N=2 Compactifications of Heterotic String on Orbifolds of $K3 \times T^2$

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Introduction and Results

Introduction

- N=2 supersymmetric heterotic string compactifications was studied in context of string dualities. The heterotic string compactified on $K3 \times T^2$ are dual to type II compactified on Calabi-Yau manifolds. [Kachru, Vafa 1995]
- These theories can be lifted to 6 dimensions. Anomaly cancellation in 6D constrains their differences between the number of hypermultiplets and vector multiplets. $N_h - N_v = 244$. [Stieberger, Walton, Font. Quevedo...]
- We study a further order N orbifold on $K3$ and a 1/N shift on T^2, cannot be lifted to 6 dimensions. So $N_h - N_v \neq 244$
- We calculate the two quantities $\mathcal{Z}_{\rm new}$ and differences between gauge threshold corrections for these compactifications.

$$\mathcal{Z}_{new} = \frac{1}{\eta^2} Tr_R((-1)^{\bar{F}} \bar{F} q^{L_0 - c/24} \bar{q}^{\bar{L_0} - \bar{c}/24})$$

Introduction: Results

- For standard embeddings we calculate the new supersymmetric index for different orbifolds of $K3$. The orbifold action g' is an order N action and has a one-to-one correspondence to the different conjugacy classes of Mathieu group M_{24}

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = -2\frac{1}{\eta^{24}}\Gamma_{3,2}^{(r,s)} \otimes E_{4,1}\left[\frac{1}{4}\alpha_{g'}^{(r,s)}E_6 - \beta_{g'}^{(r,s)}f_{g'}^{(r,s)}(\tau)E_4\right].$$

- $\alpha_{g'}^{(r,s)}$ and $\beta_{g'}^{(r,s)}f_{g'}^{(r,s)}(\tau)$ can be read off from the twisted elliptic genus of $K3$ which is given by,

$$F^{r,s}(\tau,z) = \frac{1}{N}Tr_{RRg'^r}[(-1)^{F_{K3}+\bar{F}_{K3}}g'^s e^{2\pi i z F_{K3}}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}]$$

Introduction: Results

- In general for standard embeddings one $E_8 \to E_7 \times U(1)$
- We calculate the difference between the one loop corrections of the gauge couplings from the two groups E_8 and E_7

$$\frac{1}{g_{E_7}^2}(T,U,V) - \frac{1}{g_{E_8}^2}(T,U,V) \quad [\text{Kaplunovsky Louis 1995}]$$

- One loop corrections to the gauge coupling for the gauge group G:

$$\Delta(T,U,V) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}(\mathcal{B}_G - b(G)),$$

$\mathcal{B}_G$ is given by $\left(\mathrm{a}\, q\partial_q - \frac{1}{8\pi\tau_2}\right) \mathcal{Z}_{\mathrm{new}}^G$

- The differences between the gauge corrections in one loop is given by the theta lift of twisted elliptic genus of $K3$.

$$\int_{\mathcal{F}} F^{r,s}_{(K3\times T^2)/g'} \otimes \Gamma^{r,s}_{3,2}$$

Introduction: Results

- For nonstandard embeddings for 2A orbifold of $K3$ $\mathcal{Z}_{\rm new}$

$$\mathcal{Z}_{\rm new} = -\frac{1}{\eta^{24}}\{\Gamma_{3,2}^{(0,0)} \otimes \frac{1}{12}[n_1 E_{4,1} E_6 + n_2 E_{6,1} E_4]$$
$$+\Gamma_{3,2}^{(0,1)} \otimes [\hat{a} E_{4,1}(E_6 + 2\mathcal{E}_2(\tau)E_4) + \hat{b}\mathcal{E}_2(\tau)^2(E_{6,1} + 2\mathcal{E}_2(\tau)E_{4,1})$$
$$+\hat{c} E_4(E_{6,1} + 2\mathcal{E}_2(\tau)E_{4,1})] + \cdots\}$$

- The differences between the gauge corrections in one loop is given by a linear combination of:
  1. theta lift of twisted elliptic genus of $K3$,
  2. theta lift of the elliptic genus of $K3$ and
  3. theta lift of elliptic genus of $K3$ with a half shift in T^2

$$\int_{\mathcal{F}} \left(F^{r,s}_{K3\times T^2(x\to x+\pi)} \otimes \Gamma^{r,s}_{3,2} + F^{r,s}_{(K3\times T^2)/2A} \otimes \Gamma^{r,s}_{3,2} + F_{K3} \otimes \Gamma_{3,2} \right)$$

Twisted Elliptic Genus

Twisted Elliptic Genus

- The twisted elliptic genus of $K3$ given by

$$F^{r,s}(\tau,z) = \frac{1}{N} Tr_{RRg'^r}[(-1)^{F_{K3}+\bar{F_{K3}}} g'^s e^{2\pi i z F_{K3}} q^{L_0 - c/24} \bar{q}^{\bar{L_0} - \bar{c}/24}]$$

  g' is the orbifold action on $K3$ and r is the twist, and the trace is on $K3$. $q = e^{2\pi i \tau}$

- The quantity is holomorphic in τ and z as the right movers have supersymmetry, the trace is taken over the Ramond sector. Like the elliptic genus (when g' = identity) the twisted elliptic genus should be independent of the realization of $K3$.

Twisted Elliptic Genus Transformation

• The transformation property of twisted elliptic genus are given by

$$F^{(r,s)}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=\exp\left(2\pi i\frac{cz^2}{c\tau+d}\right)F^{(cs+ar,ds+br)}(\tau,z),$$

with $a,b,c,d\in SL_2(Z)$

$$a,b,c,d\in\mathbb{Z},\qquad ad-bc=1.$$

The indices $cs+ar$ and $ds+br$ are taken to be mod N where N is the order of g'.

