XI SERC School on Experimental High-Energy Physics
National Institute of Science Education and Research

11th November 2017

STANDARD MODEL & BEYOND

Tutorial

Sreerup Raychaudhuri

TIFR, Mumbai

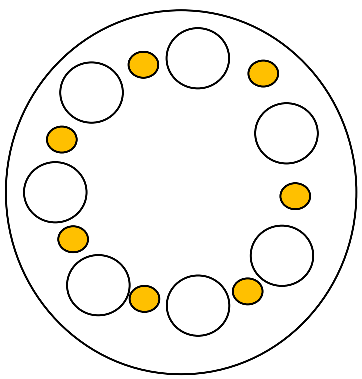
Spontaneous symmetry breaking

This arises when the action of a system has a particular symmetry, but the ground state does not...

Spontaneous symmetry breaking

This arises when the action of a system has a particular symmetry, but the ground state does not...

Example 1: Salam's banquet



People are sitting to dinner at a round table. Each has a plate in front and a glass on either hand. Before the meal, there is perfect symmetry between left glasses and right glasses.

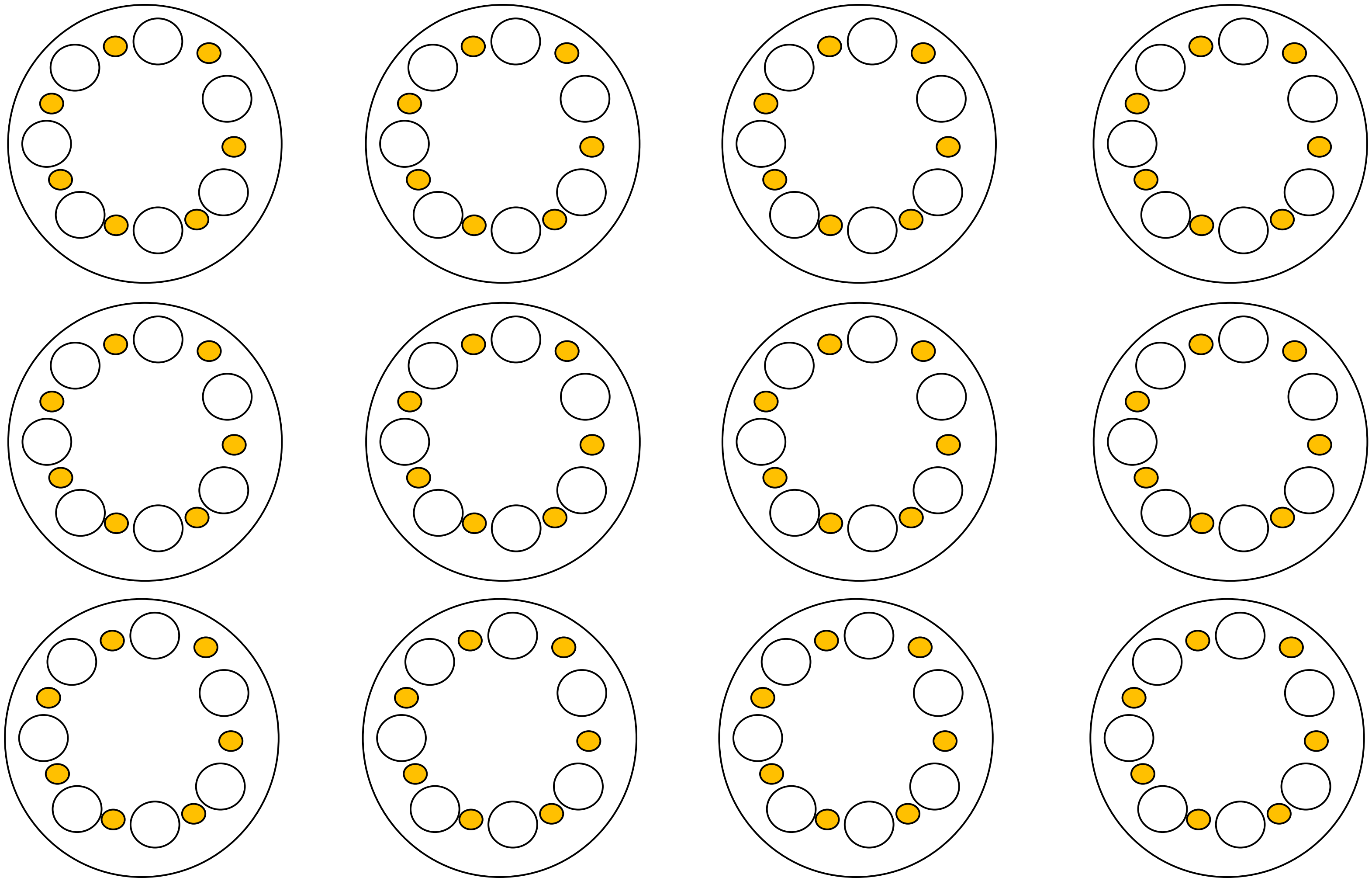
The first person to pick up a glass makes a random choice, say the left glass...

