Siegel modular forms

Historical origin

modular forms in 1 variable were applied with great success to the study of representation numbers of quadratic forms:

$S = (s_{ij})$ a positive definite integral quadratic form, described by a symmetric matrix S of size m

For $t \in \mathbb{N}$ we put

$$A(S,t) = \# \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \vec{x} \in \mathbb{Z}^m \;\middle|\; \vec{x}^{\,t} S x = \sum_{i,j} s_{ij} x_i x_j = t \right\}$$

Then the Fourier series

$$\sum_{t=0}^{\infty} A(S,t)\, e^{2\pi i t \tau}$$

defines a modular form of weight $\frac{m}{2}$ and one can study $A(S,t)$ by studying this modular form (called theta series); for m odd one needs the notion of modular form of half-integral weight.

A natural generalization is representations of forms by forms, i.e. we take another (positive semidefinite) quadratic form T of size n and study representation numbers

$$A(S,T) :=$$

$$\# \{ X \in \mathbb{Z}^{(m,n)} \mid X^t S X = T \}$$

Siegel modular forms were created to have an <u>analytic tool</u> generalizing theta series in order to study representation numbers $A(S,T)$

<u>Literature</u>:

A. Andrianov
E. Freitag
H. Klingen

$\Big\}$ books on Siegel modular forms

article by v. d. Geer in the book "1-2-3 of modular forms"

article by R. Godement "Generalités sur les formes modulaires" Seminaire Cartan 10 (1958 !)

I Generalities

Dfn R commutative ring, $n \geq 1$

$$Sp(n,R) := \{ M \in GL(2n,R) \mid \overbrace{J_n[M]}^{M^t J_n M} = J_n \}$$

with $J_n := \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$

(= isometry group for a free R-module with alternating form J_n)

We decompose $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ into $n \times n$ blocks, then

$$Sp(n,R) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n,R) \;\middle|\; \begin{array}{l} A^tC,\ B^tD \text{ symm.} \\ A^tD - C^tB = 1_n \end{array} \right\}$$

in particular,

$$M^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$$

and $$Sp(1,R) = Sl_2(R)$$

Dfn 'Siegel upper half space'

$$\mathbb{H}_n := \{ Z = Z^t = X + iY \mid Y > 0 \}$$

this can be viewed as open subspace of $\mathbb{C}^{\frac{n(n+1)}{2}}$,

simply connected (convex)

<u>Proposition</u> We can define a <u>transitive</u> <u>action</u> of $Sp(n,\mathbb{R})$ on $\mathbb{H}_n$ by

$$\begin{cases} Sp(n,\mathbb{R}) \times \mathbb{H}_n \longrightarrow \mathbb{H}_n \\ (M, Z) \longmapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \end{cases}$$

one has to check that

$CZ+D$ is invertible ?

$(AZ+B)(CZ+D)^{-1}$ symmetric ?

$\mathrm{Im}\left((AZ+B)(CZ+D)^{-1}\right)$ positive definite ?

follows from formula

$\mathrm{Im}\,(AZ+B)(CZ+D)^{-1} = (CZ+D)^{-t}\, Y\, (C\bar{Z}+D)^{-1}$

Transitivity follows from

$$\begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix}\begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix}\langle i1_n\rangle = \begin{pmatrix} A^t & A^tS \\ 0 & A^{-1} \end{pmatrix}\langle i1_n\rangle = A^tA\,i1_n + A^tSA$$

$A \in GL(n,\mathbb{R})$

$S = S^t \in \mathbb{R}^{(n,n)}$

usefull remarks:

(*) all biholomorphic maps $\mathbb{H}_n \to \mathbb{H}_n$ are of the form

$Z \mapsto M\langle Z\rangle, \quad M \in Sp(n,\mathbb{R})$

(**) $\mathrm{Stabilizer}(i\cdot 1_n) = Sp(n,\mathbb{R}) \cap O(2n,\mathbb{R})$

$$= \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in Sp(n,\mathbb{R}) \right\} \underset{\uparrow}{\simeq} U(n,\mathbb{C})$$

by $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto A + iB$

(***) $d\omega_n := \dfrac{dX\,dY}{\det(Y)^{n+1}}$ defines <u>invariant</u> measure on $\mathbb{H}_n$

Reduction theory

we need a higher-dimensional analogue of

 $(Sl_2(\mathbb{Z}))$.

