PHY201: Classical Mechanics I

Problem Set IV

1. A particle is subjected to the potential $V(x) = -kx$, where k is a constant. The particle travels from $x = 0$ to $x = a$ in a time interval t_0. Assume the motion of the particle can be expressed in the form $x(t) = A + Bt + Ct^2$.

   (a) Calculate the action S.

   (b) Find the values of A, B, and C such that action is a minimum.

2. The term generalized mechanics has come to designate a variety of classical mechanics in which the Lagrangian contains time derivatives of q_i higher than the first. By applying the methods of the calculus of variations, show that if there is a Lagrangian of the form $L(q_i, \dot{q}_i, \ddot{q}_i, t)$ and Hamiltons principle holds with the zero variation of both q_i and $\dot{q}_i$ at the end points, then corresponding Euler-Lagrange equations are

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, ..., n.$$

   Apply this result to the Lagrangian

$$L = -\frac{m}{2} q\ddot{q} - \frac{k}{2} q^2.$$

   Do you recognize the equations of motion?

3. A particle of mass m, total energy E, and angular momentum J slides on the inner side of a paraboloid of revolution whose axis of revolution coincides with z-axis, without any friction. The paraboloid can by described by $\rho^2 = az$ in cylindrical coordinates. Use the Lagrangian multiplier method to find the magnitude of the constraint force as a function of ρ.

4. Consider a uniform ladder of length L and mass M which has one end against a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is initially at rest at an angle 0 to the floor. We will be interested in studying the forces of constraints, so we will handle the constraints with Lagrange multipliers.

5. Consider a uniform ladder of length L and mass M which has one end against a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is initially at rest at an angle θ_0 to the floor. We will be interested in studying the forces of constraints, so we will handle the constraints with Lagrange multipliers.

   (a) Use the COM coordinates (x, y) for the ladder, and angle θ that the ladder makes to the floor. The kinetic energy is given by a term for motion of the COM, plus a term for rotation about the COM. What is the moment of inertia of the ladder for rotations about the COM? Find the full Lagrangian for the ladder and the equations for any constraints.

   (b) Derive the equations of motion using the method of Lagrange multipliers.

   (c) Solve for the angle $\theta(t)$ of the ladder to the floor by finding a formula $t = t(\theta)$ (which may involve an integral that you leave undone).

(d) At what angle θ does the ladder leave the wall? What is the height of the upper end of the ladder at this angle?