Art Gallery Problem

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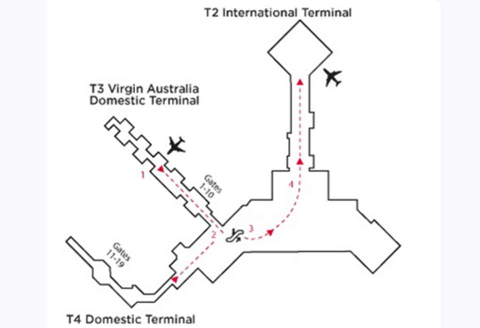
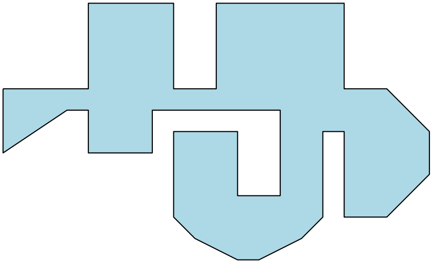
[1]Slide ideas borrowed from Marc van Kreveld and Subhash Suri

Art Gallery Theorem



- The floor plan of an art gallery/museum/airport modeled as a simple polygon with n vertices.
- Objective is to secure the interior of the polygon by placing guards.
- Each guard is stationed at a fixed point, has 360°vision, and cannot see through the walls.
- How many guards needed to see the whole room?



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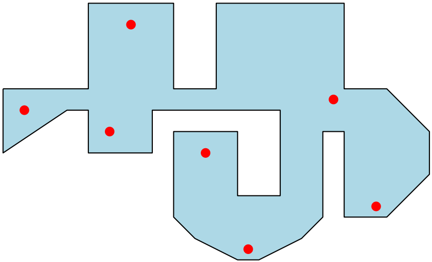

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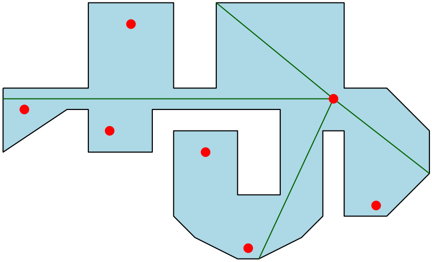

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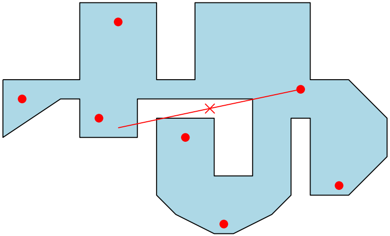

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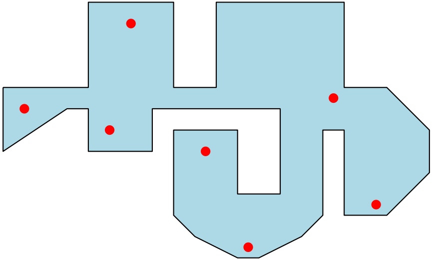

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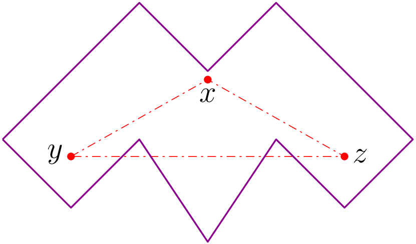
Formulation



- Visibility: p, q visible if $pq \in P$.
- x is visible from y and z. But y and z not visible to each other.
- $g(P) =$ min. number of guards to see P
- $g(n) = \max_{|V(P)|=n} g(P)$ where maximum is taken over all simple polygons with n vertices
- Art Gallery Theorem asks for bounds on function $g(n)$: what is the smallest $g(n)$ that always works for any n-gon?