Dfn a group G acts on topol. space X by $\cdot$

a fundamental domain for G is a closed subset $\mathcal{F} \subset X$

- $\mathcal{F}$ contains representatives of all G-orbits, i.e.
  $\forall x \in X : \exists\, g \in G : \; g \cdot x \in \mathcal{F}$
- overlapping happens only on boundary, i.e.
  $\forall x, y \in \mathcal{F} : \forall g \in G : \; g \cdot x = y \Rightarrow g = id$ or $x, y \in \partial \mathcal{F}$.

[• $\partial \mathcal{F}$ is of measure zero
this makes sense only if measure is available on X]

existence of such fundamental domain is not obvious
(and $\mathcal{F}$ is not unique : $X = \mathbb{R}$, $G = \mathbb{Z}$, then
$\mathcal{F} := [0,1]$ or $\mathcal{F} = [0, \frac{1}{2}] \cup [\frac{3}{2}, 2]$ is ok)

We take for granted the reduction theory of Minkowski
this is concerned with

$$X = \mathcal{P}_n := \{ Y = Y^t > 0 \mid Y \in \mathbb{R}^{(n,n)} \}$$ with action of $GL(n, \mathbb{Z})$

$$(g, Y) \mapsto g^t Y g$$

there exists a "good" fundamental domain for $\mathcal{P}_n$ under $GL(n, \mathbb{Z})$; we call Y reduced, if it is in such domain

Dfn

$$\mathcal{F}_n := \left\{ \overset{X+iY}{\overset{\shortparallel}{Z}} \in \mathbb{H}_n \;\middle|\; \begin{array}{l} Y \text{ reduced} \\ |x_{ij}| \le \frac{1}{2} \\ |\det(CZ+D)| \ge 1 \quad \text{for all } \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in Sp(n,\mathbb{Z}) \end{array} \right\}$$

Thm $\mathcal{F}_n$ is fundamental domain for action of $Sp(n,\mathbb{Z})$ on $\mathbb{H}_n$; moreover $\exists\, \alpha > 0$ s.th.

$$\forall\, Z = X+iY \in \mathcal{F}_n : \quad Y \ge \alpha \cdot 1_n$$

and $\mathcal{F}_n$ has finite volume, i.e.

$$\int_{\mathcal{F}_n} \frac{dX\,dY}{\det(Y)^{n+1}} < \infty .$$

remark the 3 conditions defining $\mathcal{F}_n$ are quite natural:

the first one comes from $\begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix} \in Sp(n,\mathbb{Z})$ for $U \in GL(n,\mathbb{Z})$

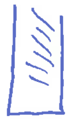
the second one comes from $\begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix} \in Sp(n,\mathbb{Z})$ for

$$S \in \mathbb{Z}^{(n,n)}, \quad S = S^t$$

the third one means that we choose Z in its orbit such that $\det(Y)$ is largest: use

$$\det \operatorname{Im}(M\langle Z\rangle) = \det(Y) \cdot |\det(CZ+D)|^{-2}$$

in degree one $(n=1)$ the existence of $\alpha > 0$ means

 can be included in 

How to define modular forms properly?

In the sequel we always use congruence groups Γ, i.e. $\exists\ N = N_\Gamma \in \mathbb{N}$ such that

$$\Gamma(N) \subseteq \Gamma \subseteq Sp(n, \mathbb{Q})$$

and $[\Gamma : \Gamma(N)] < \infty$

Remark: For $\gamma \in Sp(n, \mathbb{Q})$ the conjugate group $\gamma^{-1}\Gamma\gamma$ is again of this type

Automorphy factor: For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, $Z \in \mathbb{H}_n$

$$J(M, Z) := CZ + D \qquad (\in GL(n, \mathbb{C}))$$

$$j(M, Z) := \det(CZ + D)$$

we have a cocycle property

$$\boxed{J(M_1 \circ M_2, Z) = J(M_1, M_2\langle Z\rangle)\, J(M_2, Z)}$$

$M_1, M_2 \in Sp(n, \mathbb{R})$

and an (right) action of $Sp(n, \mathbb{R})$ on functions on $\mathbb{H}_n$ by

$$(F|_k M)(Z) := j(M, Z)^{-k}\, F(M\langle Z\rangle) = \det(CZ+D)^{-k}\, F(M\langle Z\rangle)$$

$k \in \mathbb{Z}$, $M \in Sp(n, \mathbb{R})$, $Z \in \mathbb{H}_n$.