...now everyone must pick up the left glass...

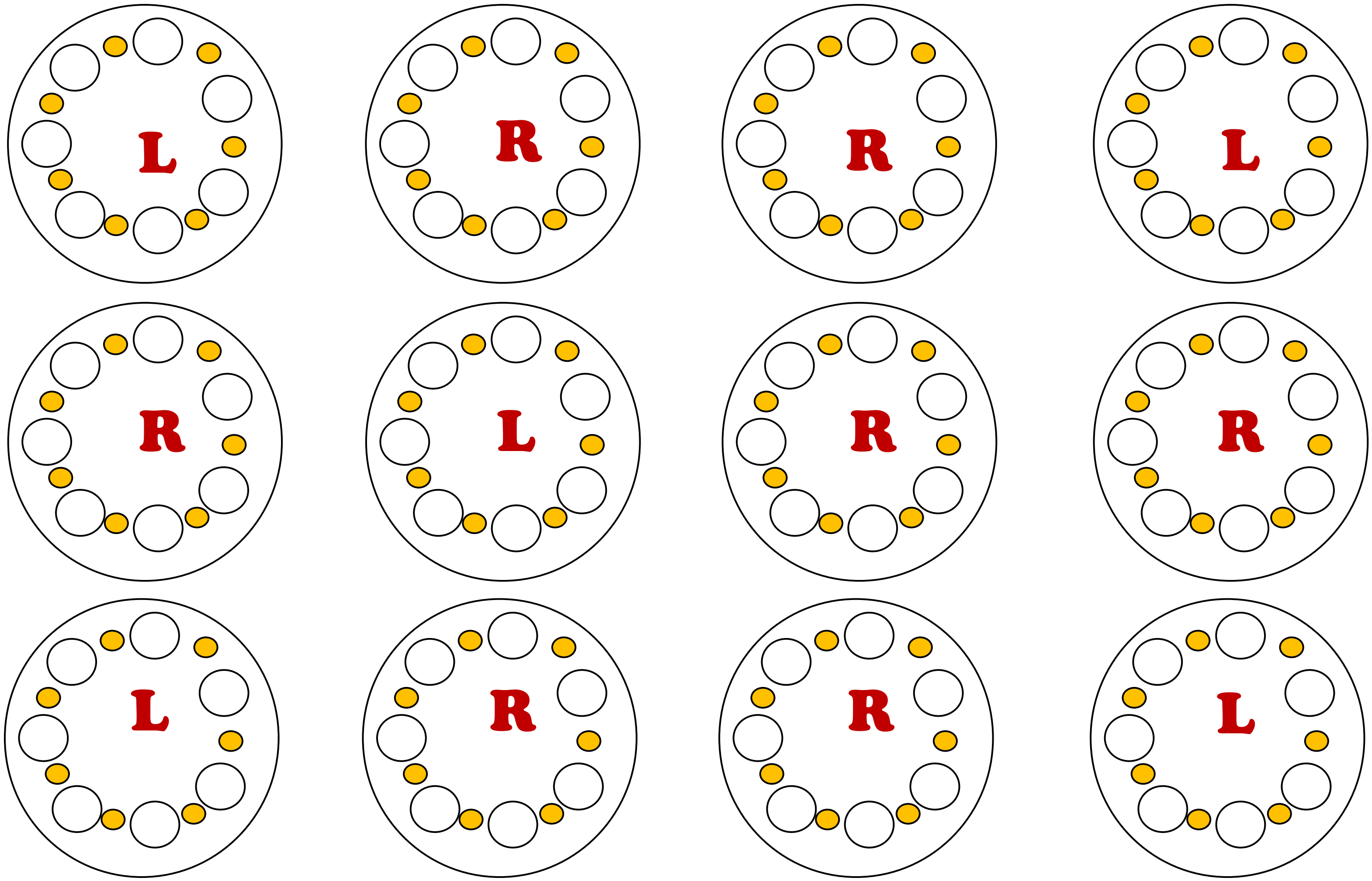
During the meal, there is no symmetry...