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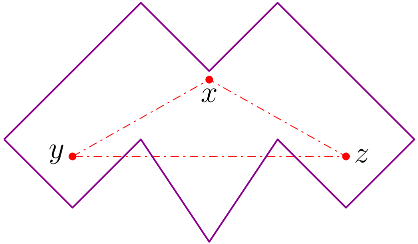

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- Problem posed to Vasek Chvatal by Victor Klee at a math conference in 1973. Chvatal solved it quickly with a complicated proof.
- Steve Fisk gave a proof from "THE BOOK".
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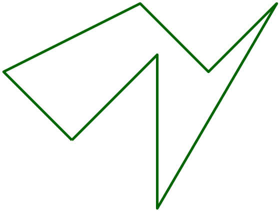
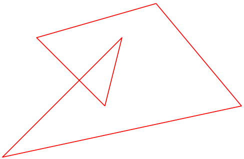
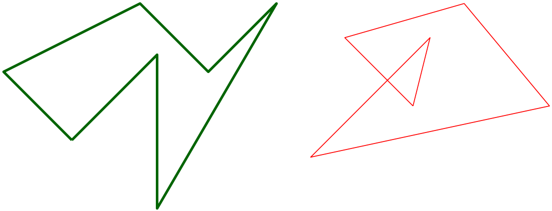


Simple Polygon



Not a simple polygon

- A simple polygon is a closed polygonal curve without self-intersection.
- By Jordan Theorem, a polygon divides the plane into interior, exterior, and boundary.
- We use polygon both for boundary and its interior; the context will make the usage clear.
- Polygons with holes are topologically different



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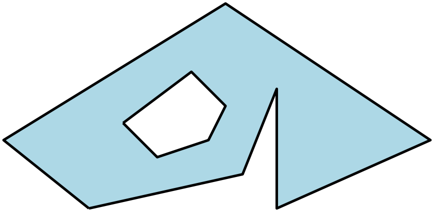

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Trying it Out

- $g(3)$??$g(4)$??$g(5)$??

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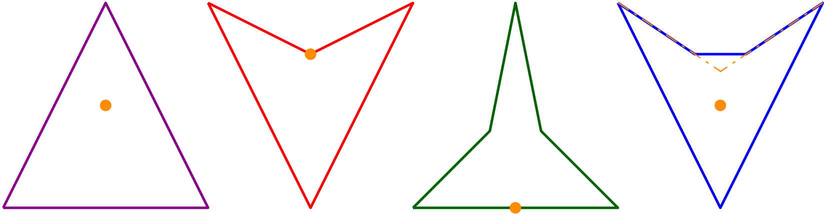

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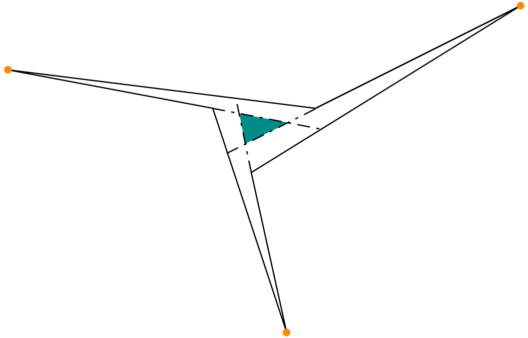

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Art Gallery Theorem

Art Gallery Theorem $g(n) = \lfloor n/3 \rfloor$

- Every n-gon can be guarded with $\lfloor n/3 \rfloor$ vertex guards.
- Some n-gons require at least $\lfloor n/3 \rfloor$ (arbitrary) guards.

