K3 theories and modular forms

Workshop on Modular Forms and Black Holes,
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Plan:

1. Calabi-Yau$_2$ theories: moduli spaces and symmetries
2. A simple class of examples: lattice CFTs
3. Reflecting a special K3 theory
4. Summary and outlook

[Nahm/W01] *A hiker's guide to K3 - Aspects of $N = (4,4)$ superconformal field theory with central charge $c = 6$*, Commun. Math. Phys. **216** (2001), 85-138; hep-th/9912067

[W01] *Consistency of orbifold conformal field theories on K3*, Adv. Theor. Math. Phys. **5** (2001), 429-456; hep-th/0010281

[Gaberdiel/Taormina/Volpato/W14] *A K3 sigma model with $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry*, JHEP **1402:022** (2014); arXiv:1309.4127 [hep-th]

[Taormina/W••] *The Conway Moonshine Module is a Reflected K3 Theory*, in preparation

Calabi-Yau D-folds

Definition

A CALABI-YAU D-FOLD Y is a compact Riemannian manifold of dimension $2D$ with holonomy group $\mathrm{Hol}(Y) \subset \mathrm{SU}(D)$.

If $D = 2$:

if $\mathrm{Hol}(Y) = \{\mathrm{id}\}$: Y is a COMPLEX 2-TORUS Y^0;

if $\mathrm{Hol}(Y) = \mathrm{SU}(2)$: Y is a K3 SURFACE Y^{16}.

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Theorem **(tertium non datur)**

If Y is a Calabi-Yau 2-fold, then Y is

either a complex 2-torus Y^0,

diffeomorphic to $T_0 = \mathbb{R}^4/\mathbb{Z}^4$ with elliptic genus $\mathcal{E}_T \equiv 0$;

or a K3 surface Y^{16},

diffeomorphic to $X_0 = \widetilde{T_0/\mathbb{Z}_2}$ with elliptic genus

for $\tau, z \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$: $$\mathcal{E}_{K3}(\tau, z) = 8 \sum_{k=2}^{4} \left(\frac{\vartheta_k(\tau,z)}{\vartheta_k(\tau,0)} \right)^2.$$

Ricci-flat metrics on Calabi-Yau 2-folds

Torelli + Calabi-Yau Theorem [Calabi57, Pjatecki-Šapiro/Šafarevič71, Looijenga/Peters81, Burns/Rapoport75, Yau78. . .]

Moduli space of Ricci-flat metrics of volume 1 on a Calabi-Yau 2-fold $Y = Y^\delta$, $\delta \in \{0, 16\}$:

$$\mathcal{M}_T^{\mathrm{Ricci-flat}} = {}^{\mathrm{O}^+(3,3;\mathbb{Z})}\backslash^{\mathrm{O}^+(3,3;\mathbb{R})}\big/_{\mathrm{SO}(3)\times\mathrm{O}(3)},$$

$$\mathcal{M}_{K3}^{\mathrm{Ricci-flat}} = {}^{\mathrm{O}^+(3,19;\mathbb{Z})}\backslash^{\mathrm{O}^+(3,19;\mathbb{R})}\big/_{\mathrm{SO}(3)\times\mathrm{O}(19)}:$$

quotients of the Grassmannians of 3-dimensional positive definite oriented $\Sigma_g \subset \mathbb{R}^{3,3+\delta} \cong H^2(Y^\delta,\mathbb{R})$, relative to the even self-dual lattice $H^2(Y^\delta,\mathbb{Z}) \subset H^2(Y^\delta,\mathbb{R})$.

Sigma models on Calabi-Yau 2-folds

[Aspinwall/Morrison94]

For CY 2-folds: encode sigma model data (Ricci-flat g, B-field B),

i.e. $(\Sigma_g \subset H^2(Y^\delta,\mathbb{R}), V \in \mathbb{R}_{>0}, B \in H^2(Y^\delta,\mathbb{R}))$;

choose $\upsilon^0 \in H^0(Y^\delta,\mathbb{Z}), \upsilon \in H^4(Y^\delta,\mathbb{Z})$ with $\langle \upsilon^0, \upsilon\rangle = 1$ and set

$$x := \operatorname{span}_{\mathbb{R}} \left\{ \sigma - \langle\sigma, B\rangle\upsilon,\ \sigma \in \Sigma_g;\ \upsilon^0 + B + \left(V - \tfrac{1}{2}\langle B, B\rangle\right)\upsilon \right\},$$

a 4-dim. oriented pos. def. subspace $x \subset H^{even}(Y^\delta,\mathbb{R}) \cong \mathbb{R}^{4,4+\delta}$.

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Definition (Calabi-Yau$_2$ theories)

An $N = (4,4)$ SCFT at $c = \overline{c} = 6$ with space-time SUSY, integral $U(1)$ charges and CFT elliptic genus $\mathcal{E}$ is called TOROIDAL, if $\mathcal{E} \equiv 0$, and a K3 THEORY, if $\mathcal{E} \equiv \mathcal{E}_{K3}$.

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Note [W00&W14]:

Tertium non datur, and toroidal theories are toroidal.

