Black holes and Modular Forms

References:

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Reviews:

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Black holes and modular forms

Black holes and modular forms are both very old subjects.

Recent developments in string theory have brought them together.

The goal of this lecture will be to give an idea of how this connection comes about.

Black holes

A black hole is a classical solution in general theory of relativity with special properties.

It is surrounded by an event horizon which acts as a one way membrane.

Nothing can escape from inside the event horizon to the outside.

In quantum theory a black hole behaves as a thermal system with finite temperature, entropy etc.

$$\mathbf{Entropy}\,\mathbf{S_{BH}} = \frac{\mathbf{A}}{\mathbf{4}}, \qquad \mathbf{Temperature}\,\mathbf{T_{BH}} = \frac{\mathbf{K}}{\mathbf{2\pi}}$$

Bekenstein, Hawking

A: Area of the event horizon,

K: surface gravity

Our units: $\hbar = c = k_B = G_N = 1$

The notion of entropy was originally introduced to describe thermal properties of matter.

However for ordinary objects,

statistical mechanics ⇒ a microscopic interpretation of S

e^S = number of quantum states of the system

Question: Does the entropy of a black hole have a similar statistical interpretation?

Task:

1. Calculate the number of quantum states N of the black hole with fixed energy

2. Compare $\ln N$ with S_{BH}

In order to answer this question we need to first find a quantum theory of gravity.

String theory is such a theory.

Key ingredient: Fundamental constituents of matter are different vibrational states of a string.

Typical size of a string $\sim 10^{-33}$ cm.

This is much smaller than the length scale that can be probed by any present day experiment

($\sim 10^{-16}$ cm.)

Thus to the present day experimentalists these fundamental constituents will appear to be point-like.

We formulate a theory of strings consistent with the principles of

1. Quantum mechanics and

2. Special theory of relativity.

One finds that one of the vibrational states of such a string has the properties expected of a graviton, – the mediator of gravitational force.

$\rightarrow$ String theory automatically contains gravity.

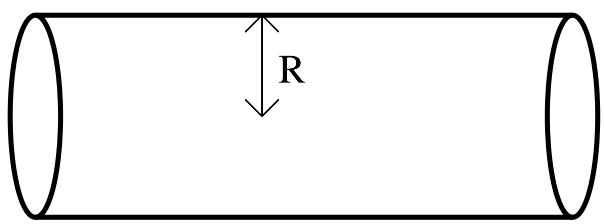
On the other hand the requirement of satisfying the laws of quantum mechanics and special theory of relativity also puts strong constraints on the theory.

Dimension of space-time = 9+1 (not 3 + 1)

The problem of having six extra dimensions can be resolved using an old idea known as

Compactification

Example: Consider the surface of a cylinder of radius R.



If R is smaller than the resolution of the most powerful microscope, no direct experiment can distinguish this world from a one dimensional world.

In string theory, take 6 of the 9 dimensions to describe a small compact space K

When the size of K is smaller that the resolution of the most powerful microscope, the space will appear to be 3 dimensional.

For suitable choices of K, the effective 3+1 dimensional theory looks very similar to the theory that describes our world.

In particular the vibrational states of the string contain mediators of not only gravity, but also of other forces of the kind which are observed in nature.

This gives us hope that string theory will eventually be able to provide us with a unified theory of the elementary constituents of matter and their interactions.

Main obstacle: Too many choices of K.

In fact we need more data to specify a solution like an element of the integer cohomology class of K

– different solutions to the equations of motion of string theory.

The physical properties of matter will depend on which solution of string theory describes the environment around us.

So far we have not been able to identify any particular solution of string theory which describes precisely the world around us.

As a result we cannot yet use string theory to make specific experimental prediction.

Nevertheless we can use string theory to address universal issues which exist in all phases, e.g. various conceptual problems in quantum gravity like understanding black holes.