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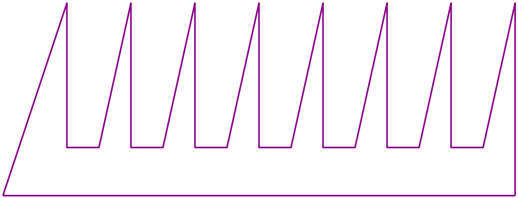
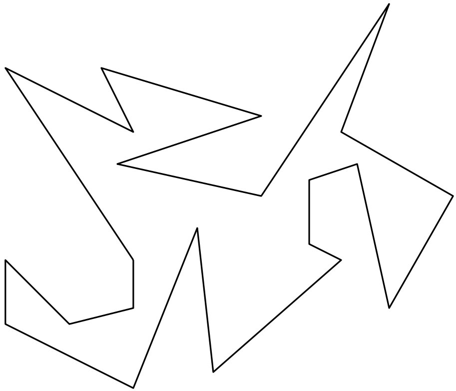
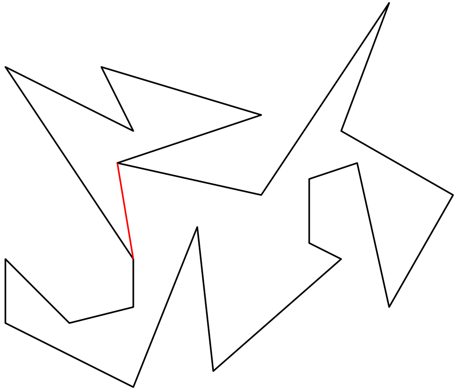




- Diagonal: Given a simple polygon, P, a diagonal is a line segment between two non-adjacent vertices that lies entirely within the interior of the polygon.



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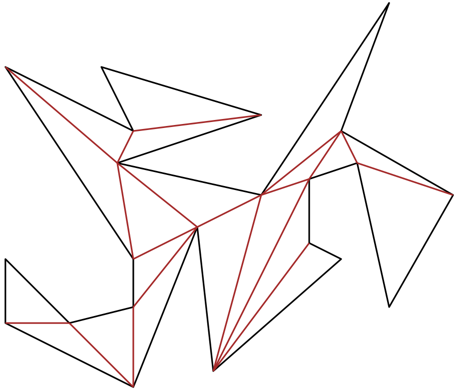
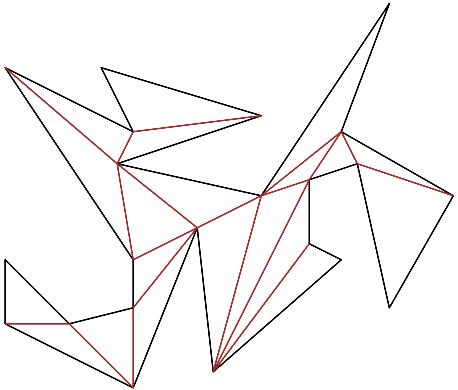
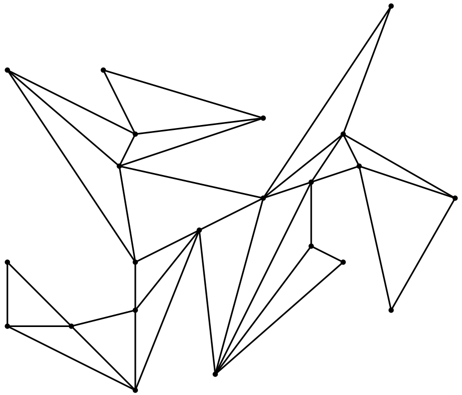


- Triangulations: Given a simple polygon P, a triangulation of P is a partition of the interior of P into triangles using diagonals.



- Observe the polygon P along with the triangulation $\mathcal{T}$ can be considered as a graph $G(P, \mathcal{T})$.
- Vertices: Polygon vertices
- Edges of the graph: Polygon edges $\bigcup$ diagonals of the triangulation



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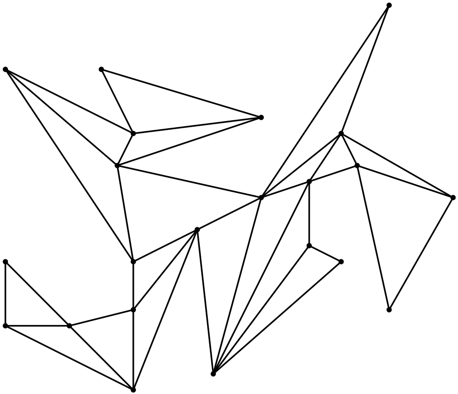
Fisk's proof from THE BOOK that $\lfloor n/3 \rfloor$ guards suffice



- Properties of the graph
- Planar $\Rightarrow$ Four colorable
- Is it three colorable?

