Moonshine

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What is Moonshine?

- Number Theory
- Representation Theory
- String Theory

The Jack Daniel's question

Theorem (Tits 1975)

If the Fischer–Griess Monster group exists, then it has order

$$|\mathbb{M}| = 2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 11^{2}\cdot 13^{3}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71 \simeq 8\times 10^{53}$$

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$X_0^+(p) := \Gamma_0(p)^+ \backslash \mathbb{H}$ *is genus* 0 *iff*

$$p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$$

The Monster characters

Fischer–Livingstone–Thorne construct the character table for the (still conjectural) Monster group (1978).

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$$\chi_1(e) = 1$$
$$\chi_2(e) = 196883$$
$$\chi_3(e) = 21296876$$
$$\chi_4(e) = 842609326$$
$$\vdots$$
$$\chi_{194}(e) = 258823477531055064045234375.$$

McKay's observation

$$196884 = 1 + 196883$$

Thompson's generalizations

Thompson further observed:

$$
\begin{aligned}
196884 &= 1 + 196883 \\
21493760 &= 1 + 196883 + 21296876 \\
864299970 &= 1 + 1 + 196883 + 196883 + 21296876 + 842609326
\end{aligned}
$$

Thompson's generalizations

Thompson further observed:

$$\underbrace{\begin{array}{r} 196884 \\ 21493760 \\ 864299970 \end{array}}_{\text{Coefficients of } j(\tau)} \begin{array}{c} = \\ = \\ = \end{array} \underbrace{\begin{array}{l} 1 + 196883 \\ 1 + 196883 + 21296876 \\ 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \end{array}}_{\text{Dimensions of irreducible representations of the Monster } \mathbb{M}}$$

Klein's j-function

Fact

Klein's j-function

$$j(\tau) - 744 = \sum_{n=-1}^{\infty} c(n) q^n$$
$$= q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

satisfies

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau) \quad \text{for every matrix } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Monster module

Conjecture (Thompson)

There is an infinite-dimensional graded $\mathbb{M}$-module

$$V^\natural = \bigoplus_{n=1}^{\infty} V_n^\natural$$

with

$$\dim(V_n^\natural) = c(n).$$

The McKay-Thompson Series

Definition (Thompson)

Assuming the conjecture, if $g \in \mathbb{M}$, then define the **McKay–Thompson series**

$$T_g(\tau) := q^{-1} + \sum_{n=1}^{\infty} c_g(n) q^n,$$

where

$$c_g(n) := \mathrm{tr}(g | V_n^{\natural})$$

Conway and Norton

Conjecture (Monstrous Moonshine)

For each conjugacy class $[g]$ of $\mathbb{M}$, there is an explicit discrete subgroup Γ_g,

$$\Gamma_0(|g|) \leq \quad \Gamma_g \quad \leq \Gamma_0(|g|)^+ < \mathrm{SL}_2(\mathbb{R})$$

so that the McKay–Thompson series

$$T_g(\tau) = q^{-1} + O(q)$$

is a Hauptmodl for Γ_g.

The proof

Theorem (Frenkel–Lepowsky–Meurman)

The "most natural" representation of the Monster is the infinite dimensional vertex operator algebra,

$$V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural \quad \textit{with} \quad \mathrm{Dim}(V_n^\natural) = c(n).$$

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Theorem (Borcherds)

The Monstrous Moonshine Conjecture is true with $V^\natural$ as defined by F–L–M.

Gauge/Gravity duality

Vertex operator algebras are now recognized as chiral halves of 2-dimensional *Conformal Field Theories*.

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Maldacena (1998)

2-dim CFTs $\Longleftrightarrow$ gravity theories in 3-dim.

Black Holes and Quantum Gravity

Conjecture (Witten (2007))

The 2-dim CFT of $V^\natural \otimes V^\natural$ *is dual to pure 3-dim quantum gravity theory for the most negative possible cosmological constant. This implies there are* 194 *"black hole states".*

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Question

How are these different black hole states distributed?