Strategy:

1. Identify a simple class of black hole solutions even if they do not describe a black hole in our environment.

2. Count the number of quantum states N.

3. Calculate Bekenstein-Hawking entropy S_{BH}

4. Compare $\ln N$ with S_{BH}.

If it fails even for one solution, we have to declare the whole string theory internally inconsistent.

Note: S_{BH} is only an approximate result for entropy valid for large area.

In principle one can systematically calculate corrections of order $\ln A$, $1/A$ etc. but so far only the first subleading corrections have been computed for some black holes.

Modular forms

– well known 'functions' of one or more complex variables.

$$f(z_1, z_2, z_3, \cdots)$$

Typically they are periodic under $z_i \to z_i + 1$ and hence admit Fourier expansion

$$f(z_1, z_2, z_3, \cdots) = \sum_{n_1, n_2, n_3} e^{2\pi i(n_1 z_1 + n_2 z_2 + n_3 z_3 + \cdots)} g(n_1, n_2, n_3, \cdots)$$

$g(n_1, n_2, n_3, \cdots)$ are integer valued functions.

It turns out that for many black holes in string theory, the number of microstates N is given by the expansion coefficients $g(n_1, n_2, n_3)$ of some modular forms.

n_1, n_2, n_3: some combination of the charges carried by the black hole.

Large area corresponds to large n_1, n_2, n_3

This leads to some concrete predictions.

1. For large $\mathbf{n_1, n_2, n_3}$

$$\ln g(n_1, n_2, n_3, \cdots) = S_{BH} + \text{corrections}$$

2. Since $\mathbf{g(n_1, n_2, n_3, \cdots)}$ given the number of states, it must be positive for every $\mathbf{n_1, n_2, n_3, \cdots}$.

If either of this fails then string theory is inconsistent.

So far it has never failed.

An example: Igusa cusp form is a well known Siegel modular form

– a function $\phi_{10}(\rho, \sigma, \mathbf{v})$ of three variables

Expand

$$\frac{1}{\phi_{10}(\rho, \sigma, \mathbf{v})} = \sum_{\mathbf{m},\mathbf{n},\mathbf{p}} (-1)^{\mathbf{p}+1} \mathbf{g}(\mathbf{m}, \mathbf{n}, \mathbf{p}) \mathbf{e}^{2\pi \mathbf{i}(\mathbf{m}\rho + \mathbf{n}\sigma + \mathbf{p}\mathbf{v})}$$

g(m,n,p) counts the number of states of a special class of black holes for

$$\mathbf{p}^2 < 4\mathbf{mn}, \quad \mathbf{0} \le \mathbf{p} \le \mathbf{2m}, \mathbf{2n}$$

Predictions:

1. For large m,n,p

$$\mathbf{g}(\mathbf{m}, \mathbf{n}, \mathbf{p}) = \pi\sqrt{\mathbf{4}\,\mathbf{m}\,\mathbf{n} - \mathbf{p}^2} + \cdots$$

– can be proved

2. For all m,n,p in the range above, $\mathbf{g(m, n, p) \geq 0}$

– proved for large m,n,p

– tested in all cases up to m,n = 5

– proved for m=1,2 and arbitrary n, p in the range

but general proof is still lacking.

Goal:

1. Identify suitable black holes in string theory for which we can count the number of quantum states N.

2. Compare $\ln \mathbf{N}$ with $\mathbf{A/4} + \cdots$

Recall: black hole solutions asymptotically approach

$$\mathbf{K \times R^{3,1}}$$

We shall use a simple class of solutions for which K is

$$\mathbf{T^6} \qquad \text{or} \qquad \mathbf{K3 \times T^2}$$

in type II string theory

These theories have several unrealistic features.

1. Supersymmetry – a symmetry that transforms fermions to bosons and vice versa

– not observed in nature.

2. Several Maxwell fields e.g. $g_{m\mu}$ where m is along a compact circle and μ is the space-time coordinate

3. Several massless scalar fields without potential e.g. g_{mn} describing the shape and size of compact spaces, and the 'dilaton' that controls the strength on interaction.

The asymptotic form of the black hole solution is characterized by arbitrary values of these scalar fields known as moduli fields

– determine shape and size of K and other data.

