holomorphic differential operators raising the weight ν is in general of dimension $1 + [\nu/2]$ for Jacobi forms on $\mathbb{H} \times \mathbb{C}$. In [4], Choie and Eholzer explicitly give a family of bilinear holomorphic differential operators using Rankin-Cohen type operators of right dimension $1 + [\nu/2]$ and also remark (in section 8) that it would be interesting to understand how their construction can be generalized to higher Jacobi forms.

Skogman [9] extended the theory of Jacobi forms over a totally real number field, known as Hilbert-Jacobi forms. In this paper, we study differential operators of Rankin-Cohen type on the space of Hilbert-Jacobi forms which give an answer to the question posed by Choie and Eholzer in [4].

The paper is organized as follows. In section 2 we recall basic facts about Hilbert-Jacobi forms and define Rankin-Cohen type operators for the Hilbert-Jacobi forms and state the main result. We develop certain tools for our proof in section 3 and give a proof of the main result in section 4. We follow the same exposition as given in [4].

2. Preliminaries and Statement of Result

Let K be a totally real number field of degree $g := [K, \mathbb{Q}]$ over $\mathbb{Q}$ with ring of its algebraic integers $\mathcal{O}_K$ and we denote its g real embedding by $\sigma_1, \cdots, \sigma_g$. We denote i-th embedding of an element $\alpha \in K$ by $\alpha^{(i)} := \sigma_i(\alpha)$ for any $1 \leqslant i \leqslant g$. An element $\alpha \in K$ is said to be

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totally positive, $\alpha > 0$, if all its embeddings $\alpha^{(i)}$ into $\mathbb{R}$ are positive. The trace and norm of $\alpha \in K$ are defined by $\mathrm{tr}(\alpha) = \sum_{i=1}^{g} \alpha^{(i)}$ and $N(\alpha) = \prod_{i=1}^{g} \alpha^{(i)}$, respectively. The trace and norm of an element $\alpha \in \mathbb{C}^g$ are given by the sum and by the product of its components, respectively. More generally, for $c = (c_1, \ldots, c_g), d = (d_1, \ldots, d_g), k = (k_1, \ldots, k_g)$ and $m = (m_1, \ldots, m_g) \in \mathbb{C}^g$, we define the following:

$$\mathrm{tr}(mz) := \sum_{i=1}^{g} m_i z_i \quad \text{and} \quad (cz+d)^k := \prod_{i=1}^{g} (c_i z_i + d_i)^{k_i}.$$

Let $\Gamma_K := SL_2(\mathcal{O}_K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathcal{O}_K, ad - bc = 1 \right\}$. We denote the Hilbert-Jacobi group as $\Gamma^J(K)$ defined by

$$\Gamma^J(K) := SL_2(\mathcal{O}_K) \ltimes (\mathcal{O}_K \times \mathcal{O}_K),$$

with the group multiplication

$$\gamma_1.\gamma_2 := \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, (\lambda_1, \mu_1) \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + (\lambda_2, \mu_2) \right),$$

where $\gamma_i := \left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, (\lambda_i, \mu_i) \right)$ for $i = 1, 2$. The Hilbert-Jacobi group $\Gamma^J(K)$ acts on the space $\mathbb{H}^g \times \mathbb{C}^g$ by

$$\begin{aligned} &\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \circ (\tau_1, \ldots, \tau_g, z_1, \ldots, z_g) \\ &= \left(\frac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \ldots, \frac{a^{(g)}\tau_g + b^{(g)}}{c^{(g)}\tau_g + d^{(g)}}, \frac{z_1 + \lambda^{(1)}\tau_1 + \mu^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \ldots, \frac{z_g + \lambda^{(g)}\tau_g + \mu^{(g)}}{c^{(g)}\tau_g + d^{(g)}} \right) \end{aligned}$$

where $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu) \right) \in \Gamma^J(K)$ and $(\tau_1, ..., \tau_g, z_1, ..., z_g) \in \mathbb{H}^g \times \mathbb{C}^g$.

For an integer $x \in \mathbb{N}_0$, we denote $\overrightarrow{x} := (x, \ldots, x) \in \mathbb{N}_0^g$. For $\nu = (\nu_1, \ldots, \nu_g) \in \mathbb{N}_0^g$, $l = (l_1, \ldots, l_g) \in \mathbb{N}_0^g$ and $z = (z_1, \ldots, z_g) \in \mathbb{C}^g$, we denote

$$|\nu| = \sum_{i=1}^{g} \nu_i, \ \ \nu! = \prod_{i=1}^{g} \nu_i! \text{ and } z^\nu = \prod_{i=1}^{g} z_i^{\nu_i}.$$

Also we denote $\nu \leqslant l$ if $\nu_j \leqslant l_j$ for all $1 \leqslant j \leqslant g$ and $e[z]$ for $e^{2\pi i z}$ for $z \in \mathbb{C}$.