The Multiplicities

Consider the Moonshine expressions

$$\begin{aligned} 196884 &= 1 + 196883 \\ 21493760 &= 1 + 196883 + 21296876 \\ 864299970 &= 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \\ &\vdots \\ c(n) &= \sum_{i=1}^{194} \mathbf{m}_i(n)\chi_i(e) \end{aligned}$$

How many '1's, '196883's, etc. show up in these equations?

The Multiplicities

Consider the Moonshine expressions

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\begin{aligned}
c_g(1) &= \chi_1(g) + \chi_2(g) \\
c_g(2) &= \chi_1(g) + \chi_2(g) + \chi_3(g) \\
c_g(3) &= 2 \cdot \chi_1(g) + 2 \cdot \chi_2(g) + \chi_3(g) + \chi_4(g) \\
&\vdots \\
c_g(n) &= \sum_{i=1}^{194} \mathbf{m}_i(n) \chi_i(g)
\end{aligned}
$$

How many 'χ_1's, 'χ_2's, etc. show up in these equations?

Some Proportions

n	$\delta(\mathbf{m}_1(n))$	$\delta(\mathbf{m}_2(n))$	$\cdots$	$\delta(\mathbf{m}_{194}(n))$
1	1/2	1/2	$\cdots$	0
2	1/3	1/3	$\cdots$	0

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100	$4.427\ldots \times 10^{-18}$	$1.077\ldots \times 10^{-16}$	$\cdots$	$0.04428\ldots$
120	$1.377\ldots \times 10^{-21}$	$5.501\ldots \times 10^{-20}$	$\cdots$	$0.04428\ldots$
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160	$2.621\ldots \times 10^{-27}$	$3.443\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
180	$1.877\ldots \times 10^{-28}$	$3.371\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
200	$1.715\ldots \times 10^{-28}$	$3.369\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
220	$1.711\ldots \times 10^{-28}$	$3.368\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
240	$1.711\ldots \times 10^{-28}$	$3.368\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$

Distribution of Moonshine

Theorem (Duncan-G-Ono)

If $1 \leq i \leq 194$, *then as* $n \to +\infty$ *we have*

$$\mathbf{m}_i(n) \sim \frac{\dim(\chi_i)}{|\mathbb{M}|} \cdot \frac{e^{4\pi\sqrt{|n|}}}{\sqrt{2}|n|^{3/4}}$$

Corollary

Corollary (Duncan-G-Ono)

We have that

$$\delta\left(\mathbf{m}_i\right) := \lim_{n\to+\infty} \frac{\mathbf{m}_i(n)}{\sum_{i=1}^{194} \mathbf{m}_i(n)}$$

is well defined, and

$$\delta\left(\mathbf{m}_i\right) = \frac{\dim(\chi_i)}{\sum_{j=1}^{194} \dim(\chi_j)} = \frac{\dim(\chi_i)}{5844076785304502808013602136}.$$

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The trivial character is extremely rare!

Further Moonshine

- Generalized Moonshine for subquotients of $\mathbb{M}$ (Norton, Queen)

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- Generalized Moonshine for subquotients of $\mathbb{M}$
  (Norton, Queen)
- Umbral Moonshine: Niemeier Lattices $\leftrightarrow$ mock MFs
  (Eguchi–Ooguri–Tachikawa, Gannon, Cheng–Duncan–Harvey, Duncan–G–Ono)
- Thompson Moonshine
  (Harvey–Rayhaun, G–Mertens)

Umbral Moonshine

Observation (Eguchi, Ooguri, Tachikawa (2010))

Using the K3 surface elliptic genus, there is a **mock modular form**

$$H(\tau) = q^{-\frac{1}{8}} \left(45q + 231q^2 + 770q^3 + 2277q^4 + 5796q^5 + \dots\right).$$

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The degrees of the irreducible repn's of the Mathieu group M_{24} are:

$$1, 23, 45, 231, 252, 253, 483, 770, 990, 1035,$$
$$1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395.$$

Mathieu Moonshine

Theorem (Gannon (2013))

There is an infinite dimensional graded M_{24}-module whose McKay-Thompson series are specific mock modular forms.

Larger Framework of Moonshine?