• For a prime N we can get all the sectors using modular transformations on $F^{(0,1)}$

Hodge Diamond

If the order of g' action which quotients the $K3$ be $N \in \{2, 3, 5, 7\}$ the Hodge diamond of $K3/\mathbb{Z}_N$ becomes

$$h_{(0,0)} = h_{(2,2)} = h_{(0,2)} = h_{(2,0)} = 1,$$
$$h_{(1,1)} = 2\left(\frac{24}{N+1} - 2\right) = 2k$$

- Different values of k forms different orbifolds.

N	$h_{(1,1)}$	k
1	20	10
2	12	6
3	8	4
5	4	2
7	2	1

Standard Embedding

Standard Embedding Definition

• The standard embedding is defined as the spin connection of $K3$ is embedded in one of the E_8 lattices.

• Consider λ^{I} λ'^{I} are 16 free Majorana Weyl fermions as coordinates of E_8, E_8', X^a as $K3$ coordinates X^i as T^2 coordinates.

The coupling of the gauge connections to the sigma model fields is given by,

$$G = \sum_{\mathrm{I,J}=1}^{4} \lambda^{\mathrm{I}} B_a^{\mathrm{IJ}} \partial X^a \lambda^{\mathrm{J}} + \sum_{\mathrm{I,J}=5}^{16} \lambda^{\mathrm{I}} A_i^{\mathrm{IJ}} \partial X^i \lambda^{\mathrm{J}} + \sum_{\mathrm{I,J}=1}^{16} \lambda'^{\mathrm{I}} A_i'^{\mathrm{IJ}} \partial X^i \lambda'^{\mathrm{J}}$$

B_a^{IJ} is the $SU(2)$ spin connection of $K3$ and A_i^{IJ}, $A_i'^{\mathrm{IJ}}$ are flat connection on T^2

Standard Embedding Definition

- Thus the E_8 lattice coupled to SU(2) spin connection of $K3$ breaks down to a $D2$ which couples to the SU(2) spin connection, and a $D6$ where the fermions couple to the flat connection of T^2.
- The fermions of the untouched E_8 lattice gets coupled to flat connection of T^2.
- The $D2$ fermions coupled to $K3$ form the super-partners of the bosons of $K3$. Along with the right movers they form a SCFT with $(c, \bar{c})$=(6,6).
- So the internal CFT splits as

$$\mathcal{H} = \mathcal{H}_{D2K3} \otimes \mathcal{H}_{D6} \otimes \mathcal{H}_{E_8} \otimes \mathcal{H}_{T^2}$$

Standard Embedding, $\mathcal{Z}_{\mathrm{new}}$

New Supersymmetric Index

The new supersymmetric index would be given as

$$\mathcal{Z}_{new} = \frac{1}{\eta^2} Tr_R(e^{\pi i F} F q^{L_0 - c/24} \bar{q}^{\bar{L_0} - \bar{c}/24})$$

where $F = F^{T^2} + F^{K3}$ is the total fermion number coming from the right movers.

This is 0 due to the zero modes of the fermions on T^2 unless

$$\mathcal{Z}_{new} = \frac{1}{\eta^2} Tr_R(e^{\pi i (F_{K3} + F_{T^2})} F_{T^2} q^{L_0 - c/24} \bar{q}^{\bar{L_0} - \bar{c}/24})$$

For any realization of $K3$, $\mathcal{Z}_{\rm new}$ is given by

$$2 \frac{(E_4 \times E_6)}{\eta^{24}} \Gamma_{2,2}$$

where, $\Gamma_{2,2}$ is the lattice sum given as $\sum q^{p_L^2/2} \bar{q}^{p_R^2/2}$ while the sum is on the winding and momentum modes of the T^2.

$\mathcal{Z}_{\rm new}$

Evaluating the trace we get,

$$\mathcal{Z}_{\rm new} = \frac{1}{\eta^2(\tau)} \frac{\Gamma^{(r,s)}_{2,2}(q,\bar{q})}{\eta^2(\tau)} \left[\frac{\theta_2^6(\tau)}{\eta^6(\tau)} \Phi_R^{(r,s)} + \frac{\theta_3^6(\tau)}{\eta^6(\tau)} \Phi_{NS^+}^{(r,s)} - \frac{\theta_4^6(\tau)}{\eta^6(\tau)} \Phi_{NS^-}^{(r,s)} \right] \frac{E_4(q)}{\eta^8(\tau)}.$$

Now we shall see $\mathcal{Z}_{\rm new}$ can be written in terms of twisted elliptic genus.

$\mathcal{Z}_{\rm new}$

- Now from the bosonic oscillators of T^2, we obtain $\frac{\Gamma_{2,2}^{(r,s)}}{\eta^2}$. The lattice sum is defined as

$$\begin{aligned}
\Gamma_{2,2}^{(r,s)}(q,\bar{q}) &= \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}}\,\bar{q}^{\frac{p_R^2}{2}}\,e^{2\pi i m_1 s/N}, \\
\frac{1}{2}p_R^2 &= \frac{1}{2T_2U_2}|-m_1U+m_2+n_1T+n_2TU|^2, \\
\frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2+m_1n_1+m_2n_2\,.
\end{aligned}$$

T, U are the complex Kahler and complex structure moduli of the torus T^2.