Given the asymptotic form of the solution, a black hole solution is characterized by several electric charges $Q_1, Q_2, \cdots$ and magnetic charges $P_1, P_2, \cdots$ and mass M.

In a finite region of space these differ from the vacuum solution

Far away they approach a vacuum solution, characterized by the constant values of the moduli fields $\{\phi_i\}$.

The area of the event horizon and hence the entropy is a function of $M, \{Q_i\}, \{P_i\}$ and the asymptotic values of $\{\phi_i\}$.

Supersymmetry puts a lower found on M:

$$M \geq f(\{Q_i\}, \{P_i\}, \{\phi_i\})$$

When the bound is saturated the black hole acquires special properties:

1. It remains invariant under some of the supersymmetry transformations -

- BPS black holes

2. Area and hence entropy becomes independent of $\{\phi_i\}$

3. Temperature goes to zero

4. Carries zero angular momentum (singlet representation of rotational SU(2) group)

5. Whether black holes or not, the quantity

$$\mathbf{index \equiv Trace_{\{Q_i\},\{P_i\}}(-1)^{2J_3}}$$

does not depend on $\{\phi_i\}$.

Trace runs over all states in the Hilbert space with fixed charges.

This suggests a strategy:

1. Pick some charges $\{\mathbf{Q_i}\}$, $\{\mathbf{P_i}\}$ and adjust $\{\phi_i\}$ so that we have a black hole solution with smooth horizon carrying these charges.

– typically requires the interaction strength, labelled by one of the $\{\phi_i\}$'s to be moderate and charges to be large.

2. For such black holes we can calculate the area A of the event horizon and calculate its entropy.

It is also possible to calculate corrections systematically like those proportional to lnA or negative powers of A, but these are difficult.

3. Since black hole carries zero angular momentum we have

$$\mathbf{Trace}_{\{\mathbf{Q_i}\},\{\mathbf{P_i}\}}(-\mathbf{1})^{\mathbf{2J_3}} = \mathbf{Trace}_{\{\mathbf{Q_i}\},\{\mathbf{P_i}\}}(\mathbf{1}) = \mathbf{exp}[\mathbf{entropy}]$$

4. Since this is independent of the $\{\phi_i\}$'s we can choose another value of $\{\phi_i\}$ to see if we can directly count the states weighted by $(-1)^{2J_3}$

5. In some cases this is possible when the interaction strength is small.

In that case we can compare the two results and see if they agree

– will provide an explanation of black hole entropy.

6. The index computed from state counting can *a priori* be either positive or negative.

But supersymmetric black hole carries zero angular momentum

$\Rightarrow$ the index computed using the black hole description is positive

$\Rightarrow$ index computed from state counting must also be positive

– an independent prediction that can be tested.

A simple example of state counting

Consider a string wound along a compact circle of radius R, carrying momentum n/R for some integral n.

n: a charge

How many states are there?

– depends on how many ways the string ca n oscillate.

Let us suppose that the string oscillates only in one direction.

Then n can arise as a single quanta carrying momentum n, or two quanta carrying momenta (1, n-1), (2, n-2) etc. or more quanta sharing the momenta.

All of them are BPS states as long as the quanta carry positive momenta.

Ex. Check that the answer is the coefficient of q^n in

$$\prod_{\mathbf{k}=\mathbf{1}}^{\infty}(\mathbf{1}-\mathbf{q}^{\mathbf{k}})^{-\mathbf{1}}$$

In actual problem string has 24 modes of vibration, giving

$$\prod_{\mathbf{k}=\mathbf{1}}^{\infty}(\mathbf{1}-\mathbf{q}^{\mathbf{k}})^{-\mathbf{24}}$$

– up to a multiplicative factor of q, this is Δ.

For systems which describe black holes with smooth horizon the counting is more complicated, but the principle is the same.

Simplest case: type II string theory compactified on T^6

This theory has 28 maxwell fields

(6 of them come from components of the metric along the internal directions, others have origin in other fields)

– a state is characterized by 28 electric and 28 magnetic charges.

