Harmonic Oscillations of a soft Massive Spring

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Let the spring have mass m and an attached mass M. Let the spring have a spring constant K. Now, a fraction f of the spring will deform by ux if the entire spring deforms by x (if the spring is horizontal, so that we may ignore its own weight). Since all parts of the spring are under the same force, the spring constant of a fraction f of the spring must be K/u.

Now let's consider the extension of the spring under its own weight m and the attached weight M. We need a coordinate to represent position along the spring. Let the fraction of the spring which is above a particular point P be represented by u, so that u = 0 at $x = 0$ and u = 1 at $x = 1$, where it is attached to the weight M.

At any given point P, the tension in the spring F is given by

$$F = Mg + (1 - u)mg$$

and $(1 - u)mg$ is the fraction of mass below a point P.

A small section of the spring du has spring constant k/du as shown below :

$$\begin{aligned} F &= \frac{K}{du}dx & (1) \\ dx &= F\frac{du}{K} & (2) \end{aligned}$$

Integrating from zero to u gives the extension of the spring at the position u, $x(u)$ as below :

$$\begin{aligned} \int_0^u dx &= \int_0^u Mg + (1-u)mg\frac{du}{K} & (3) \\ x(u) &= \frac{Mg}{K}u + \frac{mg}{K}u - \frac{u^2}{2}\frac{mg}{K} & (4) \\ &= \frac{g}{K}(Mu + mu - m\frac{u^2}{2}) & (5) \\ &= \frac{g}{K}(Mu + m * (u - \frac{u^2}{2})) & (6) \end{aligned}$$

The displacement x is measured relative to the corresponding location if there were no load on the spring. So, if the spring has a natural length l, then the position relative to the anchor point is $x + ul$.

Therefore at rest, the spring is stretched by,

$$x(1) = \frac{g}{K}(M + \frac{m}{2}) \tag{7}$$

Now let's consider the spring in motion. If we look at a small section of the spring du, then the force on that section is its own weight $mgdu$ plus any difference in the tension in the spring below the section and the tension in the spring above the section. We find

$$F = mgdu + d\tau$$

.

We know that the tension in the spring is proportional to its extension

$$\tau = K\frac{dx}{du}$$

Therefore,

$$d\tau = K\frac{d^2x}{du^2}$$

We can now write the governing differential equation,

$$F \quad = \quad mgdu + K\frac{d^2x}{du^2}du$$

Therefore we get,

$$m\frac{d^2x}{dt^2}du \quad = \quad mgdu + K\frac{d^2x}{du^2}du \tag{8}$$

The solution must also satisfy the boundary conditions at $u = 0$ and $u = 1$.

$$\tau(1) = k\frac{dx}{du}\mid_{u=1} \quad = \quad MgM\frac{d^2x}{dt^2}\mid_{u=1} \tag{9}$$

$$\tau(0) = 0 \tag{10}$$

The solution is additive. $x = x_s + x_d$, where x_s is the static portion given above, and x_d is the dynamic part. Using,

$$x_s(u) = [Mu + (u - u^2/2)m]g/K \tag{11}$$

we can remove the static components from the differential equation and the boundary conditions. We find

$$K\frac{d^2x_d}{du^2} = m\frac{d^2x_d}{dt^2} \tag{12}$$

We can show this as below :

$$m\frac{d^2x_d}{dt^2} = K\frac{g}{K}(-\frac{m}{2})$$

$$\frac{d^2x}{du^2} = \frac{d^2}{du^2}(x_s(u) + x_d(u)) \tag{13}$$

$$= \frac{g}{K}\frac{d^2}{du^2}(Mu + mu - m\frac{u^2}{2}) \tag{14}$$

$$= \frac{g}{K}\frac{d}{du}(M + m - mu) \tag{15}$$

$$= \frac{g}{K}(-m) + \frac{d^2x_d}{du^2} \tag{16}$$

Therefore, from equation 8, and using equation 16,

$$m\frac{d^2x}{dt^2}du = mgdu + K(\frac{g}{K}(-m) + \frac{d^2x_d}{du^2})du \tag{17}$$

$$= K(\frac{d^2x_d}{du^2})du \tag{18}$$

Therefore,

$$m\frac{d^2x_d}{dt^2} = K\frac{d^2x_d}{du^2} \tag{19}$$

Now, $x_d(0) = 0$.

$$K\frac{dx_d}{du}\mid_{u=1} = -M\frac{d^2x_d}{dt^2}\mid_{u=1}$$

Let's solve the above differential equation. We expect to have a solution which is oscillatory, so we try $x_d = f(u)e^{i\omega t}$.