Dfn (first version) Γ as above, $k \in \mathbb{Z}$

$$M_k^n(\Gamma) = \left\{ F : \mathbb{H}_n \to \mathbb{C} \;\middle|\; \begin{array}{l} F \text{ holomorphic} \\ \forall g \in \Gamma : F|_k g = F \end{array} \right\}$$

$n = 1$: condition in cusps

this is not general enough, as one can see from switching to functions on $Sp(n, \mathbb{R})$

we ignore Γ and holomorphy for a moment and put

$$\mathcal{K} = \text{Stabilizer}(i 1_n) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in Sp(n, \mathbb{R}) \right\} \simeq U(n, \mathbb{C}),$$

explicitly

$$U(n, \mathbb{C}) \ni \underset{\overset{\parallel}{A+iB}}{k} \longleftrightarrow \tilde{k} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

we observe that

$$j(\tilde{k}, z) = \det(Bz + A) \text{ equals } \det(Bi + A) \text{ for } z = i\cdot 1_n$$

$$\Rightarrow \det(Bi + A) = j(\tilde{k}, i 1_n) = \underset{\substack{\uparrow \\ \text{a character (linear) of } \mathcal{K} \simeq U(n, \mathbb{C})}}{\chi(k)}$$

We consider two function spaces

$$\{ F : \mathbb{H}_n \to \mathbb{C} \mid F \text{ any fctn} \} \underset{S}{\overset{T}{\rightleftarrows}} \left\{ \varphi : Sp(n, \mathbb{R}) \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi(gk) = \chi(k)^{\ell} \varphi(g) \\ k \in \mathcal{K} \end{array} \right\}$$

they are isomorphic by

$$F \longmapsto T(F), \quad T(F)(g) := j(g, i)^{-\ell} F(g\langle i 1_n \rangle)$$

$$\varphi \longmapsto S(\varphi), \quad S(\varphi)(z) := j(g, i)^{\ell} \varphi(g)$$

for any $g \in Sp(n, \mathbb{R})$, $g\langle i 1_n \rangle = z$

the maps T and S are <u>equivariant</u>, i.e.

$$\boxed{T(F|_{\ell} h)(g) = T(F)(h \cdot g)} \qquad g, h \in Sp(n, \mathbb{R})$$

$\uparrow$ action defined by $|_\ell$

$\uparrow$ action defined by changing argument from g to $h \cdot g$

The problem is that (unlike in the case $n=1$) the group $U(n,\mathbb{C})$ is nonabelian and therefore one should consider not only homomorphisms $\chi: g \mapsto \det(g)^\ell$ but more generally representations $\rho: U(n,\mathbb{C}) \to \mathrm{Aut}(V_\rho)$; they can be identified with rational representations (also denoted by ρ)

$$\rho: GL(n,\mathbb{C}) \to \mathrm{Aut}(V_\rho)$$

One should then rewrite the isomorphisms S and T from above for V_ρ-valued functions. Then the more appropriate definition is that of vector-valued modular forms

Dfn (final version): $\rho: GL(n,\mathbb{C}) \to \mathrm{Aut}(V_\rho)$ a rational (finite-dim) representation; modular forms with automorphy factor ρ are then defined by

$$\boxed{M^n_\rho(\Gamma) := \left\{ F: \mathbb{H}_n \to V_\rho \;\middle|\; \begin{array}{l} F \text{ holomorphic} \\ \forall \gamma \in \Gamma: F|_\rho \gamma = F \\ n=1: \text{condition in cusps.} \end{array} \right\}}$$

here $(F|_\rho M)(Z) := \rho(CZ+D)^{-1} F(M\langle Z\rangle)$.

if $\rho = \det^\ell$, we write again $M^n_\ell(\Gamma)$ instead of $M^n_{\det^\ell}(\Gamma)$

attention the notion "vector-valued modular form" has several different meanings in the literature !!!

