S-duality, Hecke groups and modular anomalies in $\mathcal{N} = 2$ SQCD

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Outline

- Review of $\mathcal{N} = 2$ supersymmetric gauge theories in $4d$: the prepotential and the Seiberg-Witten solution.
- The non-perturbative S-duality group of conformal SQCD ($N_f = 2N$).

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- S-duality action on variables in the Seiberg-Witten effective action.
- Appearance of Hecke groups, hauptmoduls and a non-perturbative relation.
- Turning on masses and modular anomaly equations.

Review of $\mathcal{N} = 2$ theories

- We study $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group SU(N). These are gauge theories with eight supercharges.
- The $\mathcal{N} = 2$ vector-multiplet: (A_μ, λ) and (ϕ, ψ).
- Each of these fit into $\mathcal{N} = 1$ superfields:

$$\Phi = \phi + \theta^\alpha \psi_\alpha + \theta^2 \, F$$

$$\mathcal{W}_\alpha = -\mathrm{i}\lambda_\alpha + \theta_\alpha D - \frac{\mathrm{i}}{2}(\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha) F_{\mu\nu} + \theta^2 (\sigma^\mu \partial_\mu \bar{\lambda})_\alpha \, .$$

  These are chiral: $\bar{D}_{\dot{\alpha}} \Phi = \bar{D}_{\dot{\alpha}} \mathcal{W}_\alpha = 0$.

- Look for a way to write an $\mathcal{N} = 2$ superfield that contains both these sets of fields: Sohnius

$$\mathcal{A} = \Phi(\tilde{y}, \theta) + \widetilde{\theta}^\alpha \mathcal{W}_\alpha(\tilde{y}, \theta) + \widetilde{\theta}^2 G(\tilde{y}, \theta)$$

  $\mathcal{A}$ is a constrained $\mathcal{N} = 2$ chiral superfield s.t.

$$G(y, \theta) = \int d^2 \bar{\theta} \Phi^\dagger e^{2V} \, .$$

Review of $\mathcal{N} = 2$ theories

- In $\mathcal{N} = 2$ superspace, the classical Lagrangian is simple:

$$\pi i \tau \int d^2\theta d^2\widetilde{\theta}\, \mathrm{Tr} \mathcal{A}^2 + c.c\,.$$

- This action, in terms of $\mathcal{N} = 1$ multiplets, is familiar:

$$\mathcal{L} \propto \int d^2\theta \mathrm{Tr} \mathcal{W}_\alpha \mathcal{W}^\alpha + \int d^2\theta d^2\bar{\theta} \mathrm{Tr} \Phi^\dagger e^{2V} \Phi\,.$$

- The gaugino λ and ψ have the same kinetic coefficient: $\mathrm{SU}(2)_R$.
- In the non-abelian case, the bosonic potential term is of the form

$$V \propto [\phi, \phi^\dagger]^2$$

- There is a continuous set of vacua, labelled by the vev of ϕ:

$$\langle \phi \rangle = \mathrm{diag}(a_1, a_2, \ldots a_N)\,.$$

- The gauge group is completely broken down to $\mathrm{U}(1)^{N-1}$: this is the Coulomb branch.

Review of $\mathcal{N} = 2$ theories

- Question: what is the low energy effective action on the Coulomb branch?
- On the Coulomb branch, the quantum effective action is that of a $\mathrm{U}(1)^{N-1}$ gauge theory.
- A most general Lagrangian for a two-derivative theory with 8 supercharges is completely specified by a single holomorphic function, called the prepotential $\mathcal{F}$.

$$\int d^2\theta d^2\widetilde{\theta}\, \mathcal{F}(\mathcal{A}_i)\,.$$

- The Lagrangian for such a theory is given by:

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im}\left[\frac{1}{2}\int d^2\theta\, W^\alpha \cdot \Omega \cdot W_\alpha + \int d^2\theta\, d^2\bar{\theta}\, \Phi^\dagger \cdot a^{\mathrm{D}}\right]$$

$$\boxed{a_k^{\mathrm{D}} = \frac{1}{2\pi\mathrm{i}}\frac{\partial \mathcal{F}}{\partial a_k}} \quad \text{and} \quad \boxed{\Omega_{km} = \frac{1}{2\pi\mathrm{i}}\frac{\partial^2 \mathcal{F}}{\partial a_k \partial a_m}}$$

- Classically

$$\mathcal{F}_{\text{class.}} = \pi\mathrm{i}\tau \sum_i {\mathcal{A}_i}^2\,.$$

Seiberg-Witten solution of $\mathcal{N} = 2$ theories

- The SW solution is given in terms of an algebraic curve Σ and an associated differential λ.
- The quantum-corrected space of vacua is the moduli space of the algebraic curve.
- For SQCD with gauge group SU(N) and $2N$ flavours,

$$\Sigma : \quad y^2 = P_N^2(x, u_k) - h \prod_{I=1}^{2N} (x - m_I) \,.$$

- The period integrals of λ encode information about F:

$$a_k = \oint_{A_k} \lambda(u_i) \quad \text{and} \quad a_\ell^{\mathrm{D}} = \oint_{B_\ell} \lambda(u_i) \equiv \frac{\partial \mathcal{F}}{\partial a_\ell} \,.$$

- Ω is the period matrix of the Riemann surface.
- Use A-integrals to write the u_k in terms of the a_ℓ. Then, do the B-integrals to calculate the prepotential F.
- For higher rank gauge theories, this is easier said than done.

What are the quantum corrections due to . . . ?