Hence, we get,

$$\frac{d^2x_d}{dt^2} = \frac{d}{dt}(f(u)i\omega e^{i\omega t})) \tag{20}$$

$$= f(u)(-\omega)^2e^{i\omega t} \tag{21}$$

$$= (-\omega)^2 f \tag{22}$$

Therefore,

$$K\frac{d^2f}{du^2} = -m\omega^2 f$$

or

$$\frac{d^2f}{du^2} + \frac{m\omega^2}{K}f = 0$$

Solution to this equation is :

$$f = A\sin pu$$

,

where $p^2 = m\omega^2/K$ and A is an arbitrary amplitude.

At this point, we have no restrictions on ω and one boundary condition left to apply.

Applying second boundary condition gives us the following

$$\left.\frac{dx_d}{du}\right|_{u=1} = \left.\frac{df}{du}\right|_{u=1} = A\cos pu \cdot p\ |_{u=1} = Ap\cos p \tag{23}$$

$$\left.\frac{d^2x_d}{dt^2}\right|_{u=1} = f(1)\frac{d}{dt^2}e^{i\omega t} = -\omega^2 f(1) \tag{24}$$

$$= -A\omega^2 \sin p \tag{25}$$

Therefore,

$$KA\ p\cos p = MA\ \omega^2 \sin p$$
$$K\cos p = M\ \omega^2 \sin p$$
$$Kp\cos p = M\ \omega^2 \sin p$$

The boundary condition requires that $K\ p\cos p = M\omega^2 \sin p$.

This condition restricts the allowed values of p and ω. Since p is the natural argument of the trigonometric functions, it makes the most sense to eliminate ω from the equation,

Substituting for $\omega = \sqrt{(K/m)}$, we get $p\tan p = m/M$.

This condition has infinitely many solutions, spaced roughly π apart. (There are both positive and negative solutions, but the negative solutions give the same answers as the positive solutions, since we have undetermined amplitudes and phases in our equations. We'll restrict our discussion to positive values of p.) The lowest-order solution is between zero and π. Let's look at two limits.

- Consider the case where M = 0. In this case, $p = \frac{\pi}{2}$, and $\omega = \frac{\pi}{2}\sqrt{\frac{K}{m}}$. There are no nodes of vibration, except the attachment point at u = 0, and there is no tension in the spring at u = 1.

- Second, consider the case where M >> m. In this case, we can expand

$$\tan p = p - \frac{p^3}{3} + o(p^5)$$

. From here, we can solve for p in terms of m/M.

$$p = \sqrt{\frac{m}{M}} \cdot \left(1 - \frac{1}{6}\left(\frac{m}{M}\right) + o\left(\frac{m}{M}\right)^2\right)$$

And since $\omega^2 = p^2\frac{K}{m}$, we get,

$$\omega^2 = \frac{m}{M}\left(1 - \frac{m}{3M} + \frac{m^2}{36M^2} + ...\right) \cdot \frac{K}{m} \tag{26}$$

$$\omega^2 = \frac{K}{M}\left(1 - \frac{m}{3M}\right) \tag{27}$$

We neglect the higer order terms. Hence, we obtain

$$\omega^2 = \frac{K}{M\left(1 - \frac{m}{3M}\right)^{-1}}$$

In the limit $M >> m$ we can write,

$$\omega^2 = \frac{K}{M\left(1 + \frac{m}{3M}\right)} \tag{28}$$

$$\omega = \left[\frac{K}{\left(M + \frac{m}{3}\right)}\right]^{1/2} \tag{29}$$

Since p is small, the spatial function f is nearly proportional to u, which is just what you would expect from a massless spring. That is, $\frac{df}{du}$ is constant and the spring expands uniformly. Interestingly, the static extension of the spring is proportional to $M+m/2$, whereas the vibration is more nearly dependent on $M + m/3$ (for small m).