To encode Fourier expansions conveniently we introduce

$\Lambda^n := \{ T = T^t \in \frac{1}{2} \mathbb{Z}^{(n,n)} \mid t_{ii} \in \mathbb{Z} \}$; then (due to symmetry of T and Z)

$\{ \operatorname{trace}(T \cdot Z) \mid T \in \Lambda^n \} = \{ \sum_{1 \le i \le j \le n} s_{ij} z_{ij} \mid s_{ij} \in \mathbb{Z} \}$

then $F \in M^n_\rho(\Gamma)$ satisfies $F(Z+S) = F(Z)$ for all $S = S^t \in N \cdot \mathbb{Z}^{(n,n)}$

if $\Gamma(N) \subseteq \Gamma$ and hence we have a Fourier expansion

$$\boxed{F(Z) = \sum_{T \in \Lambda^n} a_F(T)\, e^{\frac{2\pi i}{N} \operatorname{trace}(T \cdot Z)}}$$ with $a_F(T) \in V_\rho$

this may be called "Fourier expansion at ∞";

if $g \in Sp(n, \mathbb{Q})$, then $F|_\rho g$ is a modular form for

$\underbrace{g^{-1} \Gamma g}$

this is again a congruence group; we get more generally

for all $g \in Sp(n, \mathbb{Q})$

$$(F|_\rho g)(Z) = \sum_{T \in \Lambda^n} a_F(T; g)\, e^{\frac{2\pi i}{N} \operatorname{trace}(TZ)}$$

with N depending on Γ and g

this may be considered either as Fourier expansion of F "at cusp g" or as Fourier expansion of $F|_\rho g$ at ∞.

Remark such Fourier expansion depends essentially only on the double coset

$\Gamma g\, Sp(n, \mathbb{Q})_\infty$ with $Sp(n, \mathbb{Q})_\infty = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{Q}) \mid C = 0 \}$

we observe that $\Gamma \backslash Sp(n, \mathbb{Q}) / Sp(n, \mathbb{Q})_\infty$ is finite

($\Rightarrow$ only finitely many Fourier expansions ....)

important property : Fourier coefficients satisfy important invariance property

$$(*) \quad \boxed{a_F(U \cdot T \cdot U^t) = \rho(U)\, a_F(T)} \quad \text{for all } U \in gl(n) \text{ s.th. } \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix} \in \Gamma.$$

Theorem "Koecher principle" (Koecher 1951) ($n > 1$ only!)

$F \in M^n_\rho(\Gamma) \;\Rightarrow\; \forall\, g \in Sp(n, \mathbb{Q})$

$$\left\| \quad (F|_\rho g)(Z) = \sum_{\substack{T \in \Lambda^n \\ T \geq 0}} a_F(T; g)\, e^{\frac{2\pi i}{N} \operatorname{trace}(TZ)} \quad \right\|$$

← this holds automatically

Remark: this means in particular that for $n > 1$ there is no analogue of the j-function!

proof (scalar-valued case, $\rho = \det^\ell$)

may assume $g = 1_{2n}$, $\Gamma = \Gamma(N)$

we assume $\exists\, T_0 \in \Lambda_n$: T_0 not positive definite and

$$a_F(T_0) \neq 0$$

by going from F to $F|_\ell \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix}$ for appropiate $U \in gl(n, \mathbb{Z})$ if necessary, we may assume right away that $t_{11} = (T_0)_{11}$ is negative we use matrices

$$V_m = \begin{pmatrix} 1 & m & & \\ 0 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in GL(n,\mathbb{Z})$$ and may choose infinitely many of these such that

$$\begin{pmatrix} V_m^t & 0_n \\ 0 & V_m^{-1} \end{pmatrix} \in \Gamma(N) \qquad (N/m)$$

The Fourier series for F converges in $Z = i\cdot 1_n \Rightarrow \exists\, C > 0$

$$\left| a_F(S)\, e^{-\frac{2\pi}{N}\operatorname{trace}(S)} \right| \leq C \quad \text{for all } S \in \Lambda^n,$$

in particular

$$|a_F(T_0)| \overset{\text{invariance }(*)}{=} |a_F(T_0[V_m])| \leq C\cdot e^{\frac{2\pi}{N}\operatorname{trace}(T_0[V_m])}$$

observe that

$$\operatorname{trace}(V_m^t T_0 V_m) = \operatorname{trace}(T_0) + \underbrace{\overset{<0}{t_{11}} m^2 + 2t_{12} m}_{\to -\infty \text{ for } m \to \infty}$$

$$\Rightarrow a_F(T_0) = 0$$

$\square$

<u>remark</u> For a Fourier series

$$F(Z) = \sum_{T \in \Lambda^n} a_F(T)\, e^{\frac{2\pi i}{N}\operatorname{trace}(TZ)}$$ the following properties are equivalent

(1) $a_F(T) = 0$ unless $T \geq 0$

(2) $F(Z)$ is bounded in any domain of type $Z = X + iY$ with $Y \geq c\cdot 1_n$, $c > 0$.