Has the symmetry really been destroyed?

No: if we consider an ensemble of systems...



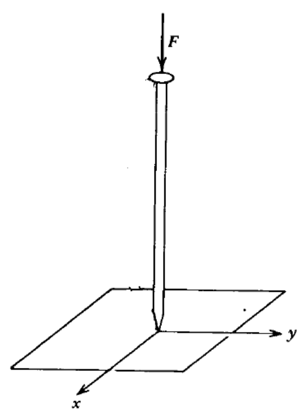
No: if we consider an ensemble of systems... the symmetry reappears!



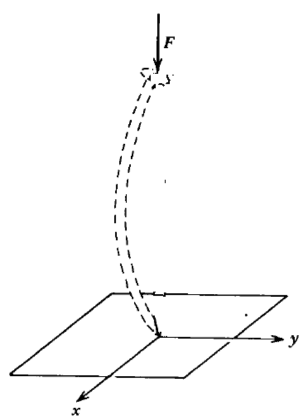
Spontaneous symmetry breaking

This arises when the action of a system has a particular symmetry, but the ground state does not...

Example 2: Bending of a knitting needle



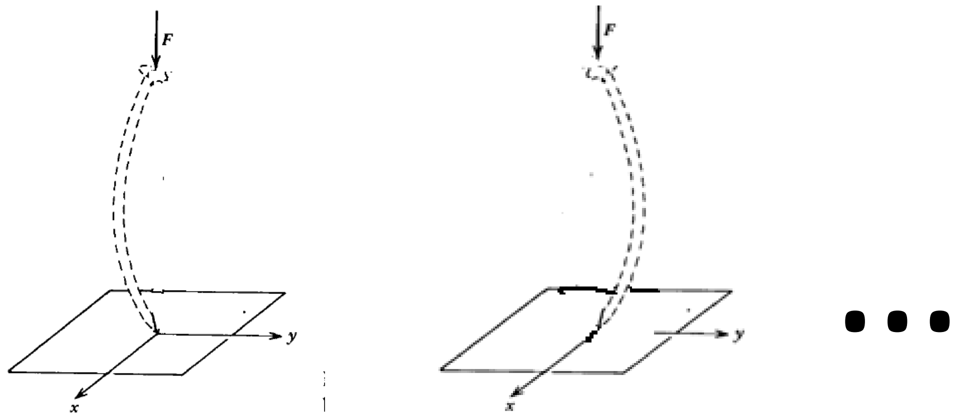
Before application of the force, the needle is erect along the z-axis... obeys azimuthal symmetry...



After application of the force, the needle bends in an arbitrary direction… azimuthal symmetry is lost…

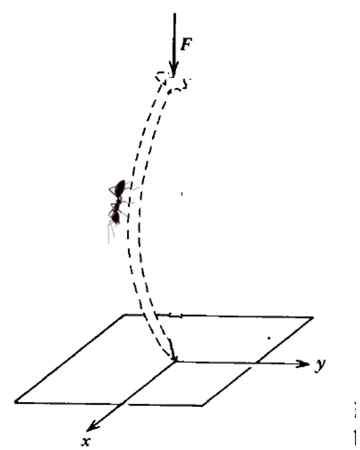
Has the symmetry really been destroyed?

No: if we consider an ensemble of systems... the symmetry reappears!



However, an ant clinging to the needle will never realise there was such a symmetry at all…