1. Construct a black hole solution carrying these charges

2. Calculate the area A of the event horizon and calculate entropy S

3. Construct quantum states at a different point in the moduli space and calculate the index

4. Compare entropy with log(index)

Black hole entropy depends on one quartic combination D of electric and magnetic charges

$$\mathbf{Entropy} = \pi\sqrt{\mathbf{D}} - \mathbf{2}\ln\mathbf{D} + \cdots$$

Index for same D is given by the Fourier expansion coefficients of a weak Jacobi form:

$$\frac{\vartheta_1(\mathbf{z}|\tau)^2}{\eta(\tau)^6} \equiv \sum_{\mathbf{k},\mathbf{l}} \mathbf{c}(\mathbf{4k} - \mathbf{l}^2)\, \mathbf{e}^{2\pi\mathbf{i}(\mathbf{k}\,\tau + \mathbf{l}\,\mathbf{z})}$$

Then

$$\mathbf{Index} = (-\mathbf{1})^{\mathbf{D}}\,\mathbf{c}(\mathbf{D})$$

This tells us that in order to match black hole result we must have

$$\ln|\mathbf{c}(\mathbf{D})| = \pi\sqrt{\mathbf{D}} - \mathbf{2}\ln\mathbf{D} + \cdots$$

– can be checked to be correct.

Furthermore, positivity of the index tells us

$$(-\mathbf{1})^{\mathbf{D}}\,\mathbf{c}(\mathbf{D}) \geq \mathbf{0} \qquad \mathbf{for}\ \mathbf{D} > \mathbf{0}$$

– can be proved (Bringmann and Murthy)

Type II on $\mathbf{K3 \times T^2}$

The result for the index is given by Fourier coefficients of the Igusa cusp form ϕ_{10}.

We shall begin by describing some properties of ϕ_{10}.

Siegel upper half plane:

$$(\rho, \sigma, \mathbf{v}) : \quad \rho, \sigma \in \mathbf{H}, \quad \mathbf{v} \in \mathbf{C}$$

$$\rho = \rho_1 + \mathbf{i}\rho_2, \quad \sigma = \sigma_1 + \mathbf{i}\sigma_2, \quad \mathbf{v} = \mathbf{v}_1 + \mathbf{i}\mathbf{v}_2$$

$$\rho_2 > 0, \quad \sigma_2 > 0, \quad \rho_2\sigma_2 - \mathbf{v}_2^2 > 0$$

Define:

$$\omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$$

Modular transformation acts as

$$\omega \to (\mathbf{A}\omega + \mathbf{B})(\mathbf{C}\omega + \mathbf{D})^{-1}$$

A, B, C, D are 2×2 **matrices with integer entries and satisfy**

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathbf{T}} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

– generates Sp(2, Z) group.

S-duality subgroup:

$$\mathbf{A} \in \mathbf{SL(2,Z)}, \quad \mathbf{A} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}, \quad \tilde{\mathbf{a}}\tilde{\mathbf{d}} - \tilde{\mathbf{b}}\tilde{\mathbf{c}} = \mathbf{1}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}$$

This gives

$$\omega \to \mathbf{A}\,\omega\,\mathbf{A^T}$$

$\phi(\omega)$ **is a modular form of weight k if**

$$\phi((\mathbf{A}\omega + \mathbf{B})(\mathbf{C}\omega + \mathbf{D})^{-\mathbf{1}}) = \mathbf{det}(\mathbf{C}\omega + \mathbf{D})^{\mathbf{k}}\phi(\omega)$$

Note: For S-duality subgroup, $\mathbf{det}(\mathbf{C}\omega + \mathbf{D}) = \mathbf{1}$**.**