Proposition $M^n_\rho(\Gamma) = \{0\}$ unless ρ is polynomial representation

if $\rho = \det^l$, it means that $M^n_l(\Gamma) = \{0\}$ unless $l \geq 0$

remark the proposition, together with the Koecher principle shows that for $n > 1$ there is NO analogue of Δ (consider $\frac{1}{\Delta}$ ....)

proof only $\rho = \det^l$, may assume $\Gamma = Sp(n, \mathbb{Z})$: $F \in M^n_l(Sp(n,\mathbb{Z}))$ $l < 0$

$$h(Z) := \underbrace{\det(Y)^{l/2}}_{\wedge} \cdot \underbrace{|F(Z)|}$$

(use $l<0$)

bounded in fundamental domain $\mathcal{F}_n$ for $Sp(n,\mathbb{Z})$

as a $Sp(n,\mathbb{Z})$-invariant fctn, $h(Z)$ is then bounded on $\mathbb{H}_n$

use formula for Fourier coeff.

$$a_F(T)\, e^{-2\pi\, \mathrm{trace}(TY)} = \int_{X = X^t \text{ modulo } 1} F(Z)\, e^{-2\pi i\, \mathrm{trace}(TX)}\, dX$$

$$\left| \;\cdot/.\; \right| \leq \sup_{X \bmod 1} |F(Z)| \leq K^{(*)} \cdot \det(Y)^{-l/2}$$

$\downarrow Y \to 0$ $\qquad\qquad\qquad\qquad$ $\downarrow Y \to 0$

$$|a_F(T)| \leq 0$$

$\Rightarrow a_F(T) = 0$ for all T $\square$

(*) K is the bound for function h from above

<u>Dfn</u> $F \in M^n_\rho(\Gamma)$ is called <u>cusp form</u> $:\Leftrightarrow$

$\forall g \in Sp(n, \mathbb{Q})$:

$$(F|_\rho g)(Z) = \sum_{\substack{T \in \Lambda^n \\ T > 0}} a_F(T; g)\, e^{\frac{2\pi i}{N} \operatorname{trace}(TZ)}$$

this defines a subvector space $S^n_\rho(\Gamma) \leq M^n_\rho(\Gamma)$

note that this condition depends only on $\Gamma \backslash Sp(n, \mathbb{Q}) / Sp(n, \mathbb{Q})_\infty$

<u>some growth properties of cusp forms</u> (scalar-valued case only)

<u>Proposition</u>: For $F \in S^n_\ell(\Gamma)$: $\exists\, a_1, a_2 > 0$:

$$|\det(Y)^{\ell/2} F(Z)| \leq a_1 \cdot e^{-a_2 \det(Y)^{1/n}} \qquad (\text{all } Z \in \mathbb{H}_n)$$

in particular, for $F, G \in S^n_\ell(\Gamma)$
the integral

$$\int_{\mathcal{F}} F(Z)\overline{G(Z)} \det(Y)^\ell \frac{dX\,dY}{\det Y^{n+1}} \quad \text{converges}$$

(Petersson product)

here $\mathcal{F}$ is a fundamental domain for Γ.

It defines a scalar product on $S^n_\ell(\Gamma)$

On $\Gamma \backslash Sp(n, \mathbb{R})$ it corresponds naturally to an integration w.r.t. Haar measure.

Remark: convergence of Petersson-integral is almost equivalent to cuspidality, more precisely, for $F \in M_\ell^n(\Gamma)$

$$\int_{\mathcal{F}} |F(z)|^2 \det Y^\ell \frac{dX\,dY}{\det Y^{n+1}} < \infty \quad \begin{cases} \Leftrightarrow F \text{ cuspidal} & \ell > n \\ \text{always convergent} & \ell \le \frac{n}{2} \end{cases}$$

for other weights it depends on ranks of non-vanishing Fourier coefficients

Proposition ("Hecke bound")

$$F = \sum a_F(T)\, e^{\frac{2\pi i}{N} \operatorname{trace}(TZ)} \in S_\ell^n(\Gamma)$$

$$\Rightarrow \exists\, c > 0 : \quad |a_F(T)| \le c \cdot \det(T)^{\ell/2}$$

what about the $a_F(T)$ with $\operatorname{rank}(T) < n$ in general ?

($n=1$ just constants); specific question for $n > 1$

Dfn $0 \le \nu \le n$, $F = \sum a_F(T)\, e^{\frac{2\pi i}{N} \operatorname{trace}(TZ)}$

$$\phi^{(\nu)}(F)(Z^*) := \sum_{T_1 \in \Lambda^{n-\nu}} a_F\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} e^{\frac{2\pi i}{N} \operatorname{trace}(T_1 Z^*)}$$

$Z^* \in \mathbb{H}_{n-\nu}$

Notation

$$C_{n,n-\nu}(\mathbb{R}) := \left\{ M \in Sp(n,\mathbb{R}) \;\middle|\; M = \begin{pmatrix} A_1 & 0 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & D_4 \end{pmatrix} \in Sp(n,\mathbb{R}) \right\}$$

A_1, B_1, C_1, D_1 of size $n-\nu$

This is a subgroup ("maximal parabolic") of $Sp(n, \mathbb{R})$ with

$$M^* := \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in Sp(n-\nu, \mathbb{R}).$$

<u>Proposition</u> For $F \in M_\ell^n(\Gamma)$

$\Phi^\nu(F)$ is a modular form (weight ℓ, degree $n-\nu$)

for $(\Gamma \cap C_{n,n-\nu})^* \leq Sp(n-\nu, \mathbb{Q})$

proof: We obtain $\Phi^{(\nu)}(F)$ by a limit process

$$Z = \begin{pmatrix} Z^* & 0 \\ 0 & i\lambda \cdot 1_\nu \end{pmatrix} \qquad \lambda \to \infty, \text{ observing that}$$

for $T \in \Lambda_n$, $T \geq 0$

$$\lim_{\lambda \to \infty} e^{\frac{2\pi i}{N} \operatorname{trace}(TZ)} = 0 \text{ unless } T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$$

Furthermore, one has to use that

$$\Phi^{(\nu)}(F|_\ell g) = \Phi^{(\nu)}(F)|_\ell g^* \text{ holds for}$$

$$g \in C_{n,n-\nu}$$

the main reason for this is that for $g = \begin{pmatrix} \begin{matrix} A_1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} B_1 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} C_1 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} D_1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$ in $C_{n,n-\nu}$, we have

$$g\left\langle \begin{pmatrix} Z^* & 0 \\ 0 & i\lambda 1_\nu \end{pmatrix} \right\rangle = \begin{pmatrix} (A_1 Z^* + B_1)(C_1 Z^* + D)^{-1} & 0 \\ 0 & \bullet \end{pmatrix}$$

something

Remark: The procedure above (called Siegel's Φ-operator) can also be applied to $F|_e \gamma$ $(\gamma \in Sp(n,\mathbb{Q}))$ to get several Φ-operators; in particular

$$F \in M_e^n(\Gamma) \text{ is cusp form} \iff \Phi^{(1)}(F|_e \gamma) = 0$$

for all $\gamma \in Sp(n,\mathbb{Q})$

$$\iff \Phi^{(1)}(F|_e \gamma) = 0 \text{ for all}$$

$$\gamma \in \underbrace{\Gamma \backslash Sp(n,\mathbb{Q}) / C_{n,n-1}(\mathbb{Q})}_{\text{finite (!)}}$$

Construction principle for modular forms

We discuss a simple construction of modular forms, which covers several important types of modular forms; we completely ignore the technical problem of convergence. For $n=1$ this principle ("averaging") was systematically investigated by Petersson.

Principle of averaging: Γ fixed throughout

$\varphi : \mathbb{H}_n \to \mathbb{C}$ holom. fctn

$\Gamma' := \{\gamma \in \Gamma \mid \varphi|_e \gamma = \varphi\}$ subgroup of Γ

$$P_\ell^n(\varphi, \Gamma') := \sum_{\gamma \in \Gamma' \backslash \Gamma} \varphi|_\ell \gamma$$

this defines an element of $M_\ell^n(\Gamma)$, provided that the series converges absolutely and locally uniformly

We describe several important example

<u>Example 0</u> "Hecke operators"

$$GSp(n, \mathbb{R})^+ := \left\{ M \in GL_{2n}(\mathbb{R}) \;\middle|\; M^t \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} M = \lambda \cdot \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}, \; \lambda > 0 \right\}$$