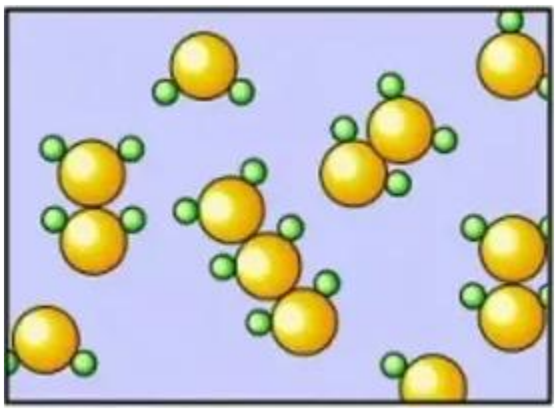
Hidden symmetry



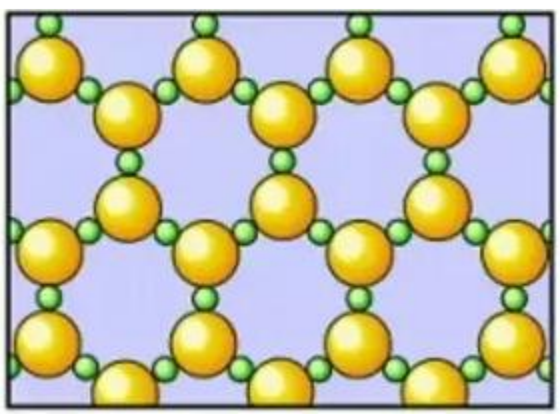
Spontaneous symmetry breaking ⇔ Hidden symmetry

This arises when the action of a system has a particular symmetry, but the ground state does not...

Example 3: Freezing of water



Above the melting point (0^0 C), all molecules are free to make random motions... obeys translation invariance...

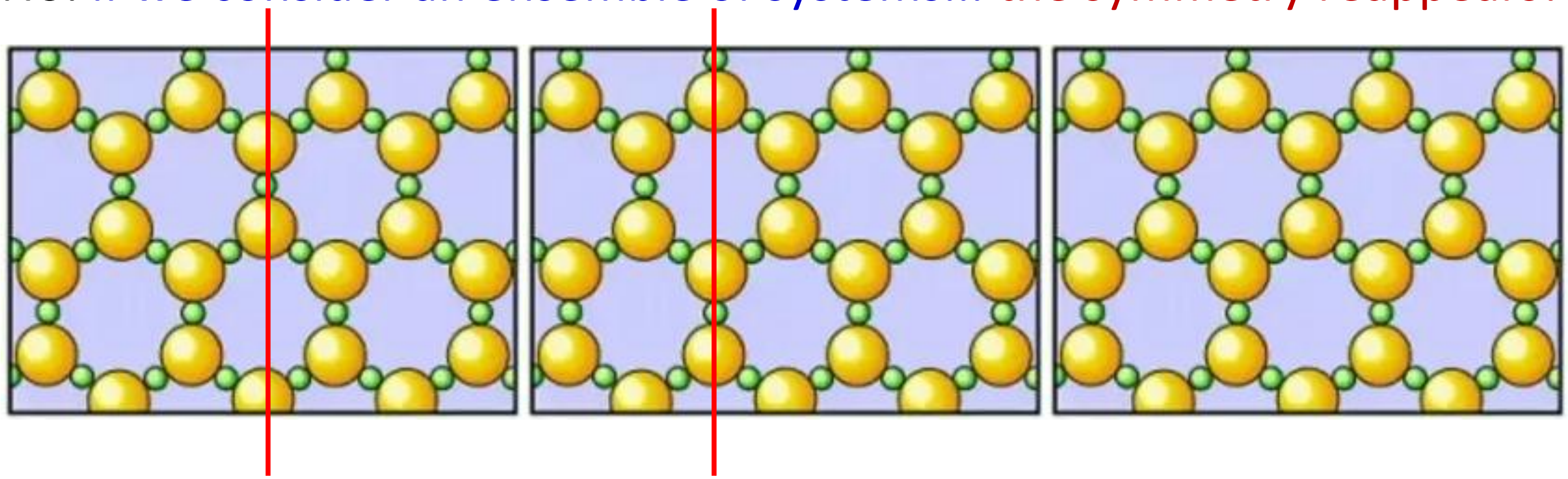


Below the melting point (0^0 C), all the molecules are fixed in the lattice sites, i.e. translation invariance is lost

…only partially, as translation by lattice constants is still possible : residual symmetry

Has the symmetry really been destroyed?

No: if we consider an ensemble of systems... the symmetry reappears!



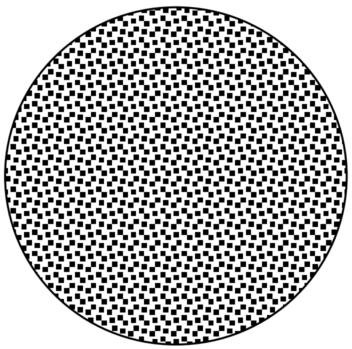
If we confine ourselves to the inside of an ice cube (Coleman's demon), then translation invariance will always be violated...