For a holomorphic function $\phi : \mathbb{H}^g \times \mathbb{C}^g \to \mathbb{C}$, we define the following two slash operators. For a fixed $k \in \mathbb{N}_0^g$ and $m \in \mathcal{O}_K$,

$$\left(\phi|_{k,m} M\right)(\tau, z) := (cz+d)^{-k} e\left[\mathrm{tr}\left(-\frac{mcz^2}{c\tau + d}\right)\right] \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (\tau, z) \right), \tag{1}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_K)$ and

$$\left(\phi|_m(\lambda, \mu)\right)(\tau, z) := e[\mathrm{tr}\left(m(\lambda^2\tau + 2\lambda z)\right)]\phi\left((\lambda, \mu) \circ (\tau, z)\right) \ \text{ for } \ (\lambda, \mu) \in \mathcal{O}_K \times \mathcal{O}_K. \tag{2}$$

Definition 2.1. *A Hilbert-Jacobi form of weight k and index m for a totally real field K is a holomorphic function $\phi : \mathbb{H}^g \times \mathbb{C}^g \to \mathbb{C}$ which satisfies the following conditions:*

(1) $\phi|_{k,m}\gamma = \phi,$ for all $\gamma \in \Gamma_K$,
(2) $\phi|_m(\lambda,\mu) = \phi,$ for all $(\lambda,\mu) \in \mathcal{O}_K \times \mathcal{O}_K$,
(3) *ϕ has a Fourier expansion of the form,*

$$\phi(\tau,z) = \sum_{\substack{n,r\in\mathcal{O}_K^*\\ 4nm-r^2\geqslant 0}} c_\phi(n,r)e[\mathrm{tr}(n\tau+rz)],$$

where $\mathcal{O}_K^* = \{\mu \in K \mid \mathrm{tr}(\mu\lambda) \in \mathbb{Z}$ for all $\lambda \in \mathcal{O}_K\}$.

We note that $\mathcal{O}_K^*$ is δ_K^{-1}, the inverse of the different ideal of the number field K. Moreover, such a form ϕ is called Hilbert-Jacobi cusp form if $c_\phi(n,r) = 0$ whenever $4nm - r^2 = 0$. Let $J_{k,m}^K$ ($J_{k,m}^{K,\mathrm{cusp}}$) denote the space of Hilbert-Jacobi forms (Hilbert-Jacobi cusp forms) of weight k and index m for the field K. For more details on the theory of Hilbert-Jacobi forms we refer to [9]. Now we define the heat operators.

Definition 2.2. *For $1 \leqslant j \leqslant g$, let e_j be j-th unit vector in $\mathbb{R}^g$. For a given $m \in \mathcal{O}_K$, we define the m-th heat operator,*

$$L_m := \prod_{j=1}^{g}\left(8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right)^{e_j}. \tag{3}$$

In the above definition, we denote "$\prod$" for the composition of operators. Now we state some properties of these operators which can be proved as in the case of Jacobi forms [2].

Lemma 2.3. *Let $\phi(\tau,z)$ be a holomorphic function on the space $\mathbb{H}^g \times \mathbb{C}^g$, $k \in \mathbb{Z}^g$ and $m \in \mathcal{O}_K$. Then*

(1) *for $X \in \mathcal{O}_K \times \mathcal{O}_K$,*
$$(L_m\phi)|_m X = L_m(\phi|_m X), \tag{4}$$
(2) *for any $\nu \in \mathbb{N}_0^g$ and $M \in SL_2(\mathcal{O}_K)$, we have*
$$L_m{}^{\nu}(\phi)|_{k+2\nu,m}M = \sum_{\substack{l\in\mathbb{N}_0^g\\ l\leqslant\nu}}\binom{\nu}{l}\frac{(8\pi imc)^{\nu-l}(\alpha+\nu-1)!}{(c\tau+d)^{\nu-l}(\alpha+l-1)!}L^l{}_m(\phi|_{k,m}M), \tag{5}$$
where $\alpha = k - \frac{1}{2}$.

We define Rankin-Cohen type differential operators on the space of Hilbert-Jacobi forms using the heat operators.

Definition 2.4. *Let $\phi,\phi' : \mathbb{H}^g \times \mathbb{C}^g \to \mathbb{C}$ be two holomorphic functions and let k,k',m,m' be complex numbers. Then for any $X \in \mathbb{C}^g$, $\nu \in \mathbb{N}_0^g$ and $l \in \mathbb{N}_0^g$ with $l_i \in \{0,1\}$ for all $1 \leqslant i \leqslant g$, define*

$$[\phi,\phi']_{X,2\nu+l}^{k,k',m,m'} = \sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant l}}(-1)^j m^{l-j}m'^j[\partial_z^j\phi, \partial_z^{l-j}\phi']_{X,2\nu}^{k,k',m,m',l}, \tag{6}$$

where for any two holomorphic functions f and f' on $\mathbb{H}^g \times \mathbb{C}^g$

$$[f,f']_{X,2\nu}^{k,k',m,m',l} := \sum_{\substack{r,s,p\in\mathbb{N}_0^g,\\ r+s+p=\nu}} A_{r,s,p}(k,k',l)(1+mX)^s(1-m'X)^r L_{m+m'}^p(L_m^r(f)L_{m'}^s(f')),$$

with

$$A_{r,s,p}(k,k',l) = \frac{(-(k+k'+l-3/2+r+s+p))_{r+s}}{r!\ s!\ p!\ (k-3/2+r)!\ (k'-3/2+s)!}.$$

Here $(x)_n = \prod_{0\leqslant i\leqslant n-1}(x-i)$.