- Non-renormalization theorems imply that the prepotential is one-loop exact. Seiberg
- However, non-perturbative corrections appear at all orders, due to instantons. The prepotential takes an additive form

$$\mathcal{F} = \mathcal{F}_{\text{cl}} + \mathcal{F}_{\text{1-loop}} + \boxed{\mathcal{F}_{\text{inst.}}}$$

- The instanton contribution takes the following form

$$\mathcal{F}_{\text{inst.}} = \sum_k q_0^k \, \mathcal{F}_k \, .$$

- Here, q_0 is a counting parameter, which for a superconformal theory is simply $e^{2\pi i \tau_0}$ and k is the instanton number, obtained from the classical action:

$$S_{\text{inst}} = -2\pi i \tau \left(\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr} F \wedge F \right)$$

- There is an alternative way to calculate the instanton contributions directly using the methods of equivariant integration over the moduli space of instantons $\mathcal{M}_{N,k}$. Nekrasov

Prepotential from instanton partition function

- A naive calculation of the instanton partition function leads to divergent results due to bosonic zero modes. We obtain $\mathrm{vol}(\mathbb{R}^4)$ for the 1-instanton answer.
- Study gauge theory on an Ω-background: best thought of as a means to regularize the instanton path integral so that

$$\mathrm{vol}(\mathbb{R}^4_{\epsilon_1,\epsilon_2}) = \frac{1}{\epsilon_1\epsilon_2}$$

- The relation to our earlier story is that the instanton partition function is related to the prepotential by the relation Nekrasov

$$Z_{\text{inst.}} = \sum_k Z_k q_0^k = e^{-\frac{1}{\epsilon_1\epsilon_2}\sum_k \mathcal{F}_k q_0^k}$$

- Comparing powers of q_0 on both sides, we can read off $\mathcal{F}_k$.
- Note that each Z_k is divergent as $\epsilon_i \to 0$ but $\mathcal{F}_k$ are regular.
- An algorithmic method to calculate $Z_{\text{inst.}}$ order by order in q_0.

M-theory curve and Gaiotto's variation

- In the Seiberg-Witten solution, there was a free parameter h, while the Nekrasov method gives a power series in q_0.
- To compare with localization calculations, we need to identify h with some function of bare coupling q_0.
- Consider the M-theory curve for SQCD in the massless limit. Witten '97, Gaiotto '09

$$\mathcal{P}(v,t) = v^N\, t^2 - (1+q_0)\, t\, P_N(v) + q_0\, v^N = 0\,.$$

  Here, q_0 is the ratio of the asymptotic positions of NS5 branes in the type IIA set-up.

- q_0 is *also* the bare coupling constant in the gauge theory.
- Further, there is a canonical choice of SW differential:

$$\lambda = v\,\frac{dt}{t}\,.$$

M-theory curve vs. SW curve

- Start with the M-theory curve and rescale $t \to (1+q_0)\,t$:

$$v^N\, t^2 - P_N(t)\, t + \frac{q_0}{(1+q_0)^2}\, v^N = 0\,.$$

Shift away the linear term in t and introduce

$$y = 2\, v^N \left(t - \frac{P_N(v)}{2v^N} \right)\,.$$

- We match the SW curve

$$y^2 = P_N^2(v) - h\, v^{2N}\,,$$

with the identification

$$\boxed{h = \frac{4q_0}{(1+q_0)^2}}\,.$$

S-duality group: Gaiotto's formulation

- Consider Gaiotto's rewriting of the M-theory curve. It is the branched cover of a punctured Riemann surface ($v = x\,t$):

$$x^N + \sum_{k=2}^{N} x^{N-k} \frac{u_k}{t^{k-1}(t-1)(t-q_0)} = 0\,.$$

- The modulus of the punctured sphere is the bare coupling q_0. More generally, the space of bare couplings = moduli space of complex structures on the Riemann surface.
- The S-transformation acts as

$$(x,t) \to (-t^2\,x\,,\frac{1}{t})\,.$$

Effectively it acts as

$$\boxed{q_0 \to \frac{1}{q_0}}\,.$$

S-duality group: the SW formualtion

- Consider the SW curve:

$$y^2 = P_N^2(x, u_i) - h\, x^{2N} \,.$$

- How does the S-duality group acts on h? Use the map:

$$h = \frac{4q_0}{(1+q_0)^2} \,.$$

- From Gaiotto's analysis, h is left invariant.
- **Claim:** The S-duality group is generated by monodromies in the h-plane around $h = 0, 1, \infty$. Argyres-Buchel '97
- The original argument is that for $h = 0, 1, \infty$, the SW curve degenerates irrespective of location on the Coulomb moduli space.

Summary of results

What we do . . .

- In our work, we use equivariant localization methods to calculate the prepotential, dual-periods and the period matrix. Nekrasov
- In the SW formulation, there is an EM duality group which acts on the period integrals and the dual periods as the symplectic group $\mathrm{Sp}(2n-2,\mathbb{Z})$.
- Embed the S-duality group in the EM duality group and obtain action on $(a_\ell, a_k^{\mathrm{D}})$. Argyres-Buchel '97
- Gaiotto's work tells us that $q_0 \to \frac{1}{q_0}$ under S-duality while there is an alternative description of the duality group generated by monodromies.
- Impose compatibility of the S-action on bare and effective couplings.

What we find . . .

- Even in the massless theory, the bare coupling gets renormalized by quantum effects.
- There are $[N/2]$ effective/renormalized couplings on a special locus of the Coulomb branch.
- We find a non-perturbative formula between the bare and renormalized couplings in terms of the j-invariant of Hecke groups.
- The action of the Hecke group on the renormalized couplings is dictated by the non-perturbative S-duality of the conformal gauge theory.
- We give an independent justification of the Argyres-Buchel claim.

