Moonshines for $L_2(11)$ and M_{12}

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(Work done with my student Sutapa Samanta)

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Introduction

Classification of Finite Simple Groups

Every finite simple group is isomorphic to one of the following groups: (Source: Wikipedia)

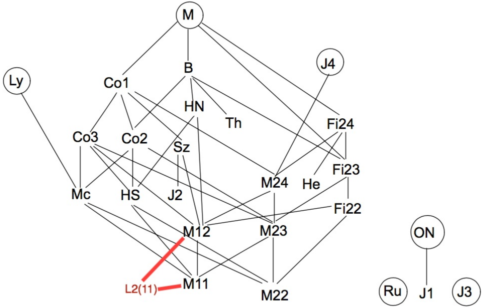
- A cyclic group with prime order;
- An alternating group of degree at least 5;
- A simple group of Lie type, including both
  - the classical Lie groups, namely the groups of projective special linear, unitary, symplectic, or orthogonal transformations over a finite field;
  - the exceptional and twisted groups of Lie type (including the Tits group which is not strictly a group of Lie type).
- The 26 sporadic simple groups.

The classification was completed in 2004 when Aschbacher and Smith filled the last gap ('the quasi-thin case') in the proof.

Fun Reading: *Symmetry and the Monster* by Mark Ronan

The sporadic simple groups

- the Mathieu groups: M_{11}, M_{12}, M_{22}, M_{23}, M_{24}; (found in 1861)
- the Janko groups: J_1, J_2, J_3, J_4; (others 1965-1980)
- the Conway groups; Co_1, Co_2, Co_3;
- the Fischer groups; Fi_{22}. Fi_{23}, Fi'_{24};
- the Higman-Sims group; HS
- the McLaughlin group: McL
- the Held group: He;
- the Rudvalis group Ru;
- the Suzuki sporadic group: Suz;
- the O'Nan group: $O'N$;
- Harada-Norton group: HN;
- the Lyons group: Ly;
- the Thompson group: Th;
- the baby Monster group: B and
- the Fischer-Griess Monster group: $\mathbb{M}$



Sources: Wikipedia and Mark Ronan

Monstrous Moonshine Conjectures

- The j-function has the followed q-series: $(q = \exp(2\pi i\tau))$

  $$j(\tau) - 744 = q^{-1} + [196883 + 1]\, q + [21296876 + 196883 + 1]\, q^2 + \cdots$$

- McKay observed that 196883 and 21296876 are the dimensions of the two smallest irreps of the Monster group.
- (Thompson, 1979): Is there an infinite dimensional $\mathbb{M}$-module

  $$V = \oplus_{m=-1}^{\infty} V_m \ ,$$

  such that $\dim(V_m)$ is the coefficient of q^m in the q-series?
- Conway and Norton, 1979: More generally, given an element g of the Monster group, let

  $$T_g(\tau) := \sum_{m=-1}^{\infty} q^m \operatorname{Tr}\left(g\big|_{V_m}\right) \ .$$

  $T_g(\tau)$, the McKay-Thompson series, is a modular form of a suitable sub-group of $SL(2,\mathbb{Z})$ at level $\mathrm{ord}(g)$.

The conjectures are now theorems

Theorem (Frenkel-Lepowsky-Meurman (1984/1985/1988))

There is a Monster module $V^\natural$ with the properties in the Thompson conjecture.

A natural representation of the Fischer-Griess Monster with the modular function J as character, Proc. Nat. Acad. Sci. U.S.A., 81(10) (1984) 3256–3260.

Theorem (Borcherds, 1992)

Suppose that $V = \oplus_{n \in \mathbb{Z}} V_n$ is the infinite dimensional graded representation of the monster simple group constructed by Frenkel, Lepowsky, and Meurman. Then for any element g of the monster the Thompson series $T_g(q) = \sum_n Tr(g|V_n)q^n$ is a Hauptmodul for a genus 0 subgroup of $SL_2(\mathbb{R})$, i.e., V satisfies the main conjecture in Conway and Norton's paper.

Monstrous moonshine and monstrous Lie superalgebras, Invent. Math., 109 (1992) 405–444.