$\mathcal{Z}_{\text{new}}$

- The partition function on the $D6$ lattice in the various sectors are given by

$$\mathcal{Z}_R(D6;q) = \frac{\theta_2^6}{\eta^6}, \quad \mathcal{Z}_{NS^+}(D6;q) = \frac{\theta_3^6}{\eta^6}, \quad \mathcal{Z}_{NS^-}(D6;q) = \frac{\theta_4^6}{\eta^6}.$$

- The indices on the combined $D2K3$, $(6,6)$ conformal field theory are given by

$$\begin{aligned}
\Phi_R^{(r,s)} &= \frac{1}{N}\mathrm{Tr}_{R\,R,g'^r}[g'^s(-1)^{F_R}q^{L_0-c/24}\bar{q}^{\bar{L_0}-\bar{c}/24}],\\
\Phi_{NS^+}^{(r,s)} &= \frac{1}{N}\mathrm{Tr}_{NS\,R,g'^r}[g'^s(-1)^{F_R}q^{L_0-c/24}\bar{q}^{\bar{L_0}-\bar{c}/24}],\\
\Phi_{NS^-}^{(r,s)} &= \frac{1}{N}\mathrm{Tr}_{NS\,R,g'^r}[g'^s(-1)^{F_R+F_L}q^{L_0-c/24}\bar{q}^{\bar{L_0}-\bar{c}/24}].
\end{aligned}$$

- The partition function of the untouched E_8 lattice gives $\frac{E_4(q)}{\eta^8(\tau)}$

$\mathcal{Z}_{\rm new}$ in terms of twisted elliptic genus

- Now the twisted elliptic genus of $K3$ which is defined as

$$F^{(r,s)}(\tau, z) = \frac{1}{N}\mathrm{Tr}_{R\, Rg'^r}[(-1)^{F_{K3}+\bar{F}_{K3}} g'^s e^{2\pi i z F_{K3}} q^{L_0 - c/24} \bar{q}^{\bar{L_0}-\bar{c}/24}].$$

- g' belongs to to automorphism related to the 26 conjugacy classes of M_{24}.
- Since this theory admits a $\mathcal{N} = 2$ spectral flow we can relate the trace over the various sectors in $\Phi^{(r,s)}$ by the following equations

$$\begin{aligned}
\Phi_R^{(r,s)} &= F^{(r,s)}(\tau, \frac{1}{2}), \\
\Phi_{NS^+}^{(r,s)} &= q^{1/4} F^{(r,s)}(\tau, \frac{\tau+1}{2}), \\
\Phi_{NS^-}^{(r,s)} &= q^{1/4} F^{(r,s)}(\tau, \frac{\tau}{2}).
\end{aligned}$$

$\mathcal{Z}_{\rm new}$ in terms of Eisenstein series

- Define Jacobi forms of index 1 $A(\tau, z)$ and $B(\tau, z)$ can be written as

$$A(\tau, z) = \frac{\theta_2^2(\tau, z)}{\theta_2^2(\tau, 0)} + \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau, 0)} + \frac{\theta_4^2(\tau, z)}{\theta_4^2(\tau, 0)}, \qquad B(\tau, z) = \frac{\theta_1^2(\tau, z)}{\eta^6(\tau)}.$$

- Twisted elliptic genus can be written as

$$F^{(0,0)}(\tau, z) = \alpha_{g'}^{(0,0)} A(\tau, z),$$

$$F^{(0,1)}(\tau, z) = \alpha_{g'}^{(0,1)} A(\tau, z) + \beta_{g'}^{(0,1)} f_{g'}^{(0,1)}(\tau) B(\tau, z),$$

$f_{g'}^{(0,1)}(\tau)$ is a weight 2 modular form of some subgroup of $SL_2(Z)$

$\mathcal{Z}_{\rm new}$ in terms of Eisenstein series

Using $\theta_2\theta_3\theta_4 = 2\eta^3$ we have

$$A(\tau, \tfrac{1}{2}) = \frac{\left(\theta_4^4\theta_2^2 + \theta_3^4\theta_2^2\right)}{4\eta^6}, \qquad B(\tau, \tfrac{1}{2}) = \frac{\theta_2^2}{\eta^6},$$

$$A(\tau, \tfrac{\tau+1}{2}) = \frac{q^{-1/4}\left(-\theta_4^4\theta_3^2 + \theta_2^4\theta_3^2\right)}{4\eta^6}, \qquad B(\tau, \tfrac{\tau+1}{2}) = \frac{q^{-1/4}\theta_3^2}{\eta^6},$$

$$A(\tau, \tfrac{\tau}{2}) = \frac{q^{-1/4}\left(\theta_3^4\theta_4^2 + \theta_2^4\theta_4^2\right)}{4\eta^6}, \qquad B(\tau, \tfrac{\tau}{2}) = -\frac{q^{-1/4}\theta_4^2}{\eta^6}.$$

using them in $\mathcal{Z}_{\rm new}$ we obtain

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = -2\frac{1}{\eta^{24}}\sum_{r,s=0}^{N-1}\Gamma_{2,2}^{(r,s)}E_4\left[\frac{1}{4}\alpha_{g'}^{(r,s)}E_6 - \beta_{g'}^{(r,s)}f_{g'}^{(r,s)}E_4\right].$$

Eisenstein series E_6, E_4 as well as $f^{(r,s)}$ are holomorphic in τ.

Gauge Threshold Corrections

Standard Embedding; Gauge Threshold

If the Wilson line is turned on in the untouched E_8 lattice from Cardoso-Curio-Lust: 1996, Stieberger:1998 and Datta, David:2015, we see that the new supersymmetric index with Wilson line becomes

$$\mathcal{Z}_{\rm new}(q,\bar q) = -2\frac{1}{\eta^{24}}\Gamma_{3,2}^{(r,s)} \otimes E_{4,1}\left[\frac{1}{4}\alpha_{g'}^{(r,s)}E_6 - \beta_{g'}^{(r,s)} f_{g'}^{(r,s)} E_4\right].$$

Presence of the Wilson line introduces an additional moduli V and with T, U. Now the lattices sums are given by

$$\begin{aligned}
\Gamma_{3,2}^{(r,s)}(q,\bar q) &= \sum_{\substack{m_1,m_2,n_2,b\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}}\,\bar q^{\frac{p_R^2}{2}}\, e^{2\pi i m_1 s/N},\\
\frac{p_R^2}{2} &= \frac{1}{4\,\mathrm{det Im}\Omega}\left|-m_1 U + m_2 + n_1 T + n_2(TU - V^2) + bV\right|^2,\\
\frac{p_L^2}{2} &= \frac{p_R^2}{2} + m_1 n_1 + m_2 n_2 + \frac{1}{4}b^2, \quad \Omega = \left(\begin{smallmatrix} U & V\\ V & T\end{smallmatrix}\right)
\end{aligned}$$

$\otimes$ product

$E_{4,1}(z)$ is Eisenstein series with the $U(1)$ chemical potential defined by

$$E_{4,1}(z) = \frac{1}{2}(\theta_3^6\theta_3^2(z) + \theta_4^6\theta_4^2(z) + \theta_2^6\theta_2^2(z))$$

Any Jacobi form of index 1, $f_{s,1}(\tau, z)$ such as $E_{4,1},\ E_{6,1}$ can be decomposed as:

$$f_{s,1}(\tau, z) = f_{s,1}^{\mathrm{even}}(\tau)\theta_{\mathrm{even}}(\tau, z) + f_{s,1}^{\mathrm{odd}}(\tau)\theta_{\mathrm{odd}}(\tau, z)$$

where $\theta_{\mathrm{even}}(z) = \theta_3(2\tau, 2z) \qquad \theta_{\mathrm{odd}}(z) = \theta_2(2\tau, 2z)$.
$\Gamma_{3,2}^{(r,s)} \otimes f_{s,1}$ is given by

$$\Gamma_{3,2}^{r,s} \otimes f_{s,1} = \Gamma_{3,2}^{r,s}(\mathrm{even}) f_{s,1}^{\mathrm{even}} + \Gamma_{3,2}^{r,s}(\mathrm{odd}) f_{s,1}^{\mathrm{odd}}\ ,$$

The even and odd parts of $\Gamma_{3,2}^{r,s}$ depends on b being even or odd.

Gauge Threshold

- The one loop corrections to the gauge coupling G is defined by the following integral over the fundamental domain

$$\Delta(T, U, V) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}(\mathcal{B}_G - b(G)),$$

$\mathcal{B}_G$ is given by $\left(\mathrm{a}\, q\partial_q - \frac{1}{8\pi\tau_2}\right) \mathcal{Z}^G_{\mathrm{new}}$. This occurs because of summing over the lattice weighted with the charge vectors.

- The constant $b(G)$ can be fixed by demanding the integral to be well defined in $\tau_2 \to \infty$ limit and a can be obtained by demanding $\mathcal{B}_G$ to be a good modular form.
- We have a similar action for the other lattice $\mathcal{B}_{G'}$

Gauge Threshold

- $\mathcal{B}_G$ thus would be given by,

$$\mathcal{B}_G = -\frac{2}{24\eta^{24}}\Gamma_{3,2}^{(r,s)} \otimes \left\{\tilde{E}_2 E_{4,1} - E_{6,1}\right\}\left[\frac{1}{4}\alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)} f_{g'}^{(r,s)} E_4\right],$$

where

$$\tilde{E}_2 = \left(E_2 - \frac{3}{\pi\tau_2}\right).$$

- Similarly for G' it becomes

$$\mathcal{B}_{G'} = -\frac{2}{24\eta^{24}}\Gamma_{3,2}^{(r,s)} \otimes E_{4,1}\left[\frac{1}{4}\alpha_{g'}^{(r,s)}\left\{\tilde{E}_2 E_6 - E_4^2\right\}\right.$$
$$\left. -\beta_{g'}^{(r,s)} f_{g'}^{(r,s)}\left\{\hat{E}_2 E_4 - E_6\right\}\right].$$

- For the second equation Ramanujan identities involving derivatives of Eisenstein series were used.

Gauge Threshold Differences

- We can use the identities

$$\begin{aligned}\frac{1}{\eta^{24}}(E_{4,1}(\tau,z)E_6 - E_{6,1}(\tau,z)E_4) &= -144B(\tau,z),\\ \frac{1}{\eta^{24}}(E_{4,1}(\tau,z)E_4^2 - E_{6,1}(\tau,z)E_6) &= 576A(\tau,z),\end{aligned}$$

- The difference in the one loop thresholds integrands gives

$$\mathcal{B}_G - \mathcal{B}_{G'} = -12\Gamma_{3,2}^{(r,s)} \otimes F^{(r,s)}.$$

Thus the difference in the one loop corrections to gauge couplings is given by

$$\Delta_G(T,U,V) - \Delta_{G'}(T,U,V) = -12\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\Gamma_{3,2}^{(r,s)} \otimes F^{(r,s)}$$

Explicit construction of 2A orbifold of $K3$

Standard Embedding; 2A conjugacy class

- It was shown by Datta, David, Lust for a the heterotic string compactified on a 2A orbifold of $K3$ and a half shift along one of the circles of T^2. The modified $\mathcal{Z}_{\rm new}$ for standard embedding was be given by,

$$\mathcal{Z}_{\rm new} = -2\frac{E_4}{\eta^{24}}[\Gamma^{0,0}2E_6 + \Gamma^{0,1}\frac{2}{3}(E_6 + 2\mathcal{E}_2(\tau)E_4)$$
$$+\Gamma^{1,0}\frac{2}{3}(E_6 - \mathcal{E}_2(\tau/2)E_4) + \Gamma^{1,1}\frac{2}{3}(E_6 - \mathcal{E}_2(\frac{\tau+1}{2})E_4)]$$
$$\mathcal{E}_N(\tau) = \frac{12i}{\pi(N-1)}\partial_\tau \log[\frac{\eta(\tau)}{\eta(N\tau)}]$$

- Here $K3$ was realized as $T^4/\mathbb{Z}_2$ and g' was a $1/2$ shift along one of the T^4 directions and one of the T^2 directions.
- Thresholds and twisted elliptic genus can be explicitly checked and they match with our general discussions

Explicit construction of 2A orbifold of $K3 \sim T^4/\mathbb{Z}_4$

2A orbifold of $K3 \sim T^4/\mathbb{Z}_4$

- Various sectors of $T^4/\mathbb{Z}_4$ would be given by,
  $g^s: \ (e^{2\pi i s/4}(y_1, y_2), \ e^{-2\pi i s/4}(y_1, y_2))$ where y_i is a bosonic coordinate of T^4
- For $T^4/\mathbb{Z}_4$ realization of $K3$, define g' as:
  $g' : (x_1, x_2, y_1, y_2, y_3, y_4) :\rightarrow (x_1 + \pi, x_2, y_1 + \pi, y_2 + \pi, y_3 + \pi, y_4 + \pi)$
  where x_1, x_2 are T^2 coordinates.
- This is a $1/2$ shift in both of the complex bosons in $K3$. works even with a shift in one of the complex bosonic directions

Twisted Elliptic genus of 2A orbifold

- Define a quantity for each sector of orbifold of T^4 and of $K3$

$$\mathcal{F}(a, r, b, s) = \frac{1}{8} Tr_{g^a, g'^r}((-1)^{F+\bar{F}} g^b g'^s e^{2\pi iz\, F} q^{L_0} \bar{q}^{\bar{L}_0})$$

- This gives the twisted elliptic genus

$$F^{r,s} = \frac{1}{8} \sum_{a,c} Tr_{g^a, g'^r}((-1)^{F+\bar{F}} g^b g'^s e^{2\pi izF} q^{L_0} \bar{q}^{\bar{L}_0})$$

- $F^{0,0}$ constitutes all the terms coming from twisted and untwisted sectors of g without any insertion of g' and from the untwisted sector of g' only so $F^{0,0} = \frac{1}{2}$(elliptic genus of $K3$)$= \frac{1}{2} Z_{K3}$.
- $F^{0,0}$ would always be proportional to Z_{K3}, for any realization N=2,3,4,6 etc.

Fixed points of 2A orbifold

To evaluate each sector of the above twisted elliptic genus we will need the fixed point under the elements $g^a g'^r$ and what elements preserve these fixed points. Example:

	Fixed points	g'	g	g^2	g^3	$g'g$	$g'g^2$	$g'g^3$
g^2	$0, \frac{(1+i)}{2}$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$
	$\frac{1}{2}, \frac{i}{2}$	$\times$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$

Table: Each row lists the property of fixed points along the y_1, y_2 direction under actions of powers of g, g'. Positions are in units of 2π Similarly y_3, y_4 direction.

From the table:

$$\mathcal{F}(2,0;1,1) = \frac{1}{2}\frac{\theta_1(z+\frac{2\tau+1}{4},\tau)\theta_1(-z+\frac{2\tau+1}{4})}{\theta_1^2(\frac{2\tau+1}{4},\tau)},$$

$$\mathcal{F}(2,0;3,1) = \frac{1}{2}\frac{\theta_1(z+\frac{2\tau+3}{4},\tau)\theta_1(-z+\frac{2\tau+3}{4})}{\theta_1^2(\frac{2\tau+3}{4},\tau)}.$$

Twisted Elliptic genus of 2A orbifold

- Observing the fixed points for different components of $\mathcal{F}$

$$
\begin{aligned}
F^{(0,1)}(\tau, z) &= \mathcal{F}(0,0;1,1) + \mathcal{F}(0,0;2,1) + \mathcal{F}(0,0;3,1) \\
&\quad + \mathcal{F}(2,0;1,1) + \mathcal{F}(2,0;3,1), \\
&= 4\frac{\theta_2^2(z,\tau)}{\theta_2^2(0,\tau)}, \\
&= \frac{4}{3}A(\tau, z) - \frac{2}{3}\mathcal{E}_2(\tau)B(\tau, z).
\end{aligned}
$$

- The second equality is due to identities involving theta functions.
- So the twisted elliptic genus of the orbifold given in belongs to the class $2A$ Eguchi, Hikami 2010, also obtained by Datta, David, Lust 2015 by $T^4/\mathbb{Z}_2$ realization of $K3$
- Use modular transformation laws to reach other sectors. David, Jatkar, Sen 2006

New supersymmetric index of 2A orbifold

Evaluating the trace, the new supersymmetric index splits into

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = -\frac{1}{2\eta^{20}(\tau)}\sum_{a,b=0}^{3}\sum_{r,s=0}^{1} e^{-\frac{2\pi iab}{16}} Z_{E_8}^{(a,b)}(\tau)\times E_4(q) \times\frac{1}{8}F(a,r,b,s;q)\Gamma_{2,2}^{(r,s)}(q,\bar{q})$$

The trace over the T^4 directions is given by

$$F(a,r,b,s;q) = \mathrm{Tr}_{g^a\, g'^r R}\left(g^b g'^s e^{i\pi F_R^{T^4}} q^{L_0}\bar{q}^{\bar{L}_0}\right).$$

It becomes:

$$F(a,r,b,s;q) = k^{(a,r,b,s)}\eta^2(\tau) q^{\frac{-a^2}{16}}\frac{1}{\theta_1^2(\frac{a\tau+b}{4},\tau)}.$$

New supersymmetric index of 2A orbifold

The coefficients $k^{(a,r,b,s)}$ for the various values of (r, s) are given by the following matrices

$$k^{(a,0,b,0)} = 16 \begin{pmatrix} 0 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad k^{(a,0,b,1)} = 16 \begin{pmatrix} 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdots\cdots$$

Only those terms are picked up where the fixed points donot move.
E_8 lattice the components

$$Z_{E_8}^{a,b} = \frac{1}{2} \sum_{\alpha,\beta} e^{-i\pi\beta a/4 \sum \gamma^i} \prod_i \theta \left[{\alpha+2\frac{a}{4}\gamma^i \atop \beta+2\frac{b}{4}\gamma^i} \right]$$

where the generalized theta functions are given by

$$\theta \left[{a \atop b} \right] (\tau) = \sum_{k \in \mathcal{Z}} q^{(k+a/2)^2/2} e^{i\pi b(k+a/2)}$$

And γ for standard shift is (1,-1,0^6)

New supersymmetric index of 2A orbifold

So using the components we can add all these sectors which gives

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = -\frac{2}{\eta^{24}(\tau)}\sum_{r,s=0}^{1}\Gamma_{2,2}^{(r,s)}E_4\left[\frac{1}{4}\alpha_{2A}^{(r,s)}E_6-\beta_{2A}^{r,s}f_{2A}^{(r,s)}(\tau)E_4\right],$$

$$\alpha_{2A}^{(0,0)}=4,\qquad \beta_{2A}^{(0,0)}=0,$$

$$\alpha_{2A}^{(0,1)}=\frac{4}{3},\qquad \beta_{2A}^{(0,1)}=-\frac{2}{3},$$

$$\alpha_{2A}^{(1,0)}=\alpha_{2A}^{(1,1)}=\frac{4}{3},\qquad \beta_{2A}^{(1,0)}=\beta_{2A}^{(1,1)}=\frac{1}{3},$$

$$f_{2A}^{(0,1)}(\tau)=\mathcal{E}_2(\tau),\quad f_{2A}^{(1,0)}(\tau)=\mathcal{E}_2(\frac{\tau}{2}),\quad f_{2A}^{(1,1)}(\tau)=\mathcal{E}_2(\frac{\tau+1}{2}).$$

This was as expected from 2A orbifold, the gauge thresholds can be calculated similarly.

New supersymmetric index of 2B orbifold

We have also calculated the new supersymmetric index for the 2B conjugacy class using $\mathrm{su}(2)^6$ realization of $K3$, the twisted elliptic genus was constructed by Gaberdiel, Volpato 2013

$$\mathcal{Z}_{\rm new}|_{(0,1)} = -2\frac{1}{\eta^{24}(\tau)}\Gamma_{2,2}^{(0,1)} \times E_4 \left[-\frac{1}{2}(\mathcal{E}_2(\tau) - 2\mathcal{E}_4(\tau))\right] E_4$$

$$\mathcal{Z}_{\rm new}|_{(0,2)} = -2\frac{1}{\eta^{24}(\tau)}\Gamma_{2,2}^{(0,2)} \times E_4 \times \left(-\frac{1}{6}E_6 + \frac{2}{3}\mathcal{E}_2(\tau)E_4\right)$$

The threshold differences again could be given by theta lift of the twisted elliptic genus.

Massless Spectrum

Massless spectrum

- Define the shift action $V = \frac{1}{4}(\gamma, \tilde{\gamma})$
  The massless states in g^n twisted sector is determined by setting left and right mass formula to zero

$$\begin{aligned} m_L^2 &= N_L + \frac{1}{2}(P + nV)^2 + E_n - 1 = 0, \\ m_R^2 &= N_R + \frac{1}{2}(r + nv)^2 + E_n - \frac{1}{2} = 0. \end{aligned}$$

  P is the $E_8 \times E_8$ lattice vector $P = (P_{E_8}; P_{E_8'})$.
- $P_{E_8}(P_{E_8'})$ may belong to vector or spinor conjugacy classes given as

$$\lambda_A = (n_1, n_2 \ldots . n_8) \qquad \lambda_B = (n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, \cdots, n_8 + \frac{1}{2}),$$

$$\text{with} \qquad \sum_{i=1}^{8} n_i = \text{even integer}.$$

Massless spectrum

- E_n is the shift in zero point energy

$$E_n = \frac{1}{4^2} n(4-n)$$

  where $n = 0, 1, 2, 3$.

- r is a $SO(8)$ weight vector with

$$\sum_{i=1}^{4} r_i = \text{ odd},$$

  v is a 4 dimensional vector given by

$$v = \frac{1}{4}(0, 0, 1, 1).$$

Massless spectrum

- The degeneracy $D(n) = \frac{1}{4}\sum_{m=0}^{3} \chi(n,m)\Delta(n,m)$ of the massless states can be obtained from Aldazabal, Font, Quevedo:1995

$$\Delta(n,m) = \exp\left\{2\pi i[(r+nv)mv - (P+nV)mV + \frac{1}{2}mn(V^2 - v^2) + m\rho]\right\}$$

- $\chi(n,m)$ is the number of fixed points in the g^n twisted sector invariant under the action of g^m. ρ is the phase by which the oscillators in the T^4 are rotated by the $\mathbb{Z}_4$ action.
- In the untwisted sector we have $\chi(0,m) = 1$, others give:

$$\chi(1,m) = \chi(3,m) = 4,$$
$$\chi(2,0) = 16, \qquad \chi(2,1) = 4, \qquad \chi(2,2) = 16, \qquad \chi(2,3) = 4.$$

Massless spectrum

- We need to see how the degeneracy changes when a further 2A action is present.
- Generically massless states don't arise from the twisted sectors of g' as all these states have half integer Kaluza-Klein modes on T^4 and therefore they are massive
- We insert the projection over g'

$$D(n; g') = \frac{1}{4} \sum_{m=0}^{3} \frac{1}{2} \left[\chi(n, m) + \chi^{(g')}(n, m) \right] \Delta(n, m),$$

where $\chi^{(g')}$ is the number for fixed points in the g^n twisted sector invariant under the action of $g^m g'$.

$$\chi^{(g')}(1, m) = \chi^{(g')}(3, m) = 0,$$
$$\chi^{(g')}(2, 0) = \chi^{(g')}(2, 2) = 0,$$
$$\chi^{(g')}(2, 1) = \chi^{(g')}(2, 3) = 4.$$

Massless spectrum

- For standard embedding the gauge group breaks from $E_8 \times E_8$ to $E_7 \times U(1) \times E_8$
- Non Abelian vector multiplets and the gravity multiplets come from the untwisted sectors of g
- Non-Abelian $\mathcal{N} = 2$ vector multiplets are in adjoints of the gauge group **133** of E_7 and **248** of E_8.
- Since $\chi^{(g')}(1, m) = \chi^{(g')}(3, m) = 0$ so degeneracy in g^1, g^3 twisted sectors just becomes half of the original $K3$.

Massless spectrum Degeneracy

Model	Shift	Sector	Matter	$N_h - N_v$
$(T^4/\mathbb{Z}_4 \times T^2)/2A$	$E_7 \times U(1) \times E_8$	g^0	$(\mathbf{56}, \mathbf{1}) + \mathbf{2}(\mathbf{1}, \mathbf{1})$	-12
		$g + g^3$	$2(\mathbf{56}, \mathbf{1}) + 16(\mathbf{1}, \mathbf{1})$	
	$\frac{1}{4}(1, -1, 0^6; 0^8)$	g^2	$3(\mathbf{56}, \mathbf{1}) + 16(\mathbf{1}, \mathbf{1})$	

Model	Shift	Sector	Matter	$N_h - N_v$
$(T^4/\mathbb{Z}_4 \times T^2)$	$E_7 \times U(1) \times E_8$	g^0	$(\mathbf{56}, \mathbf{1}) + \mathbf{2}(\mathbf{1}, \mathbf{1})$	244
		$g + g^3$	$4(\mathbf{56}, \mathbf{1}) + 32(\mathbf{1}, \mathbf{1})$	
	$\frac{1}{4}(1, -1, 0^6; 0^8)$	g^2	$5(\mathbf{56}, \mathbf{1}) + 32(\mathbf{1}, \mathbf{1})$	

Massless spectrum $N_h - N_v$

- The massless $N_h - N_v$ values is obtained from the new supersymmetric index as [Harvey Moore 1995]

$$N_h - N_v = \frac{1}{4}\eta^4 \left(\sum_{s=0}^{N-1} \mathcal{Z}_{\rm new}^{(0,s)} \right) \Bigg|_{q^{1/6}} ,$$

- This is -12 for 2A orbifold of $K3$ where $K3 = T^4/\mathbb{Z}_2$ or $T^4/\mathbb{Z}_4$
- The value of $N_h - N_v$ was 244 for all embeddings of $K3$ without any orbifold action as expected from anomaly cancellation in 6D.
- Since the g' action also produces a shift in one of the T^2 directions so the theory cannot be lifted to 6D.

Non-Standard Embeddings of 2A orbifold of $K3$

Non-Standard Embeddings

The shifts γ^i, $\tilde{\gamma}^i$ were given by Stieberger 1998.

γ	$\tilde{\gamma}$	Type	$N_h - N_v$
(1,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	Type 0	-12
(1,1,0,0,0,0,0,0)	(2,2,0,0,0,0,0,0)		
(3,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)		
(1,1,0,0,0,0,0,0)	(4,0,0,0,0,0,0,0)		
(3,1,0,0,0,0,0,0)	(4,0,0,0,0,0,0,0)	Type 1	52
(3,1,0,0,0,0,0,0)	(2,2,0,0,0,0,0,0)		
(2,1,1,0,0,0,0,0)	(2,0,0,0,0,0,0,0)		
(1,1,0,0,0,0,0,0)	(1,1,1,1,1,1,1,-1)	Type 2	84
(2,1,1,0,0,0,0,0)	(2,2,2,0,0,0,0,0)		
(3,1,1,1,1,1,0,0)	(2,0,0,0,0,0,0,0)	Type 3	116
(3,1,1,1,1,1,0,0)	(2,2,2,0,0,0,0,0)		
(1,1,1,1,1,1,-1)	(3,1,0,0,0,0,0,0)		

Table: Lattice shifts in the $2A$ orbifold with $K3 = T^4/\mathbb{Z}_4$ and $N_h - N_v$

Non-standard embeddings of 2A orbifold

- For any of the shifts we observed $\mathcal{Z}_{\rm new}$ can be written as,

$$
\begin{aligned}
\mathcal{Z}_{\rm new} &= -\frac{1}{\eta^{24}}\Big\{2\Gamma_{2,2}^{(0,0)}E_4E_6 \\
&\quad +\Gamma_{2,2}^{(0,1)}\left[(E_6+2\mathcal{E}_2(\tau)E_4)\left(\hat{b}\mathcal{E}_2^2(\tau)+(\frac{2}{3}-\hat{b})E_4\right)\right]+\cdots\Big\}
\end{aligned}
$$

- The value of $\hat{b}$ depends upon the difference between numbers of hypers and vectors and the relation is $N_h - N_v = 144\hat{b} - 12$.
- There are 4 different values of $\hat{b}$ for all the 12 non standard shifts noted by Stieberger.

Presence of Wilson line

If the Wilson line is present $\mathcal{Z}_{\rm new}$ for all the embeddings can be summarized as:

$$\mathcal{Z}_{\rm new} = -\frac{1}{\eta^{24}}\{\Gamma^{(0,0)}_{3,2} \otimes \frac{1}{12}[n_1 E_{4,1} E_6 + n_2 E_{6,1} E_4]$$
$$+\Gamma^{(0,1)}_{3,2} \otimes [\hat{a} E_{4,1}(E_6 + 2\mathcal{E}_2(\tau)E_4) + \hat{b}\mathcal{E}_2(\tau)^2(E_{6,1} + 2\mathcal{E}_2(\tau)E_{4,1})$$
$$+\hat{c}E_4(E_{6,1} + 2\mathcal{E}_2(\tau)E_{4,1})] + \cdots\}$$

$\hat{a}, \hat{c}$ depend on the instanton numbers n_1 of the embedding and the value of $\hat{b}$ by

$$\hat{a} = \frac{n_1}{36} - \frac{\hat{b}}{2}$$
$$\hat{c} = \frac{2}{3} - \hat{a} - \hat{b}$$

For standard embedding $\hat{b} = 0,\ n_1 = 24$.

List of embeddings classified by $\hat{b}$

Type	γ	$\tilde{\gamma}$	(n_1, n_2)	$\hat{a}$	$\hat{b}$	$\hat{c}$
Type 0	(1,-1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	(24,0)	2/3	0	0
	(1,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	(24,0)	2/3	0	0
	(3,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	(24,0)	2/3	0	0
	(1,1,0,0,0,0,0,0)	(2,2,0,0,0,0,0,0)	(12,12)	1/3	0	1/3
Type 1	(1,1,0,0,0,0,0,0)	(4,0,0,0,0,0,0,0)	(16,8)	2/9	4/9	0
	(3,1,0,0,0,0,0,0)	(4,0,0,0,0,0,0,0)	(16,8)	2/9	4/9	0
	(3,1,0,0,0,0,0,0)	(2,2,0,0,0,0,0,0)	(20,4)	1/3	4/9	-1/9
Type 2	(2,1,1,0,0,0,0,0)	(2,0,0,0,0,0,0,0)	(12,12)	0	2/3	0
	(1,1,0,0,0,0,0,0)	(1,1,1,1,1,1,1,-1)	(6,18)	-1/6	2/3	1/6
Type 3	(2,1,1,0,0,0,0,0)	(2,2,2,0,0,0,0,0)	(12,12)	-2/9	8/9	0
	(3,1,0,0,0,0,0,0)	(1,1,1,1,1,1,1,-1)	(14,10)	-1/18	8/9	-1/6
	(3,1,1,1,1,1,0,0)	(2,0,0,0,0,0,0,0)	(12,12)	-1/9	8/9	-1/9
	(3,1,1,1,1,1,0,0)	(2,2,2,0,0,0,0,0)	(12,12)	-1/9	8/9	-1/9

Table: Lattice shifts for $((T^4/\mathbb{Z}_4) \times T^2)/g'$ and their $\hat{a}, \hat{b}, \hat{c}$ values

Gauge Thresholds Differences

Proceeding similarly as in standard embeddings we can get the gauge threshold differences

$$\Delta_G(T,U,V) - \Delta_{G'}(T,U,V) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\{\mathcal{B}_G - \mathcal{B}_{G'}\}$$

$$\begin{aligned}
= \quad & \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\Big\{\Gamma^{(0,0)} \otimes 2(n_2 - n_1)A(z) \\
& -\Gamma^{0,1} \otimes \left[24A(z)(\frac{n_1-12}{18}) - 12B(z)\mathcal{E}_2(\tau)(\frac{2}{3} - \frac{\hat{b}}{2})\right] \\
& -\Gamma^{(1,0)} \otimes \left[24A(z)(\frac{n_1-12}{18}) + 6B(z)\mathcal{E}_2(\frac{\tau}{2})(\frac{2}{3} - \frac{\hat{b}}{2})\right] \\
& -\Gamma^{(1,1)} \otimes \left[24A(z)(\frac{n_1-12}{18}) + 6B(z)\mathcal{E}_2(\frac{\tau+1}{2})(\frac{2}{3} - \frac{\hat{b}}{2})\right]\Big\},
\end{aligned}$$

Gauge Thresholds Differences

- The threshold integral would give

$$\Delta_G(T,U,V) - \Delta_{G'}(T,U,V) = 48 \Bigg((\frac{1}{2} - \frac{3\hat{b}}{8}) \log(\det(\mathrm{Im}(\Omega))^6 \, |\Phi_6(\mathrm{U},\mathrm{T},\mathrm{V})|^2) + (\frac{n_1}{72} - \frac{1}{3} + \frac{\hat{b}}{8}) \log(\det \left(\mathrm{Im}(\Omega)\right)^{10} |\Phi_{10}(\mathrm{U},\mathrm{T},\mathrm{V})|^2) + (\frac{n_1}{72} - \frac{1}{3} + \frac{\hat{b}}{8}) \log(\det \left(\mathrm{Im}(\Omega)\right)^{10} |\Phi_{10}(2\mathrm{U},\mathrm{T}/2,\mathrm{V})|^2) \Bigg)$$

- Φ_{10} is the unique cusp form of weight 10 under $Sp(2,\mathbb{Z})$
- Φ_6 is the Siegel modular form of weight 6, It is obtained from the theta lift of the elliptic genus of $K3$ twisted by the $2A$ orbifold action.
- As expected for the standard embedding $\hat{b} = 0, n_1 = 24$ the threshold integral reduces to only Φ_6.

Conclusions and Future Directions

Conclusions

1. We observed that for standard embeddings for any orbifold on $K3$ belonging to conjugacy classes of M_{24} the difference between one loop correction of the gauge couplings can be written as the theta lift of the twisted elliptic genus of $K3$
2. It was demonstrated by explicit construction of $K3$ as $T^4/\mathbb{Z}_4$ for $g' \in 2A$ and also for $K3 \sim \mathrm{su}(2)^6$ $g' \in 2B$ action was implemented
3. $\mathcal{Z}_{\mathrm{new}}$ and the difference between one loop correction of the gauge thresholds were also studied for the 2A orbifold on non-standard embeddings on $T^4/\mathbb{Z}_4$ and here also the structure of twisted elliptic genus was present
4. All nonstandard embeddings for $K3 \sim T^4/\mathbb{Z}_4$ can be classified into 4 types according to their $N_h - N_v$ values
5. We performed a consistency check by computing $N_h - N_v$ values from the the spectrum of these non-standard embeddings as well as from the new supersymmetric index and the results were in agreement.

Future Directions

1. We can generalize the study of non-standard embedding to all the orbifold limits of $K3$, here only the limits $T^4/\mathbb{Z}_2$ and $T^4/\mathbb{Z}_4$ were considered
2. Type II duals can be studied which would teach us more on S-duality and also involve study of new Calabi Yau manifolds
3. Consider compactifications of string theory of type II on $(K3 \times T^2)/g'$. g' in the conjugacy class pA, $p = 1, 2, 3, 5, 7$ was studied in David, Jatkar and Sen 2006, Dabholkar Nampuri 2007, Dijkgraaf, Verlinde, Verlinde 1997.... where degeneracy of 1/4 BPS dyons were computed.
4. It will be interesting to generalize the results to all the 26 conjugacy classes of M_{24}. This may teach us about black hole degeneracies in $\mathcal{N} = 4$ string theory and its relation to the symmetry M_{24}.

THANK YOU