Example: Igusa cusp form of weight 10 $\phi_{10}(\rho,\sigma,v)$

$$\mathbf{e^{2\pi i(\rho+\sigma+v)}} \prod_{\substack{\mathbf{k,\ell,j\in Z} \\ \mathbf{k,\ell\geq 0, j<0 \text{ if } k=\ell=0, 4k\ell-j^2\geq -1}}} \left(\mathbf{1-e^{2\pi i(k\rho+\ell\sigma+jv)}}\right)^{\mathbf{c(4k\ell-j^2)}}$$

c(s) is defined via

$$\mathbf{8}\sum_{\mathbf{i=2}}^{\mathbf{4}} \frac{\vartheta_{\mathbf{i}}(\mathbf{z}|\tau)^{\mathbf{2}}}{\vartheta_{\mathbf{i}}(\mathbf{0}|\tau)^{\mathbf{2}}} = \sum_{\mathbf{n,j\in Z}} \mathbf{c(4n-j^2)e^{2\pi i(n\tau+jz)}}$$

c(-1)=2

⇒ k =ℓ=0 term gives

$$(\mathbf{1-e^{-2\pi i v}})^{\mathbf{2}}$$

– double zero at v=0

Other zeroes of ϕ_{10} **come from the Sp(2,Z) images of this.**

Consider the function

$$\mathbf{Z(\rho.\sigma, v) = 1/\phi_{10}(\rho, \sigma, v)}$$

– has double poles at v=0 and its images.

The Fourier coefficients of Z give the index for type II on $\mathbf{K3 \times T^2}$.

But there are subtleties.

For large $\rho_2, \sigma_2, |v_2|$ can we get a convergent Fourier expansion for Z?

$$\mathbf{Z = \sum_{m,n,p} (-1)^{p+1} g(m,n,p) e^{2\pi i(m\rho + n\sigma + pv)}}$$

This is ambiguous.

The k=ℓ=0, j=-1 term gives

$$(1 - e^{-2\pi i v})^{-2} = \sum_{n \geq 0} (n+1) e^{-2\pi i n v}$$

– good for $v_2 < 0$, i.e. $e^{-2\pi i v} < 1$.

For $v_2 > 0$ we need a different expansion:

$$(1 - e^{-2\pi i v})^{-2} = e^{4\pi i v}(1 - e^{2\pi i v})^{-2} = \sum_{n \geq 2} (n-1) e^{2\pi i n v}$$

These two different methods of expansion will give different g(m,n,p).

Each term $(1 - e^{2\pi i(k\rho + \ell\sigma + jv)})$ must be examined.

Elementary algebra: For $4k\ell - j^2 \geq 0$,

$$k\rho_2 + \ell\sigma_2 + jv_2 > 0$$

– expansion is unambiguous.

For $4k\ell - j^2 = -1$, $k\rho_2 + \ell\sigma_2 + jv_2$ may switch sign

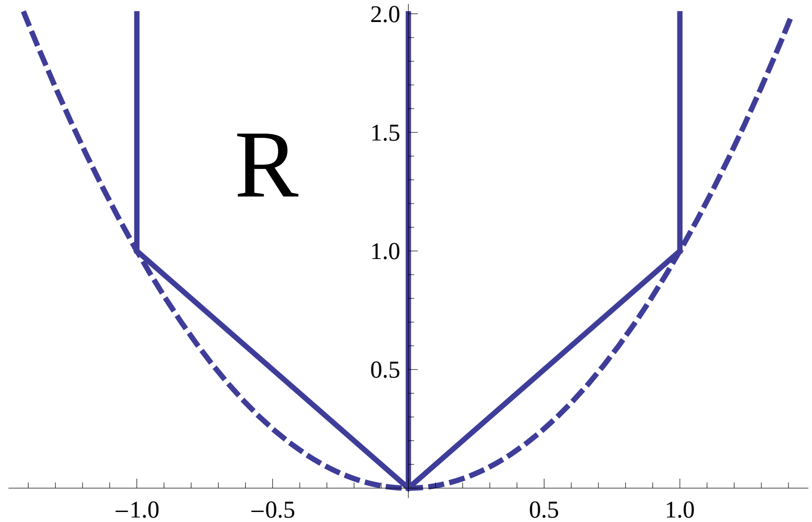
$\Rightarrow$ walls in the (ρ_2, σ_2, v_2) plane at

$$k\rho_2 + \ell\sigma_2 + jv_2 = 0, \quad 4k\ell - j^2 = -1, \quad k, \ell, j \in Z$$