Remark

There are well known connections with even unimodular positive definite rank 24 lattices:

$$\mathbb{M} \longleftrightarrow \textbf{Leech lattice}$$

$$M_{24} \longleftrightarrow A_2^{12} \text{ lattice.}$$

Umbral Moonshine

Conjecture (Cheng, Duncan, Harvey (2013))

For each of the 24 Niemeier lattices, L^X there is an associated **umbral group** *which exhibits Moonshine.*

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For each of the 24 Niemeier lattices, L^X there is an associated **umbral group** *which exhibits Moonshine.*

The associated McKay-Thompson series are all weight $1/2$ mock modular forms.

Theorem (Duncan-G-Ono)

The Umbral Moonshine Conjecture is true.

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- For the Leech lattice we recover Monstrous Moonshine! $(\pm)$
- For $X = A_2^{12}$ we have $G^X = M_{24}$ and Gannon's Theorem.
- There are 22 other isomorphism classes.

Mock theta functions

Example

The M_{12} case includes three of Ramanujan's mock thetas:

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

$$\phi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$

$$\chi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}$$

Thompson Moonshine

Conjecture (Harvey–Rayhaun)

There exists a graded Th-supermodule

$$W = \bigoplus_{\substack{m=-3 \\ m\equiv 0,1 \pmod 4}}^{\infty} W_m,$$

such that for each $g \in Th$ the McKay–Thompson series

$$\mathcal{T}_{[g]}(\tau) := \sum_{\substack{m=-3 \\ m\equiv 0,1 \pmod 4}}^{\infty} (-1)^m \mathrm{tr}(g|W_m) q^m$$

is an explicit weakly holomorphic modular form of weight $\frac{1}{2}$.

Thompson Moonshine

Theorem (G–Mertens)

The Th-supermodule W exists.

Main Tools

Moonshine implies we can write the McKay–Thompson coefficients

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where the $\mathbf{m}_\chi(n)$ are **non-negative integers.**

Orthogonality of characters

Fact

If G is a group and $g, h \in G$, then

$$\sum_{\chi_i} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} |C_G(g)| & \text{If } g \text{ and } h \text{ are conjugate} \\ 0 & \text{otherwise,} \end{cases}$$

where $C_G(g)$ is the centralizer of g in G.

$T_\chi(\tau)$

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$$T_\chi(\tau) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)}\, T_g(\tau).$$
- The orthogonality of characters gives
$$T_\chi(\tau) = q^{-1} + \sum_{n=1}^{\infty} \mathbf{m}_\chi(n) q^n.$$
- We have the inverse relation
$$T_g(\tau) = \sum_{\chi} \chi(g)\, T_\chi(\tau).$$

Rademacher Sums

Coefficients of the T_g can be given as "Rademacher sums":

$$c_g(n) = C_{g,n} \sum_{\substack{e \,|\, |g| \\ W_e \in \Gamma_g}} \sum_{c=1}^{\infty} \frac{K_g(-1, n, |g|c)}{|g|c} I_1\left(\frac{\pi\sqrt{ne}}{|g|c}\right).$$

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Growth comes from the Bessel function, and the initial term with

$$|g| = c = e = 1$$

dominates.

The Selberg-Kloosterman zeta function

Similar formulas hold for Umbral and Thompson moonshine, but the Rademacher sums do not converge absolutely!

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Proposition

The modified Selberg-Kloosterman zeta function

$$Z_g^*(m,n;s) := \sum_{c=1}^{\infty} \frac{K_g(m,n,4|g|c)}{(4|g|c)^{2s}}$$

converges conditionally at $s = 3/4$ and satisfies explicit bounds.

Hooley’s method

- “Unevaluate” quadratic gauss sums.

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- Becomes sum over quadratic forms

$$[|g|\alpha, \beta, \gamma/h]$$

of fixed discriminant with $\gamma \equiv \pm\alpha \pmod{h = \mathrm{GCD}(|g|, 24)}$.

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A finite calculation shows $\mathbf{m}_\chi(n)$ are all non-negative.

Integrality

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- Integrality implies a system of congruences on modular forms.
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- For Thompson Moonshine, the bound obtained is at least a few 100 million.
- Strategy:
  Prove simpler congruences, one prime at a time.
- We prove a *complete* system of congruences for both Thompson and Monstrous Moonshine.

Thank you