What we find . . .

- With the masses turned on, this modular structure survives.
- The constraints from S-duality, combined with the explicit answers for the instanton expansion allows us to completely resum the expansion in terms of modular/meromorphic forms.

Monodromies in a special vacuum

- Consider the special locus on the Coulombs branch s.t.

$$u_k := \langle \mathrm{Tr}\Phi^k \rangle = 0 \quad \forall \quad k \neq N\,.$$

- There is a $\mathbb{Z}_N$ symmetry on this one-dimensional locus.
- The SW curve takes the form

$$y^2 = (x^N - u_N)^2 - h\,x^{2N}\,.$$

- Monodromies that generate the S-duality group are now embedded in the EM duality group. Argyres-Buchel '97
- The monodromies around $h = 0, 1, \infty$ generate transformations of the cycles of the hyperelliptic curve.

Action of S-duality on the observables

- These are naturally elements of $\mathrm{Sp}(2N-2,\mathbb{Z})$:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

- These act on the vector $(a_k^{\mathrm{D}}, a_\ell)$ as matrix multiplication.
- On the period matrix:

$$M \cdot \Omega = (A\Omega + B) \cdot (C\Omega + D)^{-1} .$$

- Explicitly, the symplectic matrices that correspond to S and T are given by (just follow the roots)

$$M_\infty := S = \begin{pmatrix} 0 & B \\ -(B^{\mathrm{t}})^{-1} & 0 \end{pmatrix} \qquad M_0 := T = \begin{pmatrix} \mathbb{I} & C \\ 0 & \mathbb{I} \end{pmatrix}$$

- B is a specific matrix which we can calculate and C is the Cartan matrix of $\mathrm{SU}(N)$.

Massless case: dual periods

- In the special vacuum, there is only one independent set of $(a, a^{\rm D})$. One can then check that (this is unchanged by quantum corrections):

$$\begin{aligned} a_k &= \omega^{k-1} a \\ a_k^{\rm D} &= -(\omega + \omega^2 + \ldots + \omega^k)\, a^{\rm D}\,. \end{aligned}$$

  We have chosen $(a_1, a_{N-1}^{\rm D})$ to be the independent pair.

- Main result of the Nekrasov analysis:

$$\begin{aligned} a^{\rm D} &= c_N \left(\tau_0 + \log d_0 + \sum_{n=1}^{\infty} d_n q_0^n \right) a \\ &= c_N\, \tau\, a. \end{aligned}$$

- This is obtained by using the equivariant localization and we typically obtain the first few terms in the power series.
- The coupling τ is interpreted as a renormalized or quantum coupling constant. This is unlike the $\mathcal{N} = 2^*$ theories, where the bare coupling is unrenormalized.

S-duality constraints on period integrals

- Using the embedding of S into the EM duality group, we find (in the special locus):

$$S(a) = a^{\mathrm{D}} \qquad S(a^{\mathrm{D}}) = -\frac{a}{\omega^2}$$

- The dual period in the massless limit is given by

$$a^{\mathrm{D}} = c_N \, a \, \tau \quad \text{with} \quad c_N = -\frac{(1-\omega)}{\omega^2}$$

- Consistency of the S-action with the above equation gives:

$$\boxed{S(\tau) = -\frac{1}{\alpha\tau} \quad \text{with} \quad \alpha = 4\sin^2\frac{\pi}{N}\,.}$$

- For $N = 2, 3, 4, 6$, this is an integer. Any such coupling with $\alpha \in \mathbb{Z}$, we denote an arithmetic coupling.

Massless case: period matrix

- At the classical level, $\Omega_{\text{class}} = \tau_0 \, C$.
- At the one-loop level, we have Minahan-Nemeschansky '96

$$\Omega_{1-\text{loop}} \sim \sum_{k=1}^{[N/2]} \log \sin \frac{k\pi}{N} \, M_k \quad \text{with} \quad \sum_k M_k = C \, .$$

- Generically there are $N(N+1)/2$ independent couplings but drastic simplification in the special locus.
- The instanton corrections do NOT change the matrix structure. Effectively

$$\Omega = \sum_{k=1}^{[N/2]} \sigma_k \, M_k \quad \text{with} \quad \sigma_k = \log q_0 + \log \sin \frac{k\pi}{N} + \dots \, .$$

- $S(\Omega)$ obtained using explicit S-transformation. It is a mess.
- Observation: Always possible to form combinations that diagonalize the S-action.

SU(4)

- Results of the localization calculations can be packaged in the following form: Minahan-Nemeschansky '96

$$\Omega = \tau_1 \mathcal{M}_1 + \tau_2 \mathcal{M}_2$$

$$\text{with} \quad \tau_1 = \log q_0 - \log 64 - \frac{3}{8} q_0 + \frac{141}{1024} q_0^2 + \cdots$$

$$\tau_2 = \log q_0 - \log 16 - \frac{1}{2} q_0 + \frac{13}{64} q_0^2 + \cdots .$$

- S-transformation in this $\mathcal{M}_k$ basis is diagonal:

$$S(\Omega) = -\frac{1}{2\tau_1} \mathcal{M}_1 - \frac{1}{4\tau_2} \mathcal{M}_2$$

- We have diagonalized the S-action, with

$$\boxed{S(\tau_k) = -\frac{1}{\alpha_k \tau_k} \quad \text{with} \quad \alpha_k = 4 \sin^2 \frac{k\pi}{4}}$$

- Is this consistent with Gaiotto's definition of S-duality?