Once again, the symmetry has got hidden from Coleman's demon…

Spontaneous symmetry breaking ⇔ Hidden symmetry

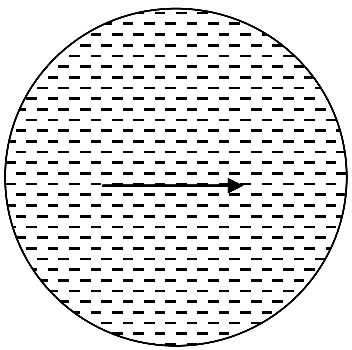
This arises when the action of a system has a particular symmetry, but the ground state does not...

Example 4: Ferromagnet below Curie temperature



Above the Curie temperature, all the domains are in random directions... obeys rotation invariance...

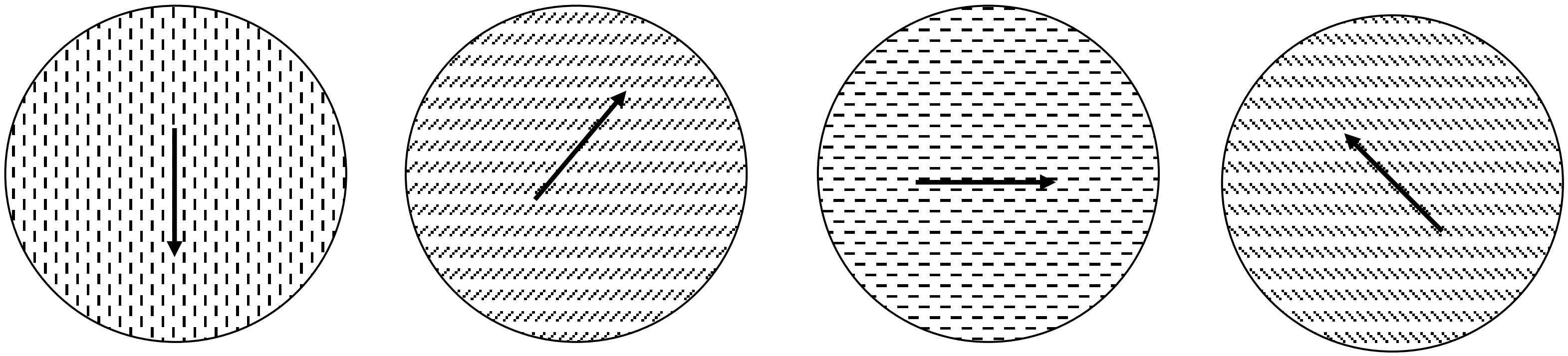
$$H = \sum_{\langle ij \rangle} J_{ij}\, \vec{S}_i . \vec{S}_j$$



Below the Curie temperature, all the domains are aligned parallel to a particular direction... magnetic field measurement will show a preferred direction, i.e. rotation invariance is lost

Has the symmetry really been destroyed?

No: if we consider an ensemble of systems... the symmetry reappears!



If we confine ourselves to the inside of a ferromagnet (Coleman's demon), then rotation invariance will always be violated...

This is always associated with a phase transition:

i.e. at some high temperature, the symmetry exists
at low temperature the symmetry disappears
in between a flip occurs... critical temperature...
$\Rightarrow$ phase transition

Lie Groups and Quantum Theory

Symmetry Group

A set of unitary transformations on a quantum system

$$|\psi\rangle \xrightarrow{T} |\psi'\rangle = \mathbb{T}|\psi\rangle$$

causes dynamical operators to transform as

$$\hat{\mathcal{O}} \xrightarrow{T} \hat{\mathcal{O}}' = \mathbb{T}\hat{\mathcal{O}}\mathbb{T}^{\dagger}$$

If the set of $\{\mathbb{T}_i\}$ transformations leaves $\hat{\mathcal{O}}$ invariant, i.e. $\hat{\mathcal{O}}' = \hat{\mathcal{O}}$, then they are called a symmetry of $\hat{\mathcal{O}}$.