For $k\rho_2 + \ell\sigma_2 + jv_2 >< 0$ expand in powers of

$$e^{\pm 2\pi i(k\rho + \ell\sigma + jv)}$$

These walls divide the $(x \equiv v_2/\rho_2, y \equiv \sigma_2/\rho_2)$ plane, restricted to the $x^2 < y$ region, into triangles.



$$\mathbf{k} + \ell\,\mathbf{y} + \mathbf{j}\,\mathbf{x} = \mathbf{0}, \quad \mathbf{4k}\ell - \mathbf{j^2} = -\mathbf{1}$$

R: expand each term in the product as given without any switch.

In each domain we have different g(m,n,p).

Walls / domains are mapped to each other under S-duality.

S-duality action on g(m,n,p):

$$\mathbf{m}\rho + \mathbf{n}\sigma + \mathbf{j}\mathbf{v} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Tr}\left[\begin{pmatrix} 2m & p \\ p & 2n \end{pmatrix}\begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}\right]$$

is invariant under S-duality if

$$\begin{pmatrix} 2m & p \\ p & 2n \end{pmatrix} \to (\mathbf{A}^{\mathbf{T}})^{-1}\begin{pmatrix} 2m & p \\ p & 2n \end{pmatrix}\mathbf{A}^{-1}$$

$\Rightarrow$ **g(m,n,p) is invariant under this transformation provided we simultaneously change (m,n,p) and the domain in which we carry out the expansion.**

Type II on $\mathbf{K3 \times T^2}$

This also has 28 Maxwell fields

– 28 electric and 28 magnetic charges

– can construct black hole solutions carrying these charges

Task:

1. Calculate BPS black hole entropy

$$\mathbf{S_{BH} = \frac{A}{4} + \cdots}$$

2. Calculate index from microscopic description.

3. Test positivity of the index and compare $\mathbf{S_{BH}}$ with ln(index).

Result for the Index:

While the index is almost independent of $\{\phi_i\}$, there are codimension one walls in the moduli space across which the index jumps discontinuously.

This happens when the state becomes marinally stable

$$\mathbf{f}(\{\mathbf{Q}_\alpha\},\{\mathbf{P}_\alpha\},\{\phi_\mathbf{i}\}) = \mathbf{f}(\{\mathbf{Q}^{(1)}_\alpha\},\{\mathbf{P}^{(1)}_\alpha\},\{\phi_\mathbf{i}\}) + \mathbf{f}(\{\mathbf{Q}^{(2)}_\alpha\},\{\mathbf{P}^{(2)}_\alpha\},\{\phi_\mathbf{i}\})$$

for some $\mathbf{Q}^{(1)}, \mathbf{P}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{P}^{(2)}$ satisfying

$$\mathbf{Q}_\alpha = \mathbf{Q}^{(1)}_\alpha + \mathbf{Q}^{(2)}_\alpha, \quad \mathbf{P}_\alpha = \mathbf{P}^{(1)}_\alpha + \mathbf{P}^{(2)}_\alpha$$

The index is ill defined on the wall of marginal stability and may jump as we cross the wall.

For given $\{\mathbf{Q}_\alpha\}, \{\mathbf{P}_\alpha\}$ there are different walls corresponding to different $\{\mathbf{Q}_\alpha^{(1)}\}, \{\mathbf{P}_\alpha^{(1)}\}$