$\left(\Rightarrow \frac{1}{\sqrt{\lambda}} M \in Sp(n, \mathbb{R}) \right.$ and we define

$$\left. F|_\ell M := F|_\ell \left(\tfrac{1}{\sqrt{\lambda}} M\right) \right)$$

For $g \in GSp(n, \mathbb{Q})^+$ we put

$$\varphi(Z) := \varphi_g(Z) = F|_\ell g \qquad (F \in M_\ell^n(\Gamma))$$

then $\Gamma' = (g^{-1} \Gamma g) \cap \Gamma$ (again congruence group)

and

$$P_\ell^n(\varphi_g)(Z) = \sum_{\gamma \in g^{-1}\Gamma g \cap \Gamma \backslash \Gamma} (F_\ell|g)|_\ell \gamma = \sum_{\delta \in \Gamma \backslash \Gamma g \Gamma} F_\ell|_\ell \delta$$

this is a <u>finite</u> sum ($\delta := g \cdot \gamma$ runs over the set indicated)

the <u>Hecke operator</u> for the double coset $\Gamma g \Gamma$

Example 1 for $g \in Sp(n, \mathbb{Q})$ define

$$\varphi_g(Z) := 1|_\ell\, g \qquad \left(= \det(CZ+D)^{-\ell} \text{ for } g = \begin{pmatrix} * & * \\ C & D \end{pmatrix} \right)$$

$$\begin{aligned} \Gamma' &= \{\gamma \in \Gamma \mid 1|_\ell\, g\gamma = 1|_\ell\, g\} \\ &= \{\gamma \in \Gamma \mid 1|_\ell (g\gamma g^{-1}) = 1\} \\ &= \{\gamma \in \Gamma \mid g\gamma g^{-1} = \begin{pmatrix} A & B \\ 0_n & D \end{pmatrix},\ \det(D)^\ell = 1\} \end{aligned}$$

if ℓ is even

$$\Gamma' = \cancel{\{\gamma \in \Gamma\}}\ \Gamma \cap g^{-1} C^{\circ}_{n,0}\, g$$

with $C^{\circ}_{n,0} := \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in C_{n,0} \mid \det D = 1\}$

$$\rightarrow P^n_\ell(\varphi_g, \Gamma') = \sum_{\gamma \in \Gamma \cap g^{-1} C^{\circ}_{n,0}\, g \backslash \Gamma} \varphi_g|_\ell\, \gamma$$

$$= \sum_{\tilde{\gamma} \in \Gamma \cap g^{-1} C_{n,0}\, g \backslash g\Gamma} 1|_\ell\, \tilde{\gamma}$$

$$= \sum_{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \cap g^{-1} C_{n,0}\, g \backslash g\Gamma} \det(CZ+D)^{-\ell}$$

"Siegel Eisenstein series attached to cusp g"

Example 2

Fix $T \in \frac{1}{N}\Lambda^n$, $T > 0$

$$\varphi(Z) = \varphi_T(Z) := e^{2\pi i \, \mathrm{trace}(TZ)}$$

Then $\Gamma' = \{ \gamma \in \Gamma \mid \varphi_T |_\ell \gamma = \varphi_T \}$

$$= \begin{pmatrix} 1_n & * \\ 0_n & 1_n \end{pmatrix} \cap \Gamma$$

and we may consider

$$P^n_\ell(\varphi_T, \Gamma') = \sum_{\gamma \in \Gamma' \backslash \Gamma} \varphi_T |_\ell \gamma = \sum_{\gamma \in \Gamma' \backslash \Gamma} \det(CZ+D)^{-\ell} e^{2\pi i \, \mathrm{trace}(T\gamma\langle Z\rangle)}$$

"Poincaré series attached to T"

Example 3

Here we start from a group $\Gamma^{(r)} \leq Sp(r, \mathbb{Q})$

$1 \leq r < n$

and we fix $f \in S^r_\ell(\Gamma^{(r)})$

$$\varphi(Z) = \varphi_f(Z) := f(Z^*) \qquad Z \in \mathbb{H}_n, \quad Z = \begin{pmatrix} Z^* & Z_2 \\ Z_2^t & Z_4 \end{pmatrix}$$

$$\Gamma' = \{ \gamma \in \Gamma \mid \gamma \in C_{n,r}, \gamma^* \in \Gamma^{(r)} \}$$

$$P^n_\ell(\varphi_f, \Gamma') = \sum_{\gamma \in \Gamma' \backslash \Gamma} \varphi_f |_\ell \gamma$$

$$= \sum_{(C\,D) = \gamma \in \Gamma' \backslash \Gamma} f(\gamma\langle Z\rangle^*) \det(CZ+D)^{-\ell}$$