Remark 2.1. In the above definition we have the following convention.

$$[\phi,\phi']_{X,2\nu}^{k,k',m,m',0} = [\phi,\phi']_{X,2\nu}^{k,k',m,m'}.$$

Remark 2.2. Note that constants $A_{r,s,p}(k,k',l)$ are different than $C_{r,s,p}(k,k')$, which appeared in [4] for the field $K=\mathbb{Q}$.

Now we state the main result.

Theorem 2.5. *Let ϕ,ϕ' be Hilbert-Jacobi forms of weight and index k,m and k',m' respectively. Then for any $X\in\mathbb{C}^g$, $\nu\in\mathbb{N}_0^g$ and $l\in\mathbb{N}_0^g$ with $l_i\in\{0,1\}$ for all $1\leqslant i\leqslant g$,*

$$[\phi,\phi']_{X,2\nu+l}^{k,k',m,m'} \tag{7}$$

is a Hilbert-Jacobi form of weight $k+k'+2\nu+l$ and index $m+m'$.

There are two known methods to prove result like Theorem 2.5. First one, by showing that $[\phi,\phi']_{X,2\nu+l}^{k,k',m,m'}$ satisfy all the required conditions to be a Hilbert-Jacobi form (see, [4, section 4]) and second one, by using generating series (see, [6, Theorem 3.2], [4, section 5]). We prove our result by using generating series. In the next section we shall develop some tools for the proof of Theorem 2.5.

3. Intermediate results

Proposition 3.1. *Let $\phi(\tau,z)\in J_{k,m}^K$ and $\alpha = k-\frac{1}{2}$. Then the formal power series associated with the Jacobi form ϕ defined by*

$$\widetilde{\phi}(\tau,z;W) := \sum_{\nu\in\mathbb{N}_0^g} \frac{L_m^\nu(\phi)(\tau,z)}{\nu!(\alpha+\nu-1)!}W^\nu, \tag{8}$$

satisfies the following functional equation,

$$\widetilde{\phi}\left(M\tau,\frac{z}{c\tau+d};\frac{W}{(c\tau+d)^2}\right) = (cz+d)^k e\left[\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right] e\left[4\mathrm{tr}\left(\frac{mcW}{c\tau+d}\right)\right]\widetilde{\phi}(\tau,z;W), \tag{9}$$

for all $M=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in SL_2(\mathcal{O}_K)$.

Proof. From the definition of $\widetilde{\phi}$, we have

$$\begin{aligned}\widetilde{\phi}\left(M\tau,\frac{z}{c\tau+d};\frac{W}{(c\tau+d)^2}\right) &= \sum_{\nu\in\mathbb{N}_0^g}\frac{L_m^\nu(\phi)\left(M\tau,\dfrac{z}{c\tau+d}\right)}{\nu!(\alpha+\nu-1)!}\frac{W^\nu}{(c\tau+d)^{2\nu}} \\ &= \sum_{\nu\in\mathbb{N}_0^g}\frac{(c\tau+d)^k e\left[\mathrm{tr}\left(\dfrac{mcz^2}{c\tau+d}\right)\right](L_m^\nu\phi)|_{k+2\nu,m}M(\tau,z)}{\nu!(\alpha+\nu-1)!}W^\nu.\end{aligned}$$

Using (5) and the assumption that $\phi \in J^K_{k,m}$, the right hand side of the above equation is equal to

$$(c\tau+d)^k e\left[\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right]\sum_{\nu\in\mathbb{N}_0^g}\frac{1}{\nu!(\alpha+\nu-1)!}\left(\sum_{\substack{l\in\mathbb{N}_0^g\\ l\leqslant\nu}}\binom{\nu}{l}\frac{(8\pi imc)^{\nu-l}(\alpha+\nu-1)!}{(c\tau+d)^{\nu-l}(\alpha+l-1)!}{L^l}_m(\phi|_{k,m}M)\right)W^\nu$$

$$=(c\tau+d)^k e\left[\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right]\sum_{\nu\in\mathbb{N}_0^g}\left(\sum_{\substack{l\in\mathbb{N}_0^g\\ l\leqslant\nu}}\frac{1}{l!(\nu-l)!(\alpha+l-1)!}\frac{(8\pi imc)^{\nu-l}}{(c\tau+d)^{\nu-l}}{L^l}_m(\phi)\right)W^\nu$$

$$=(cz+d)^k e\left[\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right]e\left[4\mathrm{tr}\left(\frac{mcW}{c\tau+z}\right)\right]\widetilde{\phi}(\tau,z,W).$$

This completes the proof. □

Let $\widetilde{f}(\tau,z;W)$ be a power series in W whose coefficients are holomorphic functions on $\mathbb{H}^g\times\mathbb{C}^g$ i.e., $\widetilde{f}(\tau,z;W)=\sum_{\nu\in\mathbb{N}_0^g}\chi_\nu(\tau,z)W^\nu$. For $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathcal{O}_K)$, we define

$$\big(\widetilde{f}|_{k,m}M\big)(\tau,z;W):=(c\tau+d)^{-k}e\left[-\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right]e\left[-4\mathrm{tr}\left(\frac{mcW}{c\tau+d}\right)\right]$$
$$\times\widetilde{f}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d};\frac{W}{(c\tau+d)^2}\right).$$

Next we show that for a given formal power series satisfying certain conditions, one can construct a family of Hilbert-Jacobi forms like in the case of Jacobi forms [Theorem 5.1, [4]].

Theorem 3.2. *Let $\widetilde{\phi}(\tau,z;W)$ be a formal power series in W, i.e.,*

$$\widetilde{\phi}(\tau,z;W)=\sum_{\nu\in\mathbb{N}_0^g}\chi_\nu(\tau,z)W^\nu,\tag{10}$$

satisfying the functional equation

$$\big(\widetilde{\phi}|_{k,m}M\big)(\tau,z;W)=\widetilde{\phi}(\tau,z;W),\quad \textit{for all } M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathcal{O}_K),\tag{11}$$

for some $k\in\mathbb{N}_0^g$ and $m\in\mathcal{O}_K$. Furthermore, assume that the coefficients $\chi_\nu(\tau,z)$ are holomorphic functions on $\mathbb{H}^g\times\mathbb{C}^g$ with Fourier expansion of the form,

$$\chi_\nu(\tau,z)=\sum_{\substack{n,r\in\mathcal{O}_K^*\\ 4nm-r^2\geqslant 0}}c(n,r)e[\mathrm{tr}(n\tau+rz)],\tag{12}$$

satisfying

$$\chi_\nu|_m Y=\chi_\nu\quad \textit{for all}\ \ Y\in\mathcal{O}_K\times\mathcal{O}_K.\tag{13}$$

Then for each $\nu\in\mathbb{N}_0^g$, the function $\xi_\nu(\tau,z)$ defined by

$$\xi_\nu(\tau,z):=\sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant\nu}}\frac{(-(k-3/2+\nu))_{\nu-j}}{j!}L_m^j(\chi_{\nu-j}),\tag{14}$$

is a Hilbert-Jacobi form of weight $k+2\nu$ and index m.

Remark 3.1. We call k and m appeared in the equation (11) is the weight and index of the power series $\widetilde{\phi}$ respectively.

Proof. We show that $\xi_\nu(\tau, z)$, defined by (14) is invariant under $SL_2(\mathcal{O}_K)$ action. For $1 \leqslant j \leqslant g$, let e_j be the j-th unit vector in $\mathbb{R}^g$. Define the j-th differential operator

$$\widetilde{L}_{k,m}^{e_j} := 8\pi i m^{(j)} \frac{\partial}{\partial \tau_j} - \frac{\partial^2}{\partial z_j^2} - (k_j - 1/2)\frac{\partial}{\partial W_j} - W_j \frac{\partial^2}{\partial W_j^2},$$

where $k = (k_1, k_2, \ldots, k_g)$ and $m \in \mathcal{O}_K$. Let $\widetilde{\mathcal{M}}_{k,m}$ be the collection of all functions $\widetilde{f}(\tau, z; W) = \sum_{\nu \in \mathbb{N}_0^g} \chi'_\nu(\tau, z) W^\nu$ which satisfy the condition:

$$\big(\widetilde{f}|_{k,m} M\big)(\tau, z; W) = \widetilde{f}(\tau, z; W), \qquad \text{for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_K).$$

We note that the constant term $\chi'_0(\tau, z)$ in the power series expansion of $\widetilde{f}(\tau, z; W) \in \widetilde{\mathcal{M}}_{k,m}$ satisfy the following:

$$\big(\chi'_0|_{k,m} M\big)(\tau, z) = \chi'_0(\tau, z), \qquad \text{for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_K). \tag{15}$$

Then using the definition of slash operator (11) one can show that

$$\widetilde{L}_{k,m}^{e_j}(\widetilde{\phi}|_{k,m} M) = (\widetilde{L}_{k,m}^{e_j} \widetilde{\phi})|_{k+2e_j, m} M,$$

for all $M \in SL_2(\mathcal{O}_K)$. We note that $\prod_{j=1}^{g} \widetilde{L}_{k,m}^{e_j}$ (the composition of all $\widetilde{L}_{k,m}^{e_j}$ for $1 \leqslant j \leqslant g$), denoted by $\widetilde{L}_{k,m}$ satisfy

$$\widetilde{L}_{k,m}(\widetilde{\phi}|_{k,m} M) = (\widetilde{L}_{k,m} \widetilde{\phi})|_{k+2,m} M, \quad \text{for all } M \in SL_2(\mathcal{O}_K).$$

In other word, $\widetilde{L}_{k,m}$ is a map from $\widetilde{\mathcal{M}}_{k,m}$ to $\widetilde{\mathcal{M}}_{k+2,m}$ which is given in terms of power series by

$$\widetilde{L}_{k,m} : \sum_{\lambda \in \mathbb{N}_0^g} \chi_\lambda(\tau, z) W^\lambda \to \sum_{\lambda \in \mathbb{N}_0^g} \Bigg(\sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant 1}} \frac{(-1)^{1+j} \binom{1}{j} (\lambda + 1 - j)! (\lambda + \alpha - j)! L_m^j(\chi_{\lambda+1-j})}{\lambda! (\lambda + \alpha - 1)!} \Bigg) W^\lambda,$$

with $\alpha = k - 1/2$. Composing the maps $\widetilde{L}_{k+i,m}$ for $1 \leqslant i \leqslant \nu - 1$,

$$\widetilde{\mathcal{M}}_{k,m} \xrightarrow{\widetilde{L}_{k,m}} \widetilde{\mathcal{M}}_{k+2,m} \xrightarrow{\widetilde{L}_{k+2,m}} \cdots \xrightarrow{\widetilde{L}_{k+2\nu-2,m}} \widetilde{\mathcal{M}}_{k+2\nu,m}$$

then it maps $\sum_{\lambda \in \mathbb{N}_0^g} \chi_\lambda(\tau, z) W^\lambda$ to

$$\sum_{\lambda \in \mathbb{N}_0^g} \Bigg(\sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant \nu}} \frac{(-1)^{\nu+j} \binom{\nu}{j} (\lambda + \nu - j)! (\lambda + 2\nu + \alpha - j - 2)! L_m^j(\chi_{\lambda+\nu-j})}{\lambda! (\lambda + \alpha + \nu - 2)!} \Bigg) W^\lambda.$$

We note that the constant term i.e., $\lambda = \overrightarrow{0}$ in the above series is $\nu!$ times ξ_ν. Hence from (15), ξ_ν is invariant under $SL_2(\mathcal{O}_K)$ action. The other conditions hold easily from given hypothesis on function $\chi_\nu(\tau, z)$. ☐

In the next two lemmas we show how the operator ∂_z behaves under the group and lattice actions.

Lemma 3.3. *Let ϕ be a Hilbert-Jacobi form of weight k and index m. For $j \in \mathbb{N}_0^g$ with $j_i \in \{0,1\}$ for all $1 \leqslant i \leqslant g$, we have*

$$\partial^j_{z/c\tau+d}\widetilde{\phi}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{W}{(c\tau+d)^2}\right) = (c\tau+d)^{k+j} e\left[\mathrm{tr}\left(\frac{mcz^2}{c\tau+d}\right)\right] e\left[4\mathrm{tr}\left(\frac{mcW}{c\tau+d}\right)\right] \sum_{\substack{a\in\mathbb{N}_0^g \\ a\leqslant j}} \left(\frac{4\pi imcz}{c\tau+d}\right)^a \partial_z^{j-a}\widetilde{\phi}(\tau, z; W). \tag{16}$$

Proof. This Lemma is an easy consequence of Proposition 3.1. ☐

Lemma 3.4. *Suppose $f(z)$ is a holomorphic function on the space $\mathbb{H}^g$ and $Y = (\lambda, \mu) \in \mathcal{O}_K \times \mathcal{O}_K$. Then for $j \in \mathbb{N}_0^g$ with $j_i \in \{0,1\}$ for all $1 \leqslant i \leqslant g$, we have*

$$(\partial_z^j f)|_m Y = \sum_{\substack{a\in\mathbb{N}_0^g \\ a\leqslant j}} (-4\pi im\lambda)^a \partial_z^{j-a}(f|_m Y). \tag{17}$$

Proof. One can prove this result using the definition of the action "$|_m Y$". ☐

4. Proof of Theorem 2.5

First we prove for case $l = \overrightarrow{0}$ and then for general case $l \neq \overrightarrow{0}$.

Case I: $l = \overrightarrow{0}$. For a fixed $X \in \mathbb{C}^g$, consider the series $F_X(\tau, z; W)$ defined by

$$F_X(\tau, z; W) = \widetilde{\phi}\big(\tau, z; (1+m'X)W\big)\ \widetilde{\phi}'\big(\tau, z; (1-mX)W\big),$$

where $\widetilde{\phi}$ and $\widetilde{\phi}'$ are defined by the equation (8). We shall show that the function $F_X(\tau, z; W)$ satisfy all the necessary conditions for Theorem 3.2 and consequently deduce the result.

Using the corresponding functional equation for $\widetilde{\phi}$ and $\widetilde{\phi}'$ given in the Proposition 3.1, one can easily show that the function $F_X(\tau, z; W)$ also satisfy the same functional equation as (11) with weight $k + k'$ and index $m + m'$.

Now we shall look the power series expansion of F_X. Replacing $\widetilde{\phi}$ and $\widetilde{\phi}'$ with their corresponding expressions (8) in F_X, we get

$$\begin{aligned} F_X(\tau, z; W) &= \left(\sum_{\nu\in\mathbb{N}_0^g} \frac{(1+m'X)^\nu L_m^\nu(\phi)}{\nu!\ (k-3/2+\nu)!} W^\nu\right)\left(\sum_{\nu\in\mathbb{N}_0^g} \frac{(1-mX)^\nu L_m^\nu(\phi')}{\nu!\ (k'-3/2+\nu)!} W^\nu\right) \\ &= \sum_{\nu\in\mathbb{N}_0^g}\left(\sum_{\substack{a\in\mathbb{N}_0^g \\ a\leqslant \nu}} \frac{(1+m'X)^a(1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!} L_m^a(\phi) L_{m'}^{\nu-a}(\phi')\right) W^\nu \\ &= \sum_{\nu\in\mathbb{N}_0^g} \chi_{\nu,F}(\tau, z) W^\nu \end{aligned}$$

where

$$\chi_{\nu,F}(\tau, z) := \sum_{\substack{a\in\mathbb{N}_0^g \\ a\leqslant \nu}} \frac{(1+m'X)^a(1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!} L_m^a(\phi) L_{m'}^{\nu-a}(\phi'). \tag{18}$$

Clearly $\chi_{\nu,F}(\tau,z)$ is holomorphic on $\mathbb{H}^g\times\mathbb{C}^g$ for all $\nu\in\mathbb{N}_0^g$. We note that if ϕ has the Fourier expansion $\phi(\tau,z)=\sum\limits_{\substack{n,r\in\mathcal{O}_K^*\\ 4nm-r^2\geqslant 0}} c_\phi(n,r)e[\mathrm{tr}(n\tau+rz)]$, then for any $t\in\mathbb{N}$, the function $L_m^t(\phi)$ has the Fourier expansion

$$L_m^t(\phi)(\tau,z)=\sum_{\substack{n,r\in\mathcal{O}_K^*\\ 4nm-r^2\geqslant 0}} c_\phi(n,r)(4nm-r^2)^t e[\mathrm{tr}(n\tau+rz)]. \tag{19}$$

Replacing ϕ and ϕ' by their Fourier expansions and using the repeated action of the heat operator from (19), we have

$$\begin{aligned}
\chi_{\nu,F}(\tau,z) &= \sum_{\substack{a\in\mathbb{N}_0^g\\ a\leqslant\nu}} \frac{(1+m'X)^a(1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!}\\
&\times\Bigg(\sum_{\substack{n,r\in\mathcal{O}_K^*\\ 4nm-r^2\geqslant 0}}(4nm-r^2)^a c_\phi(n,r)e[\mathrm{tr}(n\tau+rz)]\Bigg)\\
&\times\Bigg(\sum_{\substack{n',r'\in\mathcal{O}_K^*\\ 4n'm'-r'^2\geqslant 0}}(4n'm'-r'^2)^{\nu-a} c_{\phi'}(n',r')e[\mathrm{tr}(n'\tau+r'z)]\Bigg)\\
&= \sum_{\substack{N,R\in\mathcal{O}_K^*\\ 4N(m+m')-R^2\geqslant 0}}\Bigg(\sum_{\substack{a\in\mathbb{N}_0^g\\ a\leqslant\nu}} \frac{(1+m'X)^a(1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!}\\
&\qquad\times\sum_{\substack{n,n',r,r'\in\mathcal{O}_K^*\\ n+n'=N,\\ r+r'=R,\\ 4nm-r^2\geqslant 0,\\ 4n'm'-r'^2\geqslant 0}}(4nm-r^2)^a(4n'm'-r'^2)^{\nu-a}c_\phi(n,r)c_{\phi'}(n',r')\Bigg)e[\mathrm{tr}(N\tau+Rz)].
\end{aligned}$$

One can check that $4N(m+m')-R^2\geqslant 0$ for the above choices of N and R and the last sum is a finite sum for a given N and R. From (4), it is clear that $\chi_{\nu,F}|_{m+m'}Y=\chi_{\nu,F}$ for all $Y\in\mathcal{O}_K\times\mathcal{O}_K$. Hence from Theorem 3.2, $\xi_{\nu,F}(\tau,z)$ is a Hilbert-Jacobi form of weight $k+k'+2\nu$ and index $m+m'$. This completes the proof in this case because $[\phi,\phi']_{X,2\nu}^{k,k',m,m'}(\tau,z)=\xi_{\nu,F}(\tau,z)$.

Case II: $l\neq\overrightarrow{0}$. For a fixed $X\in\mathbb{C}^g$, consider the function $G_X(\tau,z;W)$ defined by

$$G_X(\tau,z;W)=\sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant l}}(-1)^j m^{l-j}m'^j\partial_z^j\widetilde{\phi}\big(\tau,z;(1+m'X)W\big)\ \partial_z^{l-j}\widetilde{\phi'}\big(\tau,z;(1-mX)W\big). \tag{20}$$

We show that the function G_X satisfy the same functional equation as (11) with weight $k+k'+l$ and index $m+m'$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_K)$. Using (20), we have

$$G_X\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{W}{(c\tau+d)^2}\right) = \sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant l}} (-1)^j m^{l-j} m'^j \partial^j_{z/c\tau+d} \widetilde{\phi}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{(1+m'X)W}{(c\tau+d)^2}\right)$$
$$\times\, \partial^{l-j}_{z/c\tau+d} \widetilde{\phi}'\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{(1-mX)W}{(c\tau+d)^2}\right).$$

Using Lemma 3.3, the above equation becomes

$$\begin{aligned}
&G_X\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{W}{(c\tau+d)^2}\right)\\
&= (c\tau+d)^{k+k'+l} e\left[\mathrm{tr}\left((m+m')\frac{cz^2}{c\tau+d}\right)\right] e\left[4\mathrm{tr}\left((m+m')\frac{cW}{c\tau+d}\right)\right]\\
&\times \sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant l}} (-1)^j m^{l-j} m'^j \Bigg(\sum_{\substack{a\in\mathbb{N}_0^g\\ a\leqslant j}} \left(\frac{4\pi i m c z}{c\tau+d}\right)^a \partial_z^{j-a} \widetilde{\phi}\big(\tau, z; (1+m'X)W\big)\\
&\times \sum_{\substack{b\in\mathbb{N}_0^g\\ b\leqslant l-j}} \left(\frac{4\pi i m' c z}{c\tau+d}\right)^b \partial_z^{l-j-b} \widetilde{\phi}'\big(\tau, z; (1-mX)W\big)\Bigg).
\end{aligned}$$

Now we split the above sum into two parts,

$$\begin{aligned}
&G_X\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; \frac{W}{(c\tau+d)^2}\right)\\
&= (c\tau+d)^{k+k'+l} e\left[\mathrm{tr}\left((m+m')\frac{cz^2}{c\tau+d}\right)\right] e\left[4\mathrm{tr}\left((m+m')\frac{cW}{c\tau+d}\right)\right]\\
&\times \Bigg(\sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant l}} (-1)^j m^{l-j} m'^j \partial_z^j \widetilde{\phi}\big(\tau, z; (1+m'X)W\big)\ \partial_z^{l-j} \widetilde{\phi}'\big(\tau, z; (1-mX)W\big)\\
&+ \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^g\\ \alpha+\beta<l}} \Bigg(\sum_{\substack{j\in\mathbb{N}_0^g\\ \alpha\leqslant j\leqslant l-\beta}} (-1)^j m^{l-j} m'^j \left(\frac{4\pi i m c z}{c\tau+d}\right)^{j-\alpha} \left(\frac{4\pi i m' c z}{c\tau+d}\right)^{l-j-\beta}\Bigg)\\
&\quad \times \partial_z^{\alpha} \widetilde{\phi}\big(\tau, z; (1+m'X)W\big) \partial_z^{\beta} \widetilde{\phi}'\big(\tau, z; (1-mX)W\big)\Bigg).
\end{aligned}$$

An easy computation shows that for any pair of $\alpha, \beta \in \mathbb{N}_0^g$ with $\alpha+\beta<l$, the coefficient of $\partial_z^\alpha \widetilde{\phi}\ \partial_z^\beta \widetilde{\phi}'$ in the second sum of the above equation is zero, which prove our claim. Now replacing the corresponding power series expression for $\widetilde{\phi}$ and $\widetilde{\phi}'$ from (8) in (20), we note

that the function G_X has power series expansion of the form

$$G_X(\tau, z; W) = \sum_{\nu \in \mathbb{N}_0^g} \chi_{\nu,G}(\tau, z) W^\nu,$$

where $\chi_{\nu,G}(\tau, z)$ is given by

$$\sum_{\substack{a \in \mathbb{N}_0^g \\ a \leqslant \nu}} \frac{(1+m'X)^a (1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!} \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j L_m^a(\partial_z^j \phi) L_{m'}^{\nu-a}(\partial_z^{l-j}\phi'). \tag{21}$$

As mentioned in the previous case one can show that for each $\nu \in \mathbb{N}_0^g$, the corresponding function $\chi_{\nu,G}(\tau, z)$ has the following Fourier expansion.

$$\chi_{\nu,G}(\tau, z) = \sum_{\substack{N, R \in \mathcal{O}_K^*, \\ 4N(m+m')-R^2 \geqslant 0}} \Bigg(\sum_{\substack{a \in \mathbb{N}_0^g \\ a \leqslant \nu}} \frac{(1+m'X)^a (1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!} \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j$$
$$\times \sum_{\substack{n, n', r, r' \in \mathcal{O}_K^* \\ n+n'=N, \\ r+r'=R, \\ 4nm-r^2 \geqslant 0, \\ 4n'm'-r'^2 \geqslant 0}} (4nm-r^2)^a (4n'm'-r'^2)^{\nu-a} r^j r'^{l-j} c_\phi(n,r) c_{\phi'}(n',r') \Bigg) e[\mathrm{tr}(N\tau + Rz)].$$

Using Theorem 3.2 one can deduce that $[\phi, \phi']_{X,2\nu+l}^{k,k',m,m'} \in J_{k+k'+2\nu+l,m+m'}^K$ as $[\phi, \phi']_{X,2\nu+l}^{k,k',m,m'} = \xi_{\nu,G}(\tau, z)$ once we prove $\chi_{\nu,G}(\tau, z)|_{m+m'} Y = \chi_{\nu,G}(\tau, z)$ for all $\nu \in \mathbb{N}_0^g$ and $Y \in \mathcal{O}_K \times \mathcal{O}_K$. From (21) we have

$$\chi_{\nu,G}(\tau, z)|_{m+m'} Y = \sum_{\substack{a \in \mathbb{N}_0^g \\ a \leqslant \nu}} \frac{(1+m'X)^a (1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!}$$
$$\times \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j (\partial_z^j(L_m^a \phi))|_m Y\ (\partial_z^{l-j}(L_{m'}^{\nu-a}\phi'))|_{m'} Y.$$

From Lemma 3.4 the right hand side of the above equation is equal to

$$\sum_{\substack{a \in \mathbb{N}_0^g \\ a \leqslant \nu}} \frac{(1+m'X)^a (1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!} \sum_{\substack{j \in \mathbb{N}_0^g \\ j \leqslant l}} (-1)^j m^{l-j} m'^j$$
$$\times \Bigg(\sum_{\substack{t \in \mathbb{N}_0^g \\ t \leqslant j}} (-4\pi i m\lambda)^t \partial_z^{j-t}((L_m^a \phi)|_m Y) \Bigg) \Bigg(\sum_{\substack{s \in \mathbb{N}_0^g \\ s \leqslant l-j}} (-4\pi i m'\lambda)^s \partial_z^{l-j-s}((L_{m'}^{\nu-a}\phi')|_{m'} Y) \Bigg).$$

Now using the assumption that ϕ and ϕ' are Hilbert-Jacobi forms and $(L_m\phi)|_m Y = L_m(\phi|_m Y)$, the above expression is equal to

$$= \sum_{\substack{a\in\mathbb{N}_0^g\\ a\leqslant\nu}} \frac{(1+m'X)^a(1-mX)^{\nu-a}}{a!\ (\nu-a)!\ (k-3/2+a)!\ (k'-3/2+\nu-a)!} \sum_{\substack{j\in\mathbb{N}_0^g\\ j\leqslant l}} (-1)^j m^{l-j} m'^j$$
$$\times \Bigg(\sum_{\substack{t\in\mathbb{N}_0^g\\ t\leqslant j}} (-4\pi i m\lambda)^t \partial_z^{j-t} L_m^a \phi \Bigg) \Bigg(\sum_{\substack{s\in\mathbb{N}_0^g\\ s\leqslant l-j}} (-4\pi i m'\lambda)^s \partial_z^{l-j-s} L_{m'}^{\nu-a} \phi' \Bigg).$$

For a fixed $a \in \mathbb{N}_0^g$ we note the following. For $\alpha, \beta \in \mathbb{N}_0^g$ with $\alpha+\beta < l$, the coefficient of $\partial_z^\alpha(L_m^a\phi)\ \partial_z^\beta(L_m^{\nu-a}\phi')$ in the above expression is zero. Thus $\chi_{\nu,G}$ is invariant under the lattice action and this completes the proof.

5. Concluding Remark

Theorem 2.5 gives justification to expect that the space of bilinear holomorphic differential operators raising the weight $\nu = (\nu_1, \ldots, \nu_g) \in \mathbb{N}_0^g$ is at least $\prod_{i=1}^g (1+[\nu_i/2])$ for the space of Hilbert-Jacobi forms over a totally real number field of degree g over $\mathbb{Q}$ on $\mathbb{H}^g \times \mathbb{C}^g$. It would be of interest to prove the generalization of the result of Böcherer [1] in case of Hilbert-Jacobi forms that the dimension is exactly equal to $\prod_{i=1}^g (1+[\nu_i/2])$.

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