Quantum couplings and S-duality

- Inverting the τ_k vs. q_0, we obtain OEIS

$$q_0 \longrightarrow 64 q_1 (1 + 24 q_1 + 300 q_1^2 + \cdots) = 64 \left(\frac{\eta(2\tau_1)}{\eta(\tau_1)} \right)^{24}$$

$$q_0 \longrightarrow 16 q_2 (1 + 8 q_2 + 44 q_2^2 + \cdots) = 16 \left(\frac{\eta(4\tau_2)}{\eta(\tau_2)} \right)^{8}$$

- Note that under the action of S,

$$\tau_k \to -\frac{1}{\alpha_k \tau_k} \quad \Rightarrow \quad q_0 \to \frac{1}{q_0} \,.$$

- Consistency with S-duality and matching the first few terms with localization calculations led to this proposal.
- Remember that the localization calculations only give q_0 expansions. The key is to find separate combinations τ_k that are inverted under S-transformation.

Quantum couplings and S-duality

- This works for (almost) all the arithmetic cases:
  - For SU(2) and SU(3), there's one coupling τ_1.
  - For SU(4), there are two couplings τ_1, τ_2.
  - For SU(6), there are three couplings τ_1, τ_2, τ_3.
- For $\alpha_k \neq 1$, the map between the bare coupling and the τ_k takes the universal form

$$q_0 = (\alpha_k)^{\frac{6}{\alpha_k - 1}} \left(\frac{\eta(\alpha_k \tau_k)}{\eta(\tau_k)} \right)^{\frac{24}{\alpha_k - 1}} , \quad \alpha_k = 4 \sin^2 \frac{k\pi}{N} .$$

- It reproduces the behaviour of q_0 under S-duality as expected from Gaiotto's analysis:

$$\tau_k \to -\frac{1}{\alpha_k \tau_k} \quad \Rightarrow \quad q_0 \to \frac{1}{q_0} .$$

Exception to the rule

- The only exception is τ_1 in the SU(6) theory, which transforms with $\alpha_1 = 1$ and has the following q-expansion:

$$q_0 = 432q_1(1 + 120q_1 + 4140q_1^2 + \cdots)$$

- No η-quotient formula for this q-expansion.
- Can we find a more inclusive formula?

Quantum couplings and the j-invariant

- **Claim**: The following formula in terms of the j-invariant reproduces the q-expansion of τ_1 Wikipedia

$$q_0 = \frac{\sqrt{j(\tau_1)} - \sqrt{j(\tau_1) - 1728}}{\sqrt{j(\tau_1)} + \sqrt{j(\tau_1) - 1728}}$$

- The j-invariant appears because $\alpha_1 = 1$. So, the full modular group acts on τ_1.
- Here, $j(\mathrm{i}) = 1728$, is the value of the the j-invariant at the fixed point of the S-action.
- Can we generalize this formula to the other arithmetic couplings?
- We need to revisit the mathematical structure of the S-duality group for the other cases when $\alpha_k \in \mathbb{Z}$.

The structure of the S-duality group

- Only for arithmetic couplings the S-duality group has a subgroup in common with $\mathrm{SL}(2,\mathbb{Z})$, which is the congruence subgroup $\Gamma_1(\alpha_k)$. This is generated by Minahan '98

$$T : \tau_k \to \tau_k + 1$$
$$S' = STS^{-1} : \tau_k \to \frac{\tau_k}{1 - \alpha_k \tau_k} \, .$$

- To generate the full S-duality group, we have to adjoin to this, the S-transformation (a Fricke involution):

$$S = \begin{pmatrix} 0 & 1/\sqrt{\alpha_k} \\ -\sqrt{\alpha_k} & 0 \end{pmatrix} \, .$$

- $S(\tau_k) = -\frac{1}{\alpha_k \tau_k}$. The group is denoted $\Gamma^*(\alpha_k)$. Zagier

j-invariants for $\Gamma^*(\alpha_k)$

- Each of these have analogs of the j-function: the *hauptmodul*. All modular invariant functions are rational functions of the hauptmodul.
- We can also evaluate j at fixed point of S-action: $\frac{1}{d_{\alpha_k}} = j(\frac{i}{\sqrt{\alpha_k}})$.

$$j_1(\tau) = \left(\frac{E_4(\tau)}{\eta^8(\tau)}\right)^3 \qquad d_1 = \frac{1}{1728}$$

$$j_2(\tau) = \left[\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{12} + 64\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{12}\right]^2 \qquad d_2 = \frac{1}{256}$$

$$j_3(\tau) = \left[\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{6} + 27\left(\frac{\eta(3\tau)}{\eta(\tau)}\right)^{6}\right]^2 \qquad d_3 = \frac{1}{108}$$

$$j_4(\tau) = \left[\left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^{4} + 16\left(\frac{\eta(4\tau)}{\eta(\tau)}\right)^{4}\right]^2 \qquad d_4 = \frac{1}{64}\,.$$

Main result

$$\boxed{q_0 = \frac{\sqrt{j_{\alpha_k}(\tau_k)} - \sqrt{j_{\alpha_k}(\tau_k) - \frac{1}{d_{\alpha_k}}}}{\sqrt{j_{\alpha_k}(\tau_k)} + \sqrt{j_{\alpha_k}(\tau_k) - \frac{1}{d_{\alpha_k}}}}}$$

- This reproduces all the η-quotient expressions we found earlier.
- In addition, this leads to a geometric definition of the S-duality group in terms of monodromies in the j-plane.
- e.g. the S-transformation is generated by

$$(j_{\alpha_k}(\tau_k) - d_{\alpha_k}^{-1}) \to e^{2\pi \mathrm{i}}((j_{\alpha_k}(\tau_k) - d_{\alpha_k}^{-1}))$$

- q_0 is inverted under this operation, as expected.
- S-duality group being generated by monodromies: doesn't this sound familiar?