Moduli spaces of CY$_2$-theories

[Narain86; Seiberg88, Cecotti90, Aspinwall/Morrison94, Nahm/W01]

$$\mathcal{M}_T = O^+(4,4;\mathbb{Z}) \backslash O^+(4,4;\mathbb{R}) / SO(4)\times O(4) \longleftrightarrow (g^T, B^T)$$

geometric interpretation (Ricci flat metric g on Y, B-field $B \in H^2(Y,\mathbb{R})$)

$$\mathcal{M}_{K3} = O^+(4,20;\mathbb{Z}) \backslash O^+(4,20;\mathbb{R}) / SO(4)\times O(20) \longleftrightarrow (g^{K3}, B^{K3})$$

[Aspinwall/Morrison94]

$Y = T^4$ or $Y = K3$

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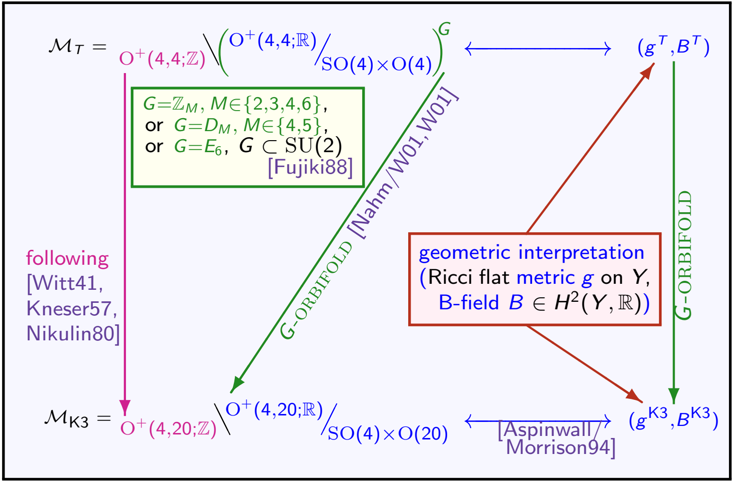

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Lattice techniques

Idea:

$$\mathrm{O}^+(4,4+\delta)\big/\,\mathrm{SO}(4)\times\mathrm{O}(4+\delta) = \text{Grassmannian } \mathcal{M}^\delta \text{ of EVEN SELF-DUAL LATTICES in } \mathbb{R}^{4,4+\delta}$$

For $p,\, p' \in \mathbb{R}^4$, $\overline{p},\, \overline{p}' \in \mathbb{R}^{4+\delta}$:

$$\binom{p}{\overline{p}} \bullet \binom{p'}{\overline{p}'} = \underbrace{p\cdot p'}_{\text{Euclidean}\,\cdot} - \underbrace{\overline{p}\cdot\overline{p}'}_{\text{Euclidean}\,\cdot}$$

relative to the positive definite

$$x_0 := \left\{ \binom{p}{0} \mid p \in \mathbb{R}^4 \right\} \subset \mathbb{R}^{4,4+\delta}$$

Geometric interpretation of $\Gamma \in \mathcal{M}^0$

– as TOROIDAL SCFT on $\mathbb{R}^4/\Lambda$ with B-field B, if

$$\Gamma = \left\{ \frac{1}{\sqrt{2}} \binom{\mu - B\lambda + \lambda}{\mu - B\lambda - \lambda} \mid \mu \in \Lambda^*,\ \lambda \in \Lambda \right\},$$

where $\Lambda \subset \mathbb{R}^4$: a rank 4 lattice, with generating matrix L,

$$\Lambda^* := \left\{ \mu \in \mathbb{R}^4 \mid \mu\cdot\lambda \in \mathbb{Z}\ \ \forall \lambda \in \Lambda \right\},$$

$$g = L^T L, \qquad B\colon \Lambda\otimes\mathbb{R} \longrightarrow \Lambda^*\otimes\mathbb{R}.$$

Toroidal SCFTs as lattice CFTs

Lattice superconformal field theory for $\Gamma \in \mathcal{M}^0$:

SPACE OF STATES: $$\mathbb{H} = \bigoplus_{\gamma\in\Gamma} \mathbb{H}_\gamma = \mathbb{H}^{\text{bos}} \oplus \mathbb{H}^{\text{ferm}},$$

a $\mathbb{C}$-vector space with
- positive definite Hermitean scalar product $\langle\cdot,\cdot\rangle$
- compatible real structure
- unitary action of
  $s\mathcal{V}\mathrm{ir}^{N=4}_{c=6} \oplus \overline{s\mathcal{V}\mathrm{ir}}^{N=4}_{\overline{c}=6} \ni L_0, J_0, \overline{L}_0, \overline{J}_0$
- modular invariant partition function:
  for $\tau, z \in \mathbb{C},\ \mathrm{Im}(\tau) > 0,\ q := e^{2\pi i\tau},\ y := e^{2\pi iz}$,
  $$Z(\tau,z) = \mathrm{Tr}_{\mathbb{H}^{\text{bos}}}\left(y^{J_0} q^{L_0-c/24} \overline{y}^{\overline{J}_0} \overline{q}^{\overline{L}_0-\overline{c}/24}\right),$$
  $$Z(\tau,z) = Z(\tau+1,z) = Z(-\tfrac{1}{\tau}, \tfrac{z}{\tau}).$$