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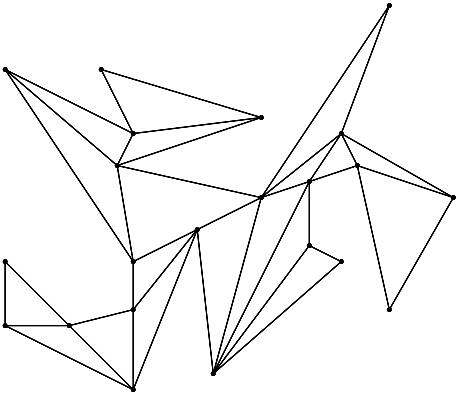

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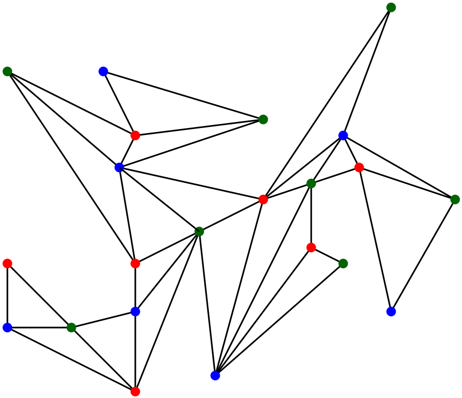

- What if the graph is three colorable
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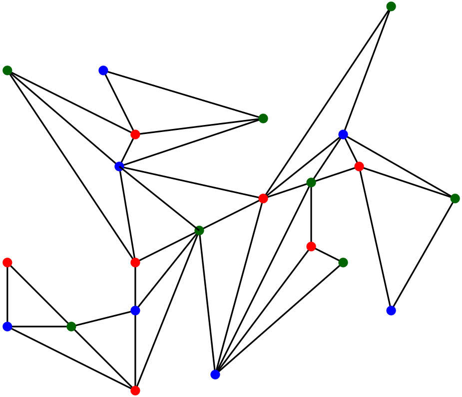

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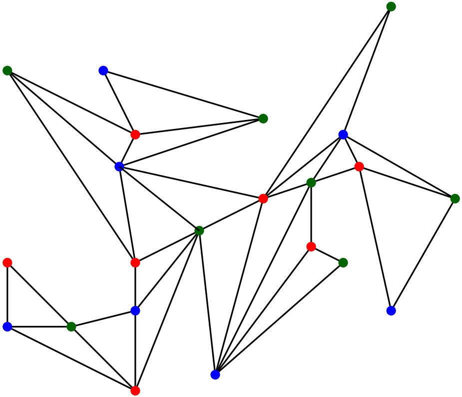

- There exist a color that is used at most $\lfloor n/3 \rfloor$ times
- Post guards at the least popular color vertices

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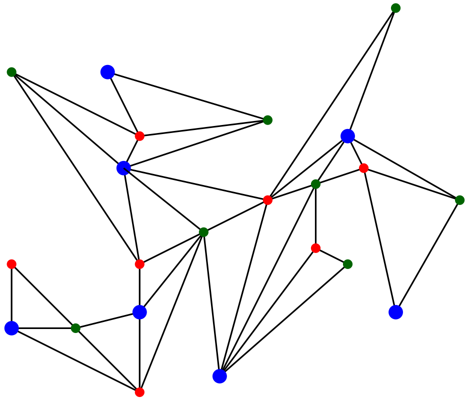

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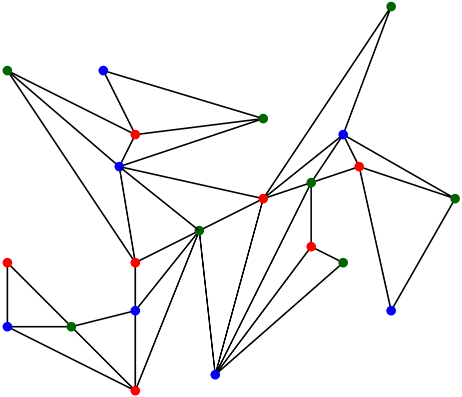

- Why $G(P, \mathcal{T})$ is three colorable?

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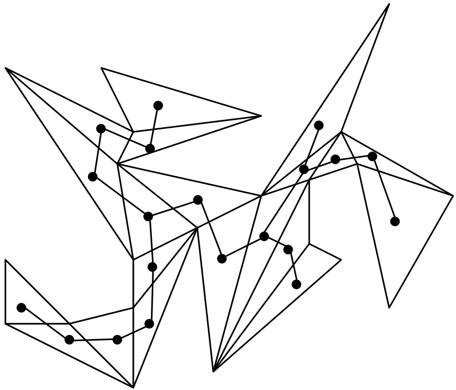
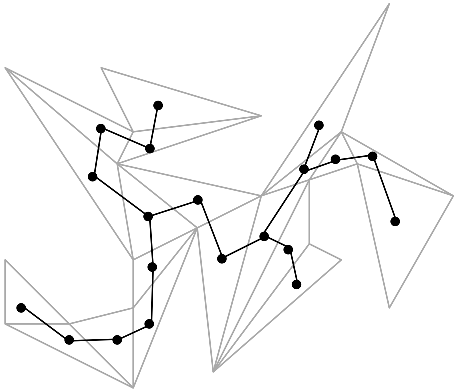


- Dual graph of a polygon: Given a polygon P and a triangulation $\mathcal{T}$ for that polygon, the dual graph is defined as $D(T) = (V, E)$, where $vi \in V$ corresponds to a specific triangle in T, and $(v_a, v_b) \in E$ if the two corresponding triangles share an edge.



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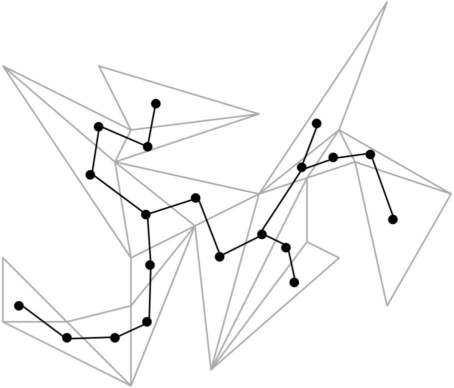
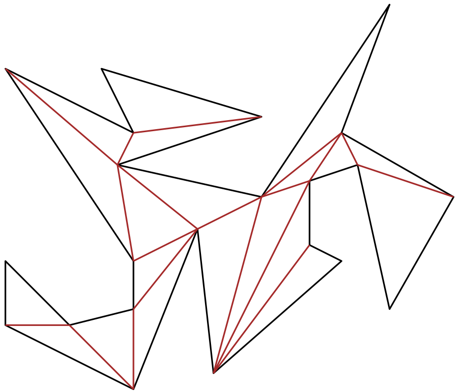


- Lemma: Dual graph of a triangulation of a simple polygon is a tree with maximum degree three.
- Edge of the dual graph corresponds to a diagonal.
- Each diagonal breaks the polygon into two disjoint pieces.



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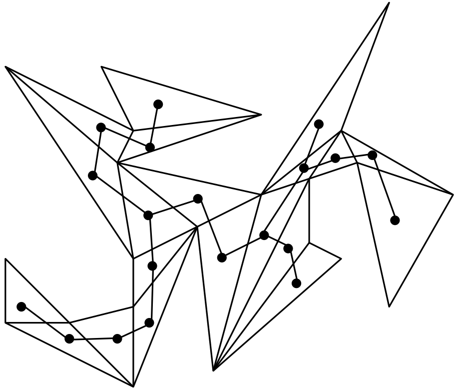
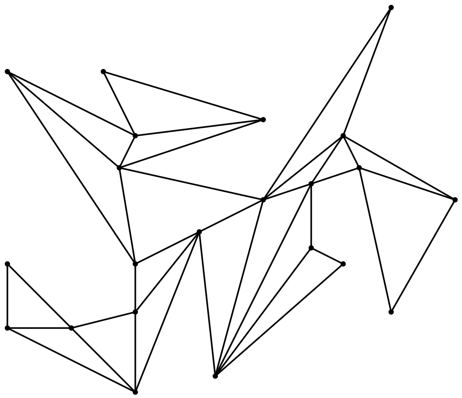


- Lemma: Dual graph of a triangulation of a simple polygon is a tree with maximum degree three.
- Deleting an edge from the dual graph breaks the graph into two connected components.
- Thus the graph is a tree.