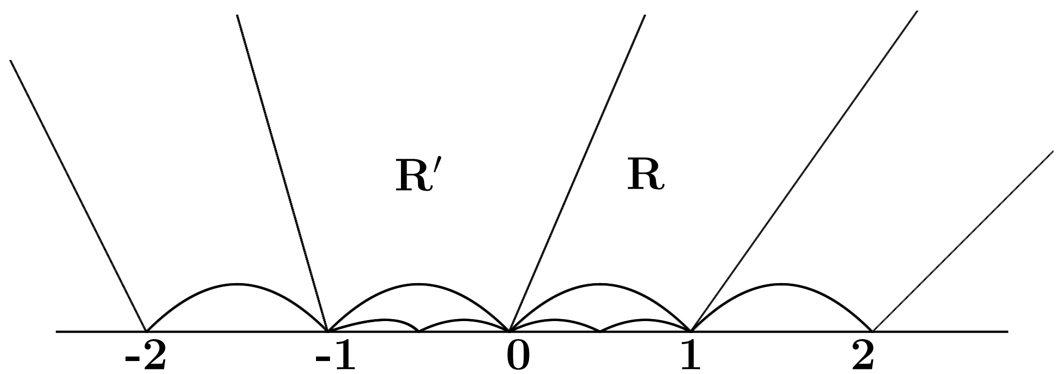
– divides the moduli space into chambers.

The index depends on the chamber of the moduli space in which we are.

We can consider the projection of the chambers into a two dimensional subspace of the 134 dimensional moduli space, labelled by

$$\tau \in \mathbf{UHP} : \mathbf{complex\ structure\ of\ T^2}$$

The walls separating different values of index are circles in τ-plane connecting rational numbers p/q to r/s with ps - qr=1, p,q,r,s$\in$ Z



Inside each chamber the index depends on three quadratic combination $\mathbf{D_1, D_2, D_3}$ of charges.

The index remains unchanged inside a chamber but jumps across the walls.

The precise equation for the walls depends on the other moduli and the charges carried by the state.

Type IIB string theory compactified on $\mathbf{K3 \times T^2}$ string theory has invariance under

$$\tau \to \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}$$

$$\begin{pmatrix} 2D_1 & D_3 \\ D_3 & 2D_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2D_1 & D_3 \\ D_3 & 2D_2 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\mathbf{ad - bc = 1}, \qquad \mathbf{a, b, c, d} \in \mathbf{Z}$$

– known as S-duality symmetry

Under this map a chamber gets mapped to a different chamber and the charges also transform

– relates $\mathbf{g(D_1, D_2, D_3)}$ in different chambers.

Recall the result for Fourier expansion of $1/\phi_{10}$

– different Fourier expansion coefficients $\mathbf{g(m,n,p)}$ in different domains in the $(\mathbf{v_2}/\rho_2, \sigma_2/\rho_2)$-plane.

Direct computation of the index shows that it is given by $\mathbf{g(D_1, D_2, D_3)}$, and that

there is one to one map between the domains in the $(\mathbf{v_2}/\rho_2, \sigma_2/\rho_2)$ plane and the chambers in the τ-plane separated by marginal stability walls

Index in one chamber in the τ-plane is given by the value of $\mathbf{g(D_1, D_2, D_3)}$ in the corresponding domain in $(\mathbf{v_2}/\rho_2, \sigma_2/\rho_2)$ plane.

Comparing the transformation on D_1, D_2, D_3 with the S-duality transformation acting on (m,n,p) we see that we can identify the two if we choose

$$(\mathbf{A^T})^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Therefore, to specify the map between domains and chambers, it is enough to specify it for one domain and the corresponding chamber.

Then rest will be fixed by S-duality map.

The domain R maps to the chamber R.

Example: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **takes $\mathbf{D_3 > 0}$ to $\mathbf{D_3 < 0}$, and maps R to R$'$ both in the τ-plane and the $(\mathbf{v_2}/\rho_\mathbf{2}, \sigma_\mathbf{2}/\rho_\mathbf{2})$-plane.**

The result for the index given above seems to be in conflict with the fact that black hole entropy is independent of $\{\phi_i\}$.

It turns out that there is a simple solution to this problem.

In the black hole description, once we fix the charges, we may get not just single centered black holes but also two centered black holes

– total index gets contribution from single and two centered black holes.

Furthermore whether two centered black hole solutions exist or not depends on $\{\phi_i\}$.

Some two centered black hole solutions that exist in one chamber ceases to exist in the neighboring chamber or vice versa.

When two centered black holes exist, they carry non-zero angular momentum, and the positivity of the index breaks down.

It turns out that this can explain the origin of the jump in the index across walls of marginal stability.

The index jumps across the wall precisely because the contribution from some two centered black holes disappear across the wall.

The jump in the index = index of two centered solutions that (dis-)appear.

We can proceed in two ways:

1. Find a chamber of the moduli space where there are no two centered black holes

– known as the attractor chamber.

Do the counting in this chamber, and compare with the results for black hole entropy.

2. Work in a general chamber but be careful to take into account the contribution from two centered black holes to the index.

We shall follow the first approach.

Attractor chamber: For given D_1, D_2, D_3, there is a special chamber in which only single centered black hole solutions exist.

For $0 \le D_3 \le 2D_1, 2D_2$, $4D_1D_2 - D_3^2 > 0$,

R is the attractor chamber.

The index must be positive in this chamber.

Furthermore the entropy of single centered black holes for large D_1, D_2, D_3 is given by:

$$\pi\sqrt{4D_1D_2 - D_3^2} - f\left(\frac{D_3}{2D_2}, \frac{\sqrt{4D_1D_2 - D_3^2}}{2D_2}\right) + \text{lower order}$$

$$f(x_1, x_2) = 12\ln(x_2) + 24\ln\eta(x_1 + ix_2) + 24\ln\eta(-x_1 + ix_2) + \cdots$$

– gives prediction for the index.

To summarize, the analysis given above predicts that

g(m,n,p) is positive in the chamber R for

$$\mathbf{0 \leq p \leq 2m, 2n, \quad 4mn - p^2 > 0}$$

– has been verified in many cases by explicit computation

– proven in the limit of large m,n,p

– proven for m=1 and 2 and all n,p

but a general proof does not exist.

Another prediction from the string theory side:

For large m,n, p

$$\mathbf{g(m,n,p) = \pi\sqrt{4mn - p^2} - f\left(\frac{p}{2n}, \frac{\sqrt{4mn-p^2}}{2n}\right) + \text{lower order}}$$

$$\mathbf{f(x_1, x_2) = 12\ln(x_2) + 24\ln\eta(x_1 + ix_2) + 24\ln\eta(-x_1 + ix_2) + \cdots}$$

– can be proven using following steps:

1. Represent g(m,n,p) as Fourier integrals:

$$\mathbf{g}(\mathbf{m},\mathbf{n},\mathbf{p}) = \int \mathbf{d}\rho \int \mathbf{d}\sigma \int \mathbf{dv}\, \mathbf{e}^{-2\pi \mathbf{i}(\mathbf{m}\rho+\mathbf{n}\sigma+\mathbf{pv})}\mathbf{Z}(\rho,\sigma,\mathbf{v})$$

along real ρ, σ, v axes situated at large values of $\rho_2, \sigma_2, \mathbf{v}_2$.

2. Deform the v contour towards small $\mathbf{v}_2$ picking up residues at poles.

The leading contribution comes from the pole at

$$\rho\sigma - \mathbf{v}^2 + \mathbf{v} = 0$$

3. After picking up residue at this pole, we evaluate the integrals over ρ and σ by saddle point method.

The result reproduces the answer given above.

There are many other examples of Siegel modular forms that appear in the counting of black hole states

– involve type II string theory on $\mathbf{T^4 \times T^2/Z_N}$ or $\mathbf{K3 \times T^2/Z_N}$

The $\mathbf{Z_N}$ acts as

1. Shift along $\mathbf{T^2}$

2. a discrete isometry of $\mathbf{T^4}$ or K3 preserving SU(2) holonomy group.

For all of these Siegel modular forms we have similar results.

The asymptotic expansion has been checked in all cases up to first subleading order.

The positivity has been experimentally verified in many cases but not proven.