with $\gamma = \binom{p}{\overline{p}} \in \Gamma$:
$$\mathrm{Tr}_{\mathbb{H}_\gamma\cap\mathbb{H}^{\text{bos}}}\left(y^{J_0} q^{L_0-c/24} \overline{y}^{\overline{J}_0} \overline{q}^{\overline{L}_0-\overline{c}/24}\right) = \frac{q^{\frac{p^2}{2}} \overline{q}^{\frac{\overline{p}^2}{2}}}{|\eta(\tau)|^8} \cdot \frac{1}{2} \sum_{k=1}^{4} \left| \frac{\vartheta_k(\tau,z)}{\eta(\tau)} \right|^4;$$

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OPERATOR PRODUCT EXPANSION

for ground states $\upsilon_\gamma \in \mathbb{H}_\gamma$, $\upsilon_{\gamma'} \in \mathbb{H}_{\gamma'}$ with $\gamma' = \binom{p'}{\overline{p}'} \in \Gamma$:
$$\pm \upsilon_\gamma(z)\upsilon_{\gamma'} \sim z^{p\cdot p'} \overline{z}^{\overline{p}\cdot\overline{p}'} \upsilon_{\gamma+\gamma'} + \cdots$$

A special K3 theory

For the $\mathbb{Z}_2$-orbifold of the D_4-torus theory, $g^T = g_*$, $B^T = B_*$:

Results:

- This theory is mirror self-dual **[W02]**.
- Its symmetry group is $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ **[Gaberdiel/Taormina/Volpato/W14]**.

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- This theory is also a lattice CFT, for a half-integral lattice $\widehat{\Gamma}$ **[Taormina/W, work in progress]**:

$\Gamma_0 := \widehat{\Gamma}^* \subset \widehat{\Gamma}$ has cosets Γ_a, $a \in \{0, 1, 2, 3\}$; with

$$\widetilde{\Gamma}_0 := \left\{ \binom{Q}{\overline{Q}} \in \mathbb{Z}^{2,2} \mid \sum_k (Q_k + \overline{Q}_k) \equiv 0(2) \right\},$$

$$\Gamma_0 \cup \Gamma_1 = \left(\widetilde{\Gamma}_0 \cup \left(\tfrac{1}{2} \binom{e_1+e_2}{e_1+e_2} + \widetilde{\Gamma}_0 \right) \right)^{\oplus 3};$$

each $\Gamma_0 \cup \Gamma_a$, $a \neq 0$, is self-dual and thus integral.

Definition of Reflection [work in progress with A. Taormina]

For certain superconformal field theories, **reflection** transforms all anti-holomorphic fields to holomorphic ones, yielding a super vertex operator algebra with an admissible module.

Properties:

- Instead of the $s\mathcal{V}\mathrm{ir}_c \oplus \overline{s\mathcal{V}\mathrm{ir}_{\overline{c}}}$ on the space of states $\mathbb{H}$, consider the diagonal $s\mathcal{V}\mathrm{ir}_{c+\overline{c}}$ action.
- In operator product expansions:

for $\upsilon, \upsilon' \in \mathbb{H}$: $\quad \upsilon(z)\,\upsilon' \sim \sum_{r,\overline{r}} \upsilon_{r,\overline{r}} z^r \overline{z}^{\overline{r}}, \quad \upsilon_{r,\overline{r}} \in \mathbb{H}.$

After reflection: $\quad \upsilon^{\text{refl.}}(z)\,\upsilon'^{\text{refl.}} \sim \sum_{r,\overline{r}} \upsilon^{\text{refl.}}_{r,\overline{r}} z^r z^{\overline{r}}.$

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Reflection Results

Results [Taormina/W, work in progress]:

- Reflection is well-defined for our special K3 theory.
- For our special K3 theory, reflection amounts to changing the quadratic form on the defining half integral lattice $\widehat{\Gamma} \subset \mathbb{R}^{6,6}$ to

$$\forall \binom{p}{\overline{p}}, \binom{p'}{\overline{p}'} \in \widehat{\Gamma}: \quad \binom{p}{\overline{p}} \tilde{\bullet} \binom{p'}{\overline{p}'} := p \cdot p' + \overline{p} \cdot \overline{p}'.$$

- The reflection of our special K3 theory agrees with John Duncan's Conway Moonshine Module ([Duncan/Mack-Crane16]: isomorphism of $\mathcal{V}\mathrm{ir}_{c+\overline{c}=12}$-representations).

Summary and outlook

Summary and outlook:

- Calabi-Yau$_2$ theories can be defined purely by means of representation theory, and their moduli spaces are known and in agreement with the expectation from string theory.
- Toroidal superconformal field theories are entirely understood, and they allow access to some K3 theories.
- Reflection builds a bridge to the theory of super vertex operator algebras.
- Reflecting special K3 theories is hoped to unveil some mysteries of Mathieu Moonshine.

The End

Thank you
for your attention!