A modular invariant SW curve

- Let us revisit the SW curve:

$$y^2 = P_N^2(x) - h\, x^{2N}\ ,$$

- Recall the the h vs. q_0 relation and compose this with the q_0 vs. j map:

$$h = \frac{4q_0}{(1+q_0)^2} = \frac{1}{1 - d_{\alpha_k} j_{\alpha_k}(\tau_k)}\ .$$

- The SW curve can be written in terms of a modular invariant j-function, the hauptmodul of $\Gamma^*(\alpha_k)$, which is identified with the S-duality group.

- We have circled back to the Argyres-Buchel claim: monodromies in the h-plane generate the S-duality group. The h-plane is just the j-plane.

What have we gained by this realization?

Rewriting in terms of the j-invariant has many features:

- Universal behaviour of j and hence h, irrespective of which τ_k is used to write the SW curve.

$$\tau_k \to \tau_A \Rightarrow h \to \infty$$
$$\tau_k \to \tau_B \Rightarrow h \to 1$$
$$\tau_k \to \tau_C \Rightarrow h \to 0\,.$$

- Earlier works in the physics literature focused on *one* of the many τ_k present. Above analysis shows they are correct but incomplete. A much more global picture emerges.

- Moreover, the hauptmodul can be expanded near any cusp. In principle, strong coupling questions can be addressed.

To summarize . . .

- The arithmetic cases SU(N), with $N = 2, 3, 4, 6$ are under control.
- The modular structure becomes manifest when the period matrix and the dual periods are written in terms of the renormalized couplings τ_k instead of the bare coupling q_0.
- What can one say about the non-arithmetic cases, still in the massless limit?

SU(5)

- For instance, for SU(5), there are two couplings:

$$\Omega = \tau_1 \mathcal{M}_1 + \tau_2 \mathcal{M}_2\,.$$

$$\tau_1 = \log \tau_0 - \log[25\sqrt{5}] - \sqrt{5}\log\left[\frac{1+\sqrt{5}}{2}\right] - \frac{8q_0}{25} + \frac{14}{125}q_0^2 + \cdots$$

$$\tau_2 = \log \tau_0 - \log[25\sqrt{5}] + \sqrt{5}\log\left[\frac{1+\sqrt{5}}{2}\right] - \frac{12q_0}{25} + \frac{24}{125}q_0^2 + \cdots$$

- Under S-duality, they transform as

$$S(\Omega) = -\frac{1}{\alpha_1 \tau_1}\mathcal{M}_1 - \frac{1}{\alpha_2 \tau_2}\mathcal{M}_2\,.$$

$$\alpha_1 = 4\sin^2\frac{\pi}{5} = 4\cos^2\frac{3\pi}{10} \qquad \alpha_2 = 4\sin^2\frac{2\pi}{5} = 4\cos^2\frac{\pi}{10}$$

- Inverting these numerically a mess. Are there j-invariants? J. Oesterle

Hecke groups

- A Hecke group $H(p)$ is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ whose generators S and T satisfy
$$S^2 = 1 \qquad (ST)^p = 1\,.$$
- These have the following action:
$$S : \tilde{\tau} \to -\frac{1}{\tilde{\tau}} \qquad T : \tilde{\tau} \to \tilde{\tau} + 2\cos\frac{\pi}{p}\,.$$
- Setting $\tilde{\tau} = 2\cos\frac{\pi}{p}\tau$, we see that on τ, $H(p)$ coincides with our S-duality group action, whenever
$$\alpha = 4\cos^2\frac{\pi}{p}\,.$$
- For arithmetic cases, this is always possible for $p \in \mathbb{Z}$.

- For the four arithmetic cases the correspondence between λ_k and p_k is summarized below:

α_k	1	2	3	4
p_k	3	4	6	∞

- Notice that these are the only cases in which both α_k and p_k are integers.

$$\frac{1}{p_k} = \frac{1}{2} - \frac{k}{N}$$

for $k = 1, \ldots, \left[\frac{N}{2}\right]$.

- For non-arithmetic cases, we find $p \in \mathbb{Q}$. As we saw, for SU(5):

$$\alpha_1 = 4\sin^2\frac{\pi}{5} = 4\cos^2\frac{3\pi}{10}$$

j-invariants for Hecke groups

- There is a hauptmodul for every Hecke group. They satisfy equations of the type

$$-2J'''J' + 3(J'')^2 = (J')^4 \left(\frac{a(p)}{J^2} + \frac{b(p)}{(1-J)^2} + \frac{c(p)}{J(J-1)} \right)$$

- Use the following ansatz near weak coupling

$$J = \frac{d}{q} + \sum_n c_n \left(\frac{d}{q} \right)^n ,$$

to solve for c_n. Local q-expansions available near each cusp for $J \equiv j_p$.

C. Doran et al. '13: Automorphic forms for triangle groups

- $d_p = j_p(\tau_A)$ can also be calculated.
- Use these results in our universal formula for q_0 in terms of the hauptmodul j_p.

j-invariants from localization

- Consider the SU(5) case again. We can check the following:

$$\log(4d_{10}) = \log[25\sqrt{5}] - \sqrt{5}\log\left[\frac{1+\sqrt{5}}{2}\right]$$
$$\log(4d_{\frac{10}{3}}) = \log[25\sqrt{5}] + \sqrt{5}\log\left[\frac{1+\sqrt{5}}{2}\right]$$

- Solving for the j-invariant and substituting in our universal q_0 vs. j formula, we find that
  - For $p = 10$, we get the expected q_0-expansion of τ_2.
  - For $p = \frac{10}{3}$, we get (unpexpectedly) the q_0-expansion of τ_1.
- For $p \in \mathbb{Q}$, we do not get a discrete group of $\mathrm{SL}(2,\mathbb{R})$ but the formulae seem to hold for all cases. Checked for $N \leq 15$.

Summary of massless results

- In the massless limit and in a *special locus* of the Coulomb branch, the period matrix of the Seiberg-Witten curve can be written as:

$$\Omega = \sum_k \tau_k \, \mathcal{M}_k$$

- The S-duality group can be embedded in the EM duality group as follows:

$$S(\Omega) = -\sum_k \frac{1}{\alpha_k \tau_k} \mathcal{M}_k \quad \text{with} \quad \alpha_k = 4 \sin^2 \frac{k\pi}{N} \, .$$

- The dual period is of the following form:

$$a^{\mathrm{D}} = c_N \, a \, \tau_1 \, .$$

Summary of massless results

- The S-duality group acts as a generalized Hecke group on each of the couplings τ_k.
- The bare coupling of the gauge theory can be expressed in terms of j-invariant of the corresponding Hecke group:

$$q_0 = \frac{\sqrt{j_p(\tau_k)} - \sqrt{j_p(\tau_k) - \frac{1}{d_p}}}{\sqrt{j_p(\tau_k)} + \sqrt{j_p(\tau_k) - \frac{1}{d_p}}}$$

  Such a relation holds for each of the τ_k even when $p \in \mathbb{Q}$.

- S-duality group is generated by monodromies in the j-plane around $j = 0$, $j = \infty$ and $j = \frac{1}{d}$ (equivalently, monodromies in the h-plane around $0, 1, \infty$).
- SW curve has uniform behaviour near cusps independent of which τ_k is chosen.

Part II: mass deformations

- Let us turn on the masses such that the $\mathbb{Z}_N$ symmetry is preserved. The SW curve is now of the form

$$y^2 = P_N^2(x) - h\,Q(x)\,.$$

- The flavour polynomial takes the form

$$Q(x) = (x^N - m^N)(x^N - \tilde{m}^N)\,.$$

  Equivalently, the mass-deformations can be parametrized in terms of the two Casimirs:

$$T_N = N(m^N + \tilde{m}^N) \quad \text{and} \quad T_{2N} = N(m^{2N} + \tilde{m}^{2N})\,.$$

Dual periods

- The dual period in the mass-deformed theory takes the form

$$a^{\mathrm{D}} = c_N \, \tau_1 \, a + \frac{c_N}{2\pi \mathrm{i}} \sum_n \frac{g_n(q_0, T_N, T_{2N})}{a^{n\,N+N-1}} \,.$$

- Define

$$\begin{aligned} X &:= a^{\mathrm{D}} - c_N \, a \, \tau_1 \\ &= \frac{c_N}{2\pi \mathrm{i}} \sum_n \frac{g_n(q_0, T_N, T_{2N})}{a^{Nn+N-1}} \end{aligned}$$

- Act with S on (a^{D}, a) and we find

$$S(X) = \frac{1}{c_N \, \omega^2 \, \tau_1} \, X \,.$$

- Apply S-duality to the second line above, we get

$$S(X) = \frac{c_N}{2\pi \mathrm{i}} \sum_n \frac{S(g_n)}{(-\omega \, a^{\mathrm{D}})^{Nn+N-1}} \,.$$

Dual periods

- Substitute the expression for a^{D} and equate the two different expressions for $S(X)$ order by order in the large-a expansion. From the leading term, we find

$$S(g_0) = \left(\mathrm{i}\sqrt{\alpha_1}\tau_1\right)^{N-2} g_0 \ .$$

- Repeat for the $S' = STS^{-1}$ transformation and we find

$$S'(g_0) = \left(1 - \alpha_1\tau_1\right)^{N-2} g_0 \ .$$

- We conclude that g_0 is a modular form of $\Gamma_1(\alpha_1)$ with weight $N-2$ and S-parity $+1$.

Dual periods

For $n > 0$ the analysis is similar. We find that

$$S(g_n) = (-1)^n (\mathrm{i}\sqrt{\alpha_1 \tau_1})^{nN+N-2} \left[g_n + \frac{nN+N-2}{4\pi \mathrm{i} \tau} \sum_m g_m g_{n-m-1} \right]$$

$$S'(g_n) = (1 - \alpha_1 \tau_1)^{nN+N-2} \left[g_n + \frac{nN+N-2}{4\pi \mathrm{i} \tau} \sum_m g_m g_{n-m-1} \right] .$$

The g_n are almost-modular forms of $\Gamma_1(\alpha)$ with weight $nN + N - 2$ and S-parity equal to n mod 2, but there is an anomaly.

Almost modular = quasi-modular

- In order to solve the constraints from S-duality, we assume that the only source of the anomalous modular transformation is the presence of $E_2(\tau)$, the second Eisenstein series.
- Recall that E_2 is a quasi-modular form of weight 2 such that

$$E_2\Big(-\frac{1}{\tau_1}\Big) = -(\mathrm{i}\tau_1)^2\left(E_2(\tau_1) + \frac{6}{\mathrm{i}\pi\tau_1}\right) .$$

- In the arithmetic cases, it is always possible to form a linear combination of E_2 and a modular form of $\Gamma_1(\alpha_1)$, which under the S transformation $\tau_1 \to -\frac{1}{\alpha_1\tau_1}$ transforms in a way similar to (44):

$$\widetilde{E}_2^{(\alpha_1)}\left(-\frac{1}{\alpha_1\tau_1}\right) = -\left(\mathrm{i}\sqrt{\alpha_1}\tau_1\right)^2\left(\widetilde{E}_2^{(\alpha_1)}(\tau_1) + \frac{6}{\mathrm{i}\pi\tau_1}\right) .$$