- Lemma: $G(P, \mathcal{T})$ is three colorable
- Proof by Induction:
- Remove a triangle which is a leaf node in the tree.

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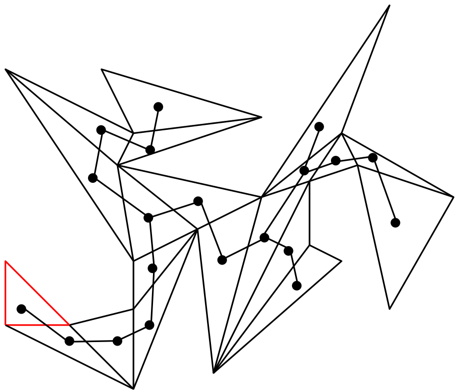

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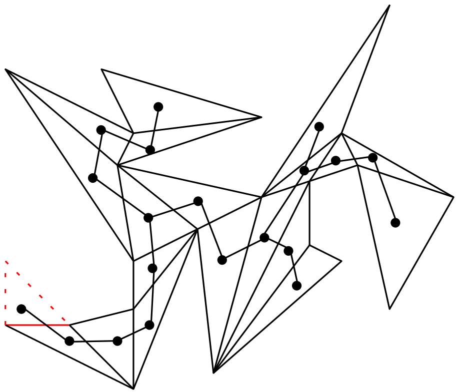

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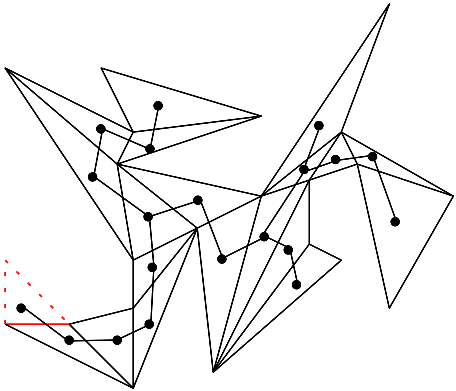

- Inductively 3-color the rest.
- Put the triangle back, coloring new vertex with the label not used by the boundary diagonal.

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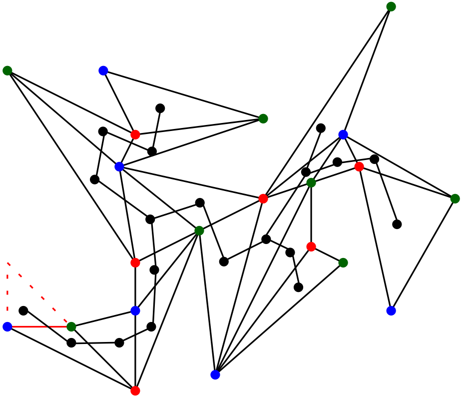

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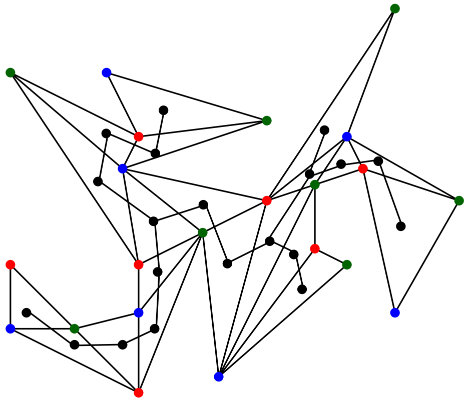

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Theorem

$\frac{n}{3}$ guards are always sufficient and sometimes necessary to guard a simple polygon with n vertices.