"Klingen-Eisenstein series of degree n attached to f"

Example 4 $\qquad Z_0 \in \mathbb{H}_n$ fixed

$$\varphi(Z) = \varphi_{Z_0}(Z) = \det(Z + Z_0)^{-\ell}$$

$$\Gamma' = \{1_{2n}\}$$

$$P^n_\ell(\varphi_{Z_0}, \Gamma')(Z) = \sum_{\gamma \in \Gamma} \det(\gamma\langle Z\rangle + Z_0)^{-\ell}\, j(\gamma, Z)^{-\ell}$$

$$= \sum_{\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma} \det(AZ + B + Z_0 C Z + Z_0 D)^{-\ell}$$

(take transpose)

$$= \sum_{\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma} \det(Z A^t + B^t + Z C^t Z_0 + D^t Z_0)^{-\ell}$$

to interchange roles of Z and Z_0, we ~~change~~

use the group $\tilde{\Gamma} = \left\{ \begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix} \;\middle|\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \right\}$

the sum above then equals

$$\sum_{\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D}\end{pmatrix} \in \tilde{\Gamma}} \det(\mathcal{A} Z_0 + \mathcal{B} + Z \tilde{\mathcal{C}} Z_0 + Z \mathcal{D})^{-\ell}$$

we get the important symmetry relation

$$P^n_{\ell,\Gamma}(\varphi_{Z_0}, \{1_{2n}\})(Z) = P^n_{\ell,\tilde{\Gamma}}(\varphi_Z, \{1_{2n}\})(Z_0)$$

(we included the group Γ in the index)

It shows that $P^n_{\ell,\Gamma}(\varphi_{Z_0}, \{1_{2n}\})$ is a modular form

in <u>both</u> variables Z and Z_0 (for Γ and $\tilde{\Gamma}$ respectively) short comment about convergence and significance of the examples

Example 1 and Example 3 : convergence $k > n+r+1$

important for inverting Siegel's Φ-operator

Example 2 convergence $\ell > 2n+1$

$$\{ P_\ell^n(\varphi_T, \Gamma') \mid T \in \Lambda^n, T > 0 \}$$

generates all of $S_\ell^n(\Gamma)$

Petersson product of $F = \sum_{\substack{S \in \Lambda^n \\ S > 0}} a_F(S)\, e^{\frac{2\pi i}{N} \operatorname{trace}(SZ)}$

against

Poincaré series $P_\ell^n(\varphi_T, \Gamma')$ gives essentially

$$a_F(T)$$

Example 4 convergence $\ell > 2n+1$

$P_\ell^n(\varphi_{Z_0}, \{1_{2n}\})(Z)$ gives a "kernel fction" for $S_\ell^n(\Gamma)$

$$P_\ell^n(\varphi_{Z_0}, \{1_{2n}\})(Z) = \text{const} \times \sum_{i=1}^{d} \frac{F_i(Z)\, \overline{F_i(-\bar{Z})}}{\| F_i \|^2},$$

where F_i goes over a orthogonal basis of $S_\ell^n(\Gamma)$

properties of this kernel ("Hilbert-Schmidt" type) allow to show that $\dim S_\ell^n(\Gamma) < \infty$.

it is important to note that in examples 2 and example 4 the convergence does not imply that the functions constructed are actually different from zero ∇

Finally we mention a completely different construction by theta series

We start from $S \in \Lambda^m$, $S > 0$, m even and define

$$\Theta_S^n(Z) = \sum_{X \in \mathbb{Z}^{(m,n)}} e^{2\pi i \operatorname{trace}(X^t S X)}$$

this defines a modular form of weight $\frac{m}{2}$ for some Γ

Its Fourier expansion appeared in the introduction:

$$\Theta_S^n(Z) = \sum_{\substack{T \in \Lambda^n \\ T \geq 0}} A(S,T)\, e^{2\pi i \operatorname{trace}(TZ)}$$