Almost modular = quasi-modular

Assuming now that g_n are also polynomials in $\widetilde{E}_2^{(\alpha_1)}$:

$$S\left(g_n\left[\widetilde{E}_2^{(\alpha_1)}\right]\right) = (-1)^n(\mathrm{i}\sqrt{\alpha_1}\tau_1)^{Nn+N-2}\, g_n\left[\widetilde{E}_2^{(\alpha_1)} + \frac{6}{\mathrm{i}\pi\tau_1}\right]$$
$$= (-1)^n(\mathrm{i}\sqrt{\alpha_1}\tau_1)^{Nn+N-2}\left(g_n\left[\widetilde{E}_2^{(\alpha_1)}\right] + \frac{6}{\mathrm{i}\pi\tau_1}\frac{\partial g_n}{\partial \widetilde{E}_2^{(\alpha_1)}}\right)$$

Comparing with action of S on the g_n we obtain a modular anomaly equation:

$$\frac{\partial g_n}{\partial E_2^{(\alpha_1)}} = \frac{nN+N-2}{24}\sum_{\ell=0}^{n-1} g_\ell g_{n-\ell-1}\,.$$

This completely fixes the anomalous behaviour of g_n.

Modular pieces fixed by comparing with the q-expansions obtained using localization.

SU(4): dual periods

- Calculate the dual periods using localization.
- In terms of the renormalized coupling q_1, we find that the dual period in the special locus is given by:

$$a^{\mathrm{D}} = (\mathrm{i}-1)\,a\,\tau_1 + \frac{(\mathrm{i}-1)}{2\pi\mathrm{i}}\sum_{n=0}^{\infty}\frac{g_n(q_1;T_4,T_8)}{a^{4n+3}}$$

- The first coefficients g_n are

$$g_0 = \frac{T_4}{12}\left(1+24\,q_1+24\,q_1^2+96\,q_1^3+\cdots\right)\ ,$$

$$g_1 = \frac{T_4^2}{4}\left(q_1+26\,q_1^2+84\,q_1^3+\cdots\right) + \frac{T_8}{56}\left(1-56\,q_1-2296\,q_1^2-13664\,q_1^3+\cdots\right)\ .$$

SU(4)

- From our general analysis, g_0 should be a modular form of $\Gamma_1(2)$ with weight 2 and S-parity $(+1)$.
- There is only one such form, namely $f^{(2)}_{2,+}$ whose weak-coupling expansion is (actually sums of ratios of η quotients):

$$f^{(2)}_{2,+} = 1 + 24q_1 + 24q_1^2 + 96q_1^3 + 24q_1^4 + 144q_1^5 \cdots \ .$$

- Comparing with (1), we are led to conclude

$$g_0 = \frac{T_4}{12}\, f^{(2)}_{2,+} \ ,$$

SU(4)

- g_1 should be a quasi-modular form of of $\Gamma_1(2)$ with weight 6 and with S-parity (-1) that solves:

$$\frac{\partial g_1}{\partial \widetilde{E}_2^{(2)}} = \frac{1}{4}\, g_0^2 \ .$$

- Integrating with respect to $\widetilde{E}_2^{(2)}$ and using the exact expression for g_0 obtained above, we find

$$g_1 = \frac{T_4^2}{576}\big(f_{2,+}^{(2)}\big)^2\, \widetilde{E}_2^{(2)} + \text{modular piece}\,,$$

- There is only one candidate modular form with the right weight and S-parity:

$$f_{2,+}^{(2)}\, f_{4,-}^{(2)} = 1 - 56q_1 - 2296q_1^2 - 13664q_1^3 + \cdots \ .$$

Comparing with the localization result (1), obtain the following exact expression

$$g_1 = \frac{T_4^2}{576}\left[\big(f_{2,+}^{(2)}\big)^2\, \widetilde{E}_2^{(2)} - \frac{3}{2}\, f_{2,+}^{(2)}\, f_{4,-}^{(2)}\right] + \frac{T_8}{56}\, f_{2,+}^{(2)}\, f_{4,-}^{(2)} \ .$$

Results for the Period matrix

- Similar analysis for the period matrix at leading order gives:

$$S\big(g_0^{(k)}\big) = \frac{\big(\mathrm{i}\sqrt{\alpha_1}\tau_1\big)^N}{\big(\mathrm{i}\sqrt{\alpha_k}\tau_k\big)^2}\, g_0^{(k)} \ ,$$

  Thus, we can infer that $g_0^{(k)}$ is a ratio of a modular form of $\Gamma_1(\lambda_1)$ with weight N and a modular form of $\Gamma_1(\lambda_k)$ with weight 2.
- At higher n, we find coupled modular anomaly equations that are integrable. The solutions are polynomials in Eisenstein series, with coefficients being meromorphic forms.
- Since an example is worth a thousand words of explanation...