Now, clearly, if we consider the product $\mathbb{T}_1\mathbb{T}_2$, then

a) $(\mathbb{T}_1\mathbb{T}_2)\widehat{\mathbb{O}}(\mathbb{T}_1\mathbb{T}_2)^{\dagger} = (\mathbb{T}_1\mathbb{T}_2)\widehat{\mathbb{O}}\left(\mathbb{T}_2^{\dagger}\mathbb{T}_1^{\dagger}\right) = \mathbb{T}_1\left(\mathbb{T}_2\widehat{\mathbb{O}}\mathbb{T}_2^{\dagger}\right)\mathbb{T}_1^{\dagger} = \mathbb{T}_1\widehat{\mathbb{O}}\mathbb{T}_1^{\dagger} = \widehat{\mathbb{O}}$

b) $\mathbb{T}_1(\mathbb{T}_2\mathbb{T}_3) = (\mathbb{T}_1\mathbb{T}_2)\mathbb{T}_3$

c) $\mathbb{1}\,\mathbb{T} = \widehat{\mathbb{T}}$

d) $\mathbb{T}^{\dagger}\mathbb{T} = \mathbb{1}$

Symmetry Group of $\hat{\mathcal{O}}$

If the set of $\{\mathbb{T}_i\}$ transformations leaves $\widehat{\mathcal{H}}$ invariant, i.e. $\widehat{\mathcal{H}}' = \widehat{\mathcal{H}}$, then they form a symmetry of the system… symmetry group of the system.

Energy levels are unchanged ⇒ spectrum is unchanged

Time evolution is unchanged ⇒ scattering effects are same

Examples:

1. One-dimensional harmonic oscillator

$$\widehat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

under parity $\hat{x} \xrightarrow{P} \mathbb{P}\hat{x}\,\mathbb{P}^\dagger = -\hat{x}$, $\hat{p} \xrightarrow{P} \mathbb{P}\hat{p}\,\mathbb{P}^\dagger = -\hat{p} \Rightarrow \widehat{\mathcal{H}}' = \widehat{\mathcal{H}}$

2. Hydrogen atom

$$\widehat{\mathcal{H}} = \frac{1}{2m}\hat{\vec{p}}^2 - \frac{e^2}{\hat{r}}$$

under rotations $\hat{\vec{p}}^2 \xrightarrow{R} \mathbb{R}\hat{\vec{p}}^2\,\mathbb{R}^\dagger = \hat{\vec{p}}^2$, $\hat{r} \xrightarrow{P} \mathbb{R}\hat{r}\,\mathbb{R}^\dagger = \hat{r} \Rightarrow \widehat{\mathcal{H}}' = \widehat{\mathcal{H}}$

Lie Group

Symmetry group $\{\mathbb{T}\}$ is continuous if $\mathbb{T} = \mathbb{T}(\theta)$ or $\mathbb{T} = \mathbb{T}(\theta_1, \theta_2, \dots, \theta_n)$

It is a Lie group if $\mathbb{T} = \mathbb{T}(\theta_1, \theta_2, \dots, \theta_n)$ is analytic in all the θ_i

Then we can expand in a Taylor series

$$\mathbb{T}(\theta) = \mathbb{T}(0) + \theta\mathbb{T}'(0) + \frac{1}{2!}\theta^2\mathbb{T}''(0) + \cdots$$

Taking $\theta = \varepsilon$ (infinitesimal)

$$\mathbb{T}(\varepsilon) = \mathbb{T}(0) + \varepsilon\mathbb{T}'(0) = \mathbb{T}(0) + i\varepsilon\{-i\mathbb{T}'(0)\} = \mathbb{1} + i\varepsilon\mathbb{G}$$

Now, for a finite transformation (unitary)

$$\mathbb{T}(\theta) = \lim_{N\to\infty}\left(\mathbb{1} + i\frac{\theta}{N}\mathbb{G}\right)^N = e^{i\theta\mathbb{G}}$$

Therefore $\mathbb{G}$ is the (Hermitian) generator of the Lie group

$$|\psi\rangle \overset{\theta}{\to} |\psi'\rangle = e^{i\theta\mathbb{G}}|\psi\rangle \qquad\qquad \hat{\mathcal{H}} \overset{T}{\to} \hat{\mathcal{H}}' = e^{i\theta\mathbb{G}}\hat{\mathcal{O}}e^{-i\theta\mathbb{G}}$$

Nöther's Theorem

For any Lie group of transformations

$$\hat{\mathcal{H}} \xrightarrow{T} \hat{\mathcal{H}}' = e^{i\theta\mathbb{G}}\hat{\mathcal{O}}e^{-i\theta\mathbb{G}}$$

Again, take $\theta = \varepsilon$ (infinitesimal)

$$\hat{\mathcal{H}} \xrightarrow{T} \hat{\mathcal{H}}' = (\mathbb{1} + i\varepsilon\mathbb{G})\hat{\mathcal{H}}(\mathbb{1} - i\varepsilon\mathbb{G})$$
$$= \hat{\mathcal{H}} + i\varepