Quark-Gluon Plasma and Heavy-Ion Collisions Phenomenology

(Tutorials of the Course)

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1 Relativistic Kinematics for RHIC and LHC

Notation: y = rapidity, η = pseudo-rapidity, η_s = spacetime-rapidity. (Do not confuse rapidity with the Cartesian coordinate y.)

Ex. 1. Prove

$$\left(y \equiv \tanh^{-1} v_z\right) \Leftrightarrow \left(y \equiv \frac{1}{2}\ln\frac{1+v_z}{1-v_z}\right) = \frac{1}{2}\ln\frac{E+p_z}{E-p_z} = \ln\frac{E+p_z}{m_T} = -\ln\frac{E-p_z}{m_T},$$

where $m_T \equiv \sqrt{p_T^2+m^2}$. Verify $E = m_T\cosh y$, $\;p_z = m_T\sinh y$. Consider the coordinate transformation $(E, p_x, p_y, p_z) \to (m_T, p_x, p_y, y)$. Derive the Jacobian.

Ex. 2. Show that in the *nonrelativistic* limit $y = v_z + \mathcal{O}(v_z^3)$.

Ex. 3. (a) Show that in the *extreme relativistic* (ER) limit (i.e., $p \equiv |\mathbf{p}| \gg m_0$), $y \simeq \eta \equiv -\ln\tan(\theta/2)$ where θ is the polar angle of the outgoing particle. ($\theta \gg 1/\gamma$ assumed. γ : Lorentz factor.) (b) Invert this relation and find θ for $\eta = 0, \pm1, \pm2, ..., \pm5$.

Ex. 4. Derive the Lorentz transformation relation for the rapidity. Do this using both the definitions of y. (This presents another reason for introducing the rapidity variable.)

Ex. 5. Consider two particles of rest masses $m_1,\ m_2$ and momenta $p_1,\ p_2$, both moving along the z-axis. (a) What are the rapidities of the two particles? (b) What is the rapidity of the CM frame? Express your answer in terms of $y_1,\ y_2$. (c) What is the rapidity of the CM frame and the rapidities of the two particles in the CM frame if $m_1 = m_2$?

Ex. 6. (a) Show explicitly that d^3p/E is invariant under a boost. (b) Show that d^3p/E can be written in a manifestly invariant form

$$2\int_{p^0=-\infty}^{+\infty} d^4p\ \theta(p^0)\ \delta(p^2-m_0^2),$$

where the integration is over p_0 only. (c) What about d^3x/t?

Ex. 7. Show that

$$d^3p/E = p_T dp_T dy d\phi = m_T dm_T dy d\phi.$$

Ex. 8. Show that

$$\frac{dN}{p_T dp_T d\eta} = \frac{p}{E}\frac{dN}{p_T dp_T dy} = \frac{p_T\cosh\eta}{\sqrt{m_0^2+p_T^2\cosh^2\eta}}\frac{dN}{p_T dp_T dy} = \sqrt{1-\frac{m_0^2}{m_T^2\cosh^2 y}}\frac{dN}{p_T dp_T dy}.$$

Ex. 9. Calculate the equivalent beam energy in lab, that would be required in a fixed-target experiment, corresponding to the maximum RHIC energy $\sqrt{s_{NN}} = 200$ GeV and the LHC energy 5.5 TeV.

Ex. 10. Recall **Ex. 1.** Prove

$$\left(\eta_s \equiv \tanh^{-1}\frac{z}{t}\right) \Leftrightarrow \left(\eta_s \equiv \frac{1}{2}\ln\frac{1+\frac{z}{t}}{1-\frac{z}{t}}\right) = \frac{1}{2}\ln\frac{t+z}{t-z} = \ln\frac{t+z}{\tau} = -\ln\frac{t-z}{\tau},$$

where $\tau \equiv \sqrt{t^2 - z^2}$. Verify $t = \tau\cosh\eta_s, \quad z = \tau\sinh\eta_s$. Consider the coordinate transformation $(t, x, y, z) \to (\tau, x, y, \eta_s)$. Derive the Jacobian.

2 Bjorken picture

Ex. 1. Show that $\delta(p_z t - Ez) = \delta(y - \eta_s)/(m_T \tau)$.

Ex. 2. Consider Cartesian coordinates (t, x, y, z) and curvilinear coordinates (τ, x, y, η_s), where $\tau = \sqrt{t^2 - z^2}$ and $\eta_s = \tanh^{-1}(z/t)$. (a) Write the components of the four-velocity $(u^\tau, u^x, u^y, u^{\eta_s})$ in terms of those of the four-velocity (u^t, u^x, u^y, u^z), and vice versa. (b) Using the definition $u^\mu = dx^\mu/ds$ where $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu$, show that $(u^t, u^x, u^y, u^z) = \gamma(1, d\mathbf{x}/dt)$ where $\gamma = [1 - (d\mathbf{x}/dt)^2]^{-1/2}$ and $(u^\tau, u^x, u^y, u^{\eta_s}) = \tilde{\gamma}(1, dx/d\tau, dy/d\tau, d\eta_s/d\tau)$ where $\tilde{\gamma} = [1-(dx/d\tau)^2-(dy/d\tau)^2-\tau^2(d\eta_s/d\tau)^2]^{-1/2}$. (c) Special case (Bjorken Picture): Ignore the x, y coordinates and let $v_z = z/t$. Show that $(u^t, u^z) = (t, z)/\tau = (\cosh\eta_s, \sinh\eta_s)$ and $(u^\tau, u^{\eta_s}) = (1, 0)$. What is the interpretation of the last equality? [Given $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1/\tau^2)$, $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -\tau^2)$, for the two sets of coordinates, respectively.]

3 Thermodynamics of QGP

Ex. 1. Show that the chemical potential of antiquarks is equal and opposite to that of quarks: $\mu_{\bar{q}} = -\mu_q$.

Ex. 2. Show that

$$\int_{-\beta\mu}^{\infty} dx \frac{(x+\beta\mu)^3}{e^x+1} + \int_{\beta\mu}^{\infty} dx \frac{(x-\beta\mu)^3}{e^x+1} = \frac{7\pi^4}{60} + \frac{\pi^2\beta^2\mu^2}{2} + \frac{\beta^4\mu^4}{4}.$$

Note that these two integrals cannot be done analytically one by one.

Ex. 3. Show that

$$\int_{-\beta\mu}^{\infty} dx \frac{(x+\beta\mu)^2}{e^x+1} - \int_{\beta\mu}^{\infty} dx \frac{(x-\beta\mu)^2}{e^x+1} = \frac{\pi^2\beta\mu}{3} + \frac{\beta^3\mu^3}{3}.$$

Note that these two integrals cannot be done analytically one by one.

Ex. 4. Let $\mu = 0$. Consider the four cases (a) $N_c = 2, N_f = 2$, (b) $N_c = 2, N_f = 3$, (c) $N_c = 3, N_f = 2$, and (d) $N_c = 3, N_f = 3$. Evaluate ε_g/π^2T^4, $(\varepsilon_q + \varepsilon_{\bar{q}})/\pi^2T^4$, and $\varepsilon_{QGP}/\pi^2T^4$. Comment on the relative contributions of gluons and quark-antiquarks in each case.

Ex. 5. Calculate explicitly the pressure due to an ideal gas of massless particles and show that it is 1/3 of its energy density.

4 Relativistic Transport Theory

Ex. 1. Show that $d^3x'd^3p' = d^3xd^3p[1 + \mathcal{O}(dt)^2]$, assuming the force to be independent of $\mathbf{p}$.

Ex. 2. (a) Derive the relativistic covariant Boltzmann equation in the presence of an external force $\mathcal{F}^\mu$, assuming that the collision term is not affected by the external force:

$$p^\mu \frac{\partial f}{\partial x^\mu} + m_0\, \mathcal{F}^\mu \frac{\partial f}{\partial p^\mu} = \mathcal{C}[f].$$

(b) Specialize to the case of the Lorentz force $\mathcal{F}^\mu = (q/m_0)F^{\mu\nu}p_\nu$, where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Find the space-time components of $\mathcal{F}^\mu$.

Ex. 3. Show that the zeroth and the first moments of the collision term $\mathcal{C}[f]$ in the Boltzmann equation $p^\mu \partial_\mu f = \mathcal{C}[f]$ vanish:

$$\int (d^3p/E)\; \mathcal{C}[f] = 0, \qquad \int (d^3p/E)\; p^\nu\; \mathcal{C}[f] = 0.$$

5 Perfect Non-Relativistic Hydrodynamics

Ex. 1. Apply Newton's second law of motion to a thin rectangular slab of fluid and derive Euler's equation. (Hint: You need to consider the time-derivative of $\vec{v}$, with respect to an observer co-moving with the fluid element, i.e., an observer at rest with respect to the fluid element.)

Ex. 2. Derive these alternate forms of Euler's equation:

(a) $\dfrac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}\left(\dfrac{H}{M}\right)$, where (H/M) is the enthalpy per unit mass.

(b) $\dfrac{\partial \vec{\Omega}}{\partial t} + \vec{\nabla} \times (\vec{\Omega} \times \vec{v}) = 0$, where $\vec{\Omega} \equiv \vec{\nabla} \times \vec{v}$ is the vorticity.

Hint: Rewrite $(\vec{v} \cdot \vec{\nabla})\vec{v}$ using the identity

$$\vec{\nabla}(\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} + (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{B} \times (\vec{\nabla} \times \vec{A}) + \vec{A} \times (\vec{\nabla} \times \vec{B}).$$

6 Perfect Relativistic Hydrodynamics

Ex. 1. Construct a projection operator $\mathcal{P}$ in three space dimensions, which when acting on an arbitrary 3-vector $\vec{A}$, annihilates that part of $\vec{A}$ which is parallel to a unit 3-vector $\hat{U}$, and projects out the part which is orthogonal to $\hat{U}$: $\mathcal{P}\hat{U} = 0, \mathcal{P}\hat{V} = \hat{V}$, where $\hat{V} \perp \hat{U}$.

Ex. 2. Verify $\Delta^{\mu\nu}\Delta_{\nu\sigma} = \Delta^\mu_\sigma$, Trace $\Delta = \Delta^\mu_\mu = 3$, $\Delta^{LRF}_{\mu\nu} = \Delta^{\mu\nu}_{LRF} = \mathrm{diag}(0, -1, -1, -1)$, $\Delta^\mu_{\nu,LRF} = \mathrm{diag}(0, 1, 1, 1)$.

Ex. 3. Consider a perfect fluid. Show that $V^\mu = 0$ even if Landau-Lifshitz's definition of U^μ is used. (Hint: First show that $V^\mu = 0$ in the LRF.)

Ex. 4. Consider a perfect fluid in its rest frame. Due to isotropy (spherical symmetry) the energy-momentum tensor in this frame is given by $T^{\mu\nu} = \mathrm{diag}(\epsilon, p, p, p)$, where ϵ is the energy density and p the usual hydrostatic pressure. Show by performing the Lorentz transformation that in a frame in constant motion with respect to this frame, $T^{\mu\nu}$ is given by $T^{\mu\nu} = (\epsilon+p)U^\mu U^\nu - pg^{\mu\nu}$, where U^μ is the hydrodynamic 4-velocity in the moving frame. (You may consider a 1+1-dimensional case for simplicity.)

Ex. 5. Discuss the other possible contractions of $T^{\mu\nu}$, such as $T^{\mu\nu}U_\nu$, $T^{\mu\nu}\Delta_{\nu\alpha}$, $T^{\mu\nu}\Delta_{\mu\nu}$, etc.

Ex. 6. (a) Verify $\Pi^{\mu\nu} = \Pi^{\nu\mu}$, $\Pi^{\mu\nu}U_\nu = 0 = U_\mu \Pi^{\mu\nu}$, and similarly for $\pi^{\mu\nu}$. (b) Thus $\pi^{\mu\nu}$ is symmetric, traceless and orthogonal to U^μ. Show that it may contain not more than 5 independent elements. (c) How many of the 4 elements of W^μ are independent? Why?