A sample case

- The period matrix Ω of the massive SU(4) theory:

$$\Omega = \widetilde{\tau}_1\,\mathcal{M}_1 + \widetilde{\tau}_2\,\mathcal{M}_2$$

- The $\widetilde{\tau}_k$ take the form

$$\widetilde{\tau}_k = \tau_k - \frac{1}{2\pi\mathrm{i}}\sum_{n=0}^{\infty}\frac{4n+3}{a^{4n+4}}\,\widehat{g}_n^{(k)}\,(q_0;T_4,T_8)\ .$$

- From localization, the first $\widehat{g}_n^{(2)}$ coefficients are

$$\widehat{g}_0^{(2)} = \frac{T_4}{12}\Big(1-\frac{1}{4}\,q_0-\frac{25}{256}\,q_0^2-\frac{29}{512}\,q_0^3+\cdots\Big)\ ,$$
$$\widehat{g}_1^{(2)} = -\frac{T_4^2}{224}\Big(q_0+\frac{7}{64}\,q_0^2+\frac{7}{512}\,q_0^3+\cdots\Big) + \frac{T_8}{56}\Big(1-q_0-\frac{5}{128}\,q_0^2-\frac{39}{512}\,q_0^3+\cdots\Big)\ .$$

Resummation with modular forms

- First, we tabulate all modular forms of $\Gamma_1(2)$ and $\Gamma_1(4)$ and their S-parity.
- The leading term is completely fixed by S-parity and modular behaviour:

$$g_0^{(2)} = \frac{T_4}{12} \frac{\left(f_{2,+}^{(2)}\right)^2}{f_{2,+}^{(4)}} \ .$$

- For g_1, we use the coupled modular anomaly equations

$$\frac{\partial g_1^{(2)}}{\partial \widetilde{E}_2^{(4)}} = \frac{3}{28} \left(g_0^{(2)}\right)^2 \qquad \text{and} \qquad \frac{\partial g_1^{(2)}}{\partial \widetilde{E}_2^{(2)}} = \frac{1}{7}\, g_0^{(2)}\, g_0 \ .$$

Resummation with quasi-modular forms

- The modular anomaly equations are solved by

$$g_1^{(2)} = \frac{3}{28}\left(g_0^{(2)}\right)^2 \widetilde{E}_2^{(4)} + \frac{1}{7}\, g_0^{(2)}\, g_0\, \widetilde{E}_2^{(2)} + \text{modular piece} \; .$$

- Use the q_0-expansion to fix the modular piece. For this case, *only* the classical and perturbative contribution needed.
- Final result:

$$g_1^{(2)} = \frac{T_4^2}{1344}\left(\frac{\left(f_{2,+}^{(2)}\right)^4 \widetilde{E}_2^{(4)}}{\left(f_{2,+}^{(4)}\right)^2} + \frac{4}{3}\,\frac{\left(f_{2,+}^{(2)}\right)^3 \widetilde{E}_2^{(2)}}{f_{2,+}^{(4)}} - \frac{9}{2}\,\frac{\left(f_{2,+}^{(2)}\right)^2 f_{4,-}^{(2)}}{f_{2,+}^{(4)}}\right) + \frac{T_8}{56}\,\frac{\left(f_{2,+}^{(2)}\right)^2 f_{4,-}^{(2)}}{f_{2,+}^{(4)}} \; .$$

More questions

- Can the analysis for the mass-deformations be repeated for the non-arithmetic cases? Possibly using automorphic forms and Eisenstein series that are also defined for Hecke groups. C. Doran et al.
- The Argyres-Seiberg analysis studies the SW curve around $h = 1$. Can we use the hautpmoduln to address questions about the theory at strong coupling?
- Can we move away from the special locus?

Further directions

- The basic mantra seems to be: a) identify the modular properties of observable of interest, b) use S-duality constraints to find the weight of a modular object and c) use localization to fix the first few terms of the power series expansion.
- Finiteness of the basis for a given weight uniquely determines the fully resummed observable.
- Are there other observables that show (quasi-)modular behaviour? [Yes: chiral correlators in $\mathcal{N} = 2^\star$ theory. 1607.08327

$$A_k = \sum_{p=0}^{[k/2]} \binom{N-k+2p}{2p} (2p-1)!! \left(\frac{m^2 E_2(\tau)}{12} \right)^p W_{k-2p} \, .$$

- Add surface operator. Can $W(z)$ also be fixed this way? [Yes! again, in $\mathcal{N} = 2^\star$ theory 1702.xxxxx]. Exhibit IR duality between codimension-2 and co-dimension-4 surface operators.
- It would be interesting to repeat this for the SQCD models.

Curiosities

α_k	$d_{\alpha_k}^{-1}$	q-expansion of j_{α_k}	$j^*_{\alpha_k} = \frac{d_{\alpha_k}}{g_0}$
1	1728	$q^{-1} + 744 + 196884q + \cdots$	$q^{-1} - 120 + 10260q + \cdots$
2	256	$q^{-1} + 104 + 4372q + \cdots$	$q^{-1} - 24 + 276q + \cdots$
3	108	$q^{-1} + 42 + 783q + \cdots$	$q^{-1} - 12 + 54q + \cdots$
4	64	$q^{-1} + 24 + 276q + \cdots$	$q^{-1} - 8 + 20q + \cdots$

The pair $(j_{\alpha_k}, j^*_{\alpha_k})$ satisfy a remarkable set of identities, called the Ramanujan-Sato identities that take the form Chan-Cooper

$$\sum_{k=0}^{\infty} s_{\alpha_k,A}(k) \frac{1}{(j_{\alpha_k}(\tau))^{k+1/2}} = \pm \sum_{k=0}^{\infty} s_{\alpha_k,B}(k) \frac{1}{(j^*_{\alpha_k}(\tau))^{k+1/2}},$$

where the $s(k)$ are integers.

Curiosities

For instance

$$s_{1,A} = \binom{2k}{k}\binom{3k}{k}\binom{6k}{k}$$

$$s_{1,B} = \sum_{j=0}^{k}\binom{2j}{j}\binom{3j}{j}\binom{6j}{3j}\binom{k+j}{k-j}(432)^{k-j}$$

There are analogous expressions for higher levels. Interpretation within gauge theory? RHS is an instanton expansion.