Mathieu moonshine

- Conjugacy classes of permutation groups can be represented by cycle shapes. For instance, the permutation (213) has one two-cycle and one one-cycle, $1^1 2^1$.
- Conway and Norton observed that the conjugacy classes of the largest Mathieu group M_{24} in its 24-dimensional permutation representation can be represented by balanced cycle shapes.
- A cycle $\rho = 1^{a_1} 2^{a_2} \cdots N^{a_N}$ is said to be balanced, if there exists an integer M (called its balancing factor) such that $\rho = \left(\frac{M}{1}\right)^{a_1} \left(\frac{M}{2}\right)^{a_2} \cdots \left(\frac{M}{N}\right)^{a_N}$. Eg. $1^2 2^1 4^1 8^2$ is a balanced with balancing factor 8.
- (Dummit-Kisilevsky-McKay and Mason, 1985):To every conjugacy class ρ of M_{24}, there exists a multiplicative eta product $\eta_\rho(\tau)$ given by the map:

$$\boxed{\rho = 1^{a_1} 2^{a_2} \cdots N^{a_N} \longrightarrow \eta_\rho := \prod_{m=1}^{N} \eta(m\tau)^{a_m}}$$

Mathieu moonshine and $\frac{1}{2}$-BPS states in string theory

- (Mukai, 1988) Symplectic automorphisms (of finite order) of $K3$ are related to elements of $M_{23} \subset M_{24}$ with ≥ 5 orbits.
- (SG and Krishna, 2009): The generating function of the degeneracy $d(n)$ of electrically charged BPS states in type II string theory on $K3 \times T^2$ twisted by a symplectic automorphism of $K3$ are given by the multiplicative eta products.

$$\sum_{n=-1}^{\infty} d(n)q^n = (\eta_\rho(\tau))^{-1} \ .$$

where ρ is the conjugacy class of automorphism.

- (Gaberdiel-Hohnegger-Volplato, 2011) More general elements (some in M_{24} and some in Co_1) arise if one includes quantum symmetries of the $K3$ sigma model.
- However, the full group M_{24} is not a symmetry of the sigma model. One can however achieve $2^8 : M_{20}$. (Gaberdiel-Taormina-Volpato-Wendland, 2013)

Mathieu moonshine and $\frac{1}{4}$-BPS states in string theory

- $\frac{1}{4}$-BPS states in type II string theory on $K3 \times T^2$ are labelled by three integers (n, ℓ, m) that correspond to T-duality invariant combination of the electric and magnetic chrage vectors.
- The three-variable generating function of the degeneracy $d(n, \ell, m)$ of electrically charged BPS states in type II string theory on $K3 \times T^2$ twisted by a symplectic automorphism of $K3$ is given by (the inverse of) a genus two Siegel modular form:

$$\sum_{(n,\ell,m)} d(n, \ell, m)\, q^n r^\ell s^m = \left(\Phi^\rho(\mathbf{Z})\right)^{-1}$$

where ρ is the conjugacy class of automorphism.
- (SG-Krishna, 2009): The Siegel modular form is given by the additive lift of a Jacobi form constructed from the multiplicative eta product. One has

$$\boxed{\rho \longrightarrow \eta_\rho(\tau) \xrightarrow{\text{Additive Lift}} \Phi^\rho(\mathbf{Z})}\ .$$

Mathieu moonshine and the elliptic genus of K3

- The elliptic genus of K3 is a Jacobi form of weight 0 and index 1.
- The 2d CFT associated with the non-linear sigma model with target space K3 has $\mathcal{N} = 4$ supersymmetry.
- (Eguchi-Ooguri-Tachikawa, 2010) Expanding the elliptic genus of K3 in terms of characters of the $\mathcal{N} = 4$ SCA, one sees the appearance of the dimensions of irreps of M_{24}.

$$Z(\tau, z) = 24\ \mathcal{C}(\tau, z) + q^{-\frac{1}{8}}\ \Sigma(\tau)\ \mathcal{B}(\tau, z)\ ,$$

where

$$\Sigma(\tau) = 2\Big(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \cdots)$$

- It is natural to look for the analog of the McKay-Thompson series in this case as well.

The McKay-Thompson series for the elliptic genus

- M_{24} has 26 conjugacy classes and the Jacobi forms for all classes were determined by several groups – Cheng, Gaberdiel et. al., Eguchi-Hikami.
- With these in hand, given a conjugacy class ρ of M_{24}, the associated Jacobi form expressed in terms of $\mathcal{N} = 4$ characters takes the form

$$Z^{\rho}(\tau, z) = \alpha^{\rho}\ \mathcal{C}(\tau, z) + q^{-\frac{1}{8}}\ \Sigma^{\rho}(\tau)\ \mathcal{B}(\tau, z)\ ,$$

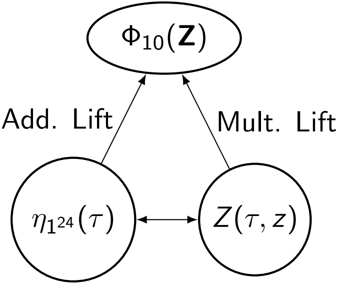
where $\alpha^{\rho} = 1 + \chi_{23}(\rho)$ and the function $\Sigma^{\rho}(\tau)$ is as follows :

$$\Sigma^{\rho}(\tau) = -2 + [\chi_{45}(\rho) + \chi_{\overline{45}}(\rho)]\ q + [\chi_{231}(\rho) + \chi_{\overline{231}}(\rho)]\ q^2 + [\chi_{770}(\rho) + \chi_{\overline{770}}(\rho)]\ q^3 + 2\chi_{2277}(\rho)\ q^4 + \cdots$$

where the subscript denotes the dimension of the irrep of M_{24}.
- An all-orders proof of the existence of such an expansion has been given by Gannon (2012).

Uniting the different Mathieu moonshines

- For the identity element of M_{24}, the generating function of $\frac{1}{4}$-BPS is the weight 10 Igusa Cusp form, $\Phi_{10}(\mathbf{Z})$.
- $\Phi_{10}(\mathbf{Z})$ is given by the additive (Maass) lift whose seed is $\phi_{-2,1}(\tau, z) \times \eta(\tau)^{24}$
- $\Phi_{10}(\mathbf{Z})$ has a product representation that is given by the 'mutliplicative lift' whose seed is the elliptic genus of K3.



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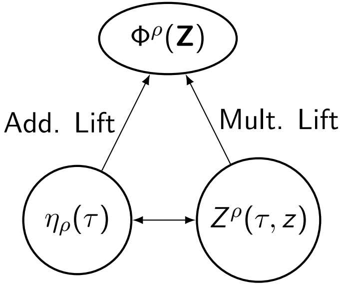

The Siegel Modular Form is not known in all cases.

BKM Lie superalgebras

- The square-root of $\Phi_{10}(\mathbf{Z})$, denoted by $\Delta_5(\mathbf{Z})$, arises as the denominator formula for a Borcherds-Kac-Moody (BKM) Lie superalgebra.
- The sum side of the denominator formula is given by the additive lift and the product side by the multiplicative lift.
- The Cartan matrix of the real simple roots is

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} .$$

- (SG and Krishna, 2008 and 2009) The square root makes sense in some cases. In all of them, the real simple roots are the same but the imaginary simple roots differ.

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- (SG and Krishna, 2008 and 2009) The square root makes sense in some cases. In all of them, the real simple roots are the same but the imaginary simple roots differ.
- Can one understand if and when the square-root makes sense?

Some finite group theory

The simple groups: $L_2(11)$ and M_{12}

- Let $\Omega = PL(11)$ denote the projective line over $\mathbb{F}_{11}$, the field of integers modulo 11.
- Ω can be taken to be the set with 12 points: $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X, \infty)$.
- Let α, β, γ and δ denote the following permutations of Ω.

$$\begin{aligned}
\alpha &= (\infty)(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X) & 1^1 11^1 \\
\beta &= (\infty)(0)(1, 3, 9, 5, 4)(2, 6, 7, X, 8) & 1^2 5^2 \\
\gamma &= (\infty, 0)(1, X), (2, 5), (3, 7)(4, 8)(6, 9) & 2^6 \\
\delta &= (\infty)(0)(1)(2, X)(3, 4)(5, 9)(6, 7)(8) & 1^4 2^4
\end{aligned}$$

- The simple groups $L_2(11)$ and M_{12} can be constructed as permutation representations as follows:

$$L_2(11)_A = \langle \alpha, \beta, \gamma \rangle \quad , \quad L_2(11)_B = \langle \alpha, \beta, \delta \rangle \quad \text{and}$$
$$M_{12} = \langle \alpha, \beta, \gamma, \delta \rangle \ .$$

The simple groups: $L_2(11)$ and M_{12}

- The representations of $L_2(11)$ are **not** related by any automorphism of M_{12}.
- $L_2(11)_A$ is a maximal sub-group of M_{12} while $L_2(11)_B$ is a maximal sub-group of M_{11}.
- 8 of the 15 conjugacy classes of M_{12} reduce to conjugacy classes of $L_2(11)$.
- For $L_2(11)_A$, the cycle shapes are

label	$1a$	$2a$	$3a$	$5a$	$5b$	$6a$	$11a$	$11b$
$\check{\rho}$	1^{12}	2^6	3^4	$1^2 5^2$	$1^2 5^2$	6^2	$1^1 11^1$	$1^1 11^1$

- For $L_2(11)_B$, the cycle shapes are

label	$1a$	$2a$	$3a$	$5a$	$5b$	$6a$	$11a$	$11b$
$\check{\rho}$	1^{12}	$1^4 2^4$	$1^3 3^3$	$1^2 5^2$	$1^2 5^2$	$1^1 2^1 3^1 6^1$	$1^1 11^1$	$1^1 11^1$

- The only conjugacy classes of M_{12} that do not appear are the ones associated with elements of order 4(a/b) and 8(a/b).

The group $M_{12}\!:\!2$

- M_{12} has an order two outer automorphism – denote it φ. The outer automorphism exchanges M_{12} conjugacy classes of elements of orders 4, 8 and 11.
- The group $M_{12}\!:\!2 = M_{12} \rtimes_\varphi \mathbb{Z}_2$ is a maximal sub-group of M_{24}.
- Some of the 26 conjugacy classes of M_{24} reduce to conjugacy classes of $M_{12}\!:\!2$.
- The 24-dimensional permutation representation of $M_{12}\!:\!2$ has two classes of elements ($g \in M_{12}$)

$$(g,e) := \begin{pmatrix} g & 0 \\ 0 & \varphi(g) \end{pmatrix} \quad \text{and} \quad (g,\varphi) := \begin{pmatrix} 0 & g \\ \varphi(g) & 0 \end{pmatrix} .$$

- Conjugacy classes, $\widetilde{\rho}$ of type (g,e) of $M_{12}\!:\!2$ descend to pairs of conjugacy classes of M_{12}. Explicitly, one has $\widetilde{\rho} = (\widehat{\rho}, \varphi(\widehat{\rho}))$, where $\widehat{\rho}$ is the conjugacy class of g in M_{12}.

A moonshine for M_{12}

A moonshine for M_{12} – the proposal

- Let $\hat{\rho}$ denote a conjugacy class of M_{12}. Then, the pair of conjugacy classes $(\hat{\rho}, \varphi(\hat{\rho}))$ become a conjugacy class $\tilde{\rho}$ of $M_{12}{:}2$ and hence M_{24}.
- Let us associate a Jacobi form $\widehat{\psi}^{\hat{\rho}}$ to M_{12} such that

$$Z^{\widetilde{\rho}}(\tau, z) = \widehat{\psi}^{\hat{\rho}}(\tau, z) + \widehat{\psi}^{\varphi(\hat{\rho})}(\tau, z) \ .$$

- Further, let $\widehat{\psi}^{\hat{\rho}}$ have the following decomposition

$$\widehat{\psi}^{\hat{\rho}}(\tau, z) = \alpha^{\hat{\rho}} \ \mathcal{C}(\tau, z) + q^{-\frac{1}{8}} \ \Sigma^{\hat{\rho}}(\tau) \ \mathcal{B}(\tau, z) \ ,$$

with $\alpha^{\hat{\rho}} = 1 + \chi_2(\hat{\rho}) = 1 + \chi_{11}(\hat{\rho})$ and

$$\hat{\Sigma}^{\hat{\rho}}(\tau) = -1 + \sum_{n=1}^{\infty} \left(\sum_{m=1}^{15} \hat{N}_m(n) \hat{\chi}_m(\hat{\rho}) \right) q^n \ .$$

with the multiplicities $\hat{N}_m(n)$ being non-negative integers.

A moonshine for M_{12} – the proposal

- The outer automorphism φ acts trivially on 9 of the 15 conjugacy classes of M_{12}. For all such cases one has $Z^{\widetilde{\rho}} = 2\ \widehat{\psi}^{\widehat{\rho}}$ and hence
$$Z^{\widetilde{\rho}}(\tau, z) = 0 \mod 2\ .$$
- For the conjugacy classes, $11a/b$, reality of $\widehat{\psi}^{\widehat{\rho}}$, needs $\widehat{\psi}^{11a} = \widehat{\psi}^{11b}$ which following if the multiplicities of the two 16-dimensional irreps of M_{12} to be equal. Hence, $Z^{(11a,11b)} = 2\ \widehat{\psi}^{11a} = 0 \mod 2$.
- Using $\Sigma^{\tilde{\rho}} = \hat{\Sigma}^{\hat{\rho}} + \hat{\Sigma}^{\varphi(\hat{\rho})}$, the multiplicities of 11 of the 15 irreps of M_{12} are determined except for two pairs of irreps $(2, 3)$ and $(9, 10)$ (using the Atlas labels). For these two, the combinations are also determined: $N_2^+(n) := (\hat{N}_2(n) + \hat{N}_3(n))$ and $N_9^+(n) := (\hat{N}_9(n) + \hat{N}_{10}(n))$.
- One needs other inputs such as modularity of $\widehat{\psi}^{\widehat{\rho}}$ to fix them. We have been unable to find solutions compatible with modularity that give rise to non-negative integers.

Two moonshines for $L_2(11)$

- The problematic conjugacy classes of M_{12} do not descend to conugacy classes of the two distinct $L_2(11)$ subgroups of M_{12}.
- This implies that there exist Jacobi forms associated with all conjugacy classes of $L_2(11)$ with non-negative multiplicities.
- These are also precisely the situation where the Fourier-Jacobi coefficients of the corresponding $M_{12}{:}2$ Jacobi form are even.

$L_2(11)$ Moonshines

- The dimensions of the irreps of $L_2(11)$ are

label	1	2	3	4	5	6	7	8
dim	1	5	5	10	10	11	12	12

- For $L_2(11)_A$, one obtains $\alpha_A = (\chi_1 + \chi_6)$ and

$$\Sigma_A = -\chi_1 + (\chi_1 + 2\chi_5 + \chi_7 + \chi_8)\, q + (\chi_2 + \chi_3 + 5\chi_4 + 2\chi_5 + 5\chi_6 + 4\chi_7 + 4\chi_8)\, q^2 + (\chi_1 + 8\chi_2 + 8\chi_3 + 9\chi_4 + 12\chi_5 + 13\chi_6 + 14\chi_7 + 14\chi_8)\, q^3 + O(q^4)$$

$L_2(11)$ Moonshines

- The dimensions of the irreps of $L_2(11)$ are

label	1	2	3	4	5	6	7	8
dim	1	5	5	10	10	11	12	12

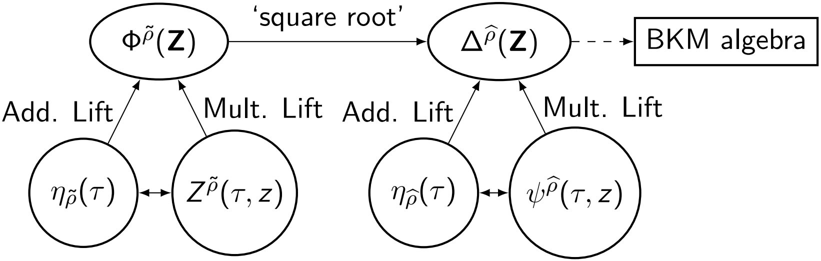
- For $L_2(11)_A$, one obtains $\alpha_A = (\chi_1 + \chi_6)$ and

$$\Sigma_A = -\chi_1 + (\chi_1 + 2\chi_5 + \chi_7 + \chi_8)\, q$$
$$+ (\chi_2 + \chi_3 + 5\chi_4 + 2\chi_5 + 5\chi_6 + 4\chi_7 + 4\chi_8)\, q^2$$
$$+(\chi_1 + 8\chi_2 + 8\chi_3 + 9\chi_4 + 12\chi_5 + 13\chi_6 + 14\chi_7 + 14\chi_8)\, q^3 + O(q^4)$$

- For $L_2(11)_B$, one obtains $\alpha_B = (2\chi_1 + \chi_5)$

$$\Sigma_B = -\chi_1 + (\chi_4 + \chi_6 + \chi_7 + \chi_8)\, q$$
$$+ (\chi_1 + 3\chi_2 + 3\chi_3 + 2\chi_4 + 4\chi_5 + 4\chi_6 + 4\chi_7 + 4\chi_8)\, q^2$$
$$+(\chi_1 + 4\chi_2 + 4\chi_3 + 15\chi_4 + 10\chi_5 + 13\chi_6 + 14\chi_7 + 14\chi_8)\, q^3 + O$$

BKM Lie superalgebras



More generally, one has

$$\Phi^{\tilde{\rho}}(\mathbf{Z}) = \Delta^{\widehat{\rho}}(\mathbf{Z}) \times \Delta^{\varphi(\widehat{\rho})}(\mathbf{Z})$$

When is there a BKM Lie superalgebra associated with $\Delta^{\widehat{\rho}}(\mathbf{Z})$?

- We need to have an additive lift that is compatible with the multiplicative lift. A necessary condition is

$$T_3 \cdot \left[\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\ \eta_{\hat{\rho}}(\tau)\right] \stackrel{?}{=} \psi^{\widehat{\rho}}(\tau,z)\left[\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\ \eta_{\hat{\rho}}(\tau)\right]$$

  This compatibility condition holds for all cases except for those indicated below in red.

- For $L_2(11)_A$,

label	1*a*	2*a*	3*a*	5*a*	5*b*	6*a*	11*a*	11*b*
$\check{\rho}$	1^{12}	2^6	3^4	$1^2 5^2$	$1^2 5^2$	6^2	$1^1 11^1$	$1^1 11^1$

- For $L_2(11)_B$,

label	1*a*	2*a*	3*a*	5*a*	5*b*	6*a*	11*a*	11*b*
$\check{\rho}$	1^{12}	$1^4 2^4$	$1^3 3^3$	$1^2 5^2$	$1^2 5^2$	$1^1 2^1 3^1 6^1$	$1^1 11^1$	$1^1 11^1$

- There is a BKM Lie superalgebra in these cases.

Concluding Remarks

- We have seen the existence of a moonshine for two distinct subgroups of M_{12} that are isomorphic to $L_2(11)$.
- The situation for M_{12} remains unsettled.
- There is a theorem of Creutzig, Hoehn and Miezaki about the evenness of the coefficients of $\Sigma^{\rho}(\tau)$. All conjugacy classes of M_{24} that descend to conjugacy classes of $M_{12}{:}2$ are even. This can be shown to follow from the existence of a moonshine for M_{12}.
- Is there a BKM Lie algebra for all $L_2(11)$ conjugacy classes? This might be possible if there is some modification to the additive lift to account for the correction factors that appear.
- The Jacobi forms associated with umbral moonshine, in some cases, are associated with BKM Lie superalgebras (SG, 2010).
- Does the outer automorphism of M_{12} have a role in string theory?