Ex. 7. Show that $(p + \Pi) \equiv -\frac{1}{3}T^{\mu\nu}\Delta_{\mu\nu}$.

Ex. 8. Consider a perfect fluid. We can write $T^{\mu\nu} = aU^\mu U^\nu + b\Delta^{\mu\nu}$, because the only second-rank tensors available to us are $U^\mu U^\nu$ and $g^{\mu\nu}$, or equivalently $U^\mu U^\nu$ and $\Delta^{\mu\nu}$. Show that $a = \epsilon$ and $b = -p$, so that

$$T^{\mu\nu} = \epsilon U^\mu U^\nu - p\Delta^{\mu\nu} = (\epsilon + p)U^\mu U^\nu - pg^{\mu\nu}$$

Ex. 9. Derive nonrelativistic limits of the above equations.

7 Flow

Ex. 1. Show that for a symmetric collision, $v_n(y)$ is an even (odd) function of y if n is even (odd); y is the rapidity.

8 Non-Relativistic (NR) limits of the perfect hydrodynamic equations

(a) Do not assume $c = 1$. Consider Lorentz transformations between the LRF of the fluid and another frame moving with respect to it. Write the components of x^μ and U^μ in the two frames.

(b) Consider successive approximations where terms up to orders $(v/c)^0$, $(v/c)^1$, $(v/c)^2$ are kept, where $\mathbf{v} = d\mathbf{x}/dt$. Write the components of U^μ to each order. Check the normalization of U^μ in each case.

(c) Show that

$$\Delta^{\mu\nu} = \begin{pmatrix} 0 & -v_x/c & -v_y/c & -v_z/c \\ -v_x/c & -1 & 0 & 0 \\ -v_y/c & 0 & -1 & 0 \\ -v_z/c & 0 & 0 & -1 \end{pmatrix} + \mathcal{O}(v^2/c^2),$$

$$\Delta_{\mu\nu} = \begin{pmatrix} 0 & v_x/c & v_y/c & v_z/c \\ v_x/c & -1 & 0 & 0 \\ v_y/c & 0 & -1 & 0 \\ v_z/c & 0 & 0 & -1 \end{pmatrix} + \mathcal{O}(v^2/c^2),$$

$$\Delta^{\mu}_{\nu} = \begin{pmatrix} 0 & v_x/c & v_y/c & v_z/c \\ -v_x/c & 1 & 0 & 0 \\ -v_y/c & 0 & 1 & 0 \\ -v_z/c & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(v^2/c^2).$$

(d) Show further that

$$\begin{aligned} \Theta &\equiv \partial_\mu u^\mu = \nabla \cdot \mathbf{v} + \mathcal{O}(v^2/c^2), \\ D &= \partial/\partial t + \mathbf{v} \cdot \nabla + \mathcal{O}(v^2/c^2), \\ \nabla^\mu &= -(\mathbf{v} \cdot \nabla/c,\ \nabla + \mathcal{O}(|\mathbf{v}|/c^2)) + \mathcal{O}(v^2/c^2). \end{aligned}$$

(e) Take NR limits of $\partial_\mu N^\mu = 0$ and $Dn + n\Theta = 0$, and show that in each case one gets the NR continuity equation $\partial\rho/\partial t + \nabla \cdot (\rho \mathbf{v}) = 0$.

(f) Show that the energy equation $D\epsilon + (\epsilon + p)\Theta = 0$ also yields, in the NR limit, the NR continuity equation for ρ. [Note: Although pressure p and energy density ϵ have the same dimensions, $p \sim c_s^2 \epsilon / c^2 \ll \epsilon$, where c_s is the speed of sound.

(g) Show that the momentum balance equation $(\epsilon + p) D u^\mu = c^2 \nabla^\mu p$, in the NR limit and for $\mu = i$, yields Euler's equation. Discuss carefully the equation resulting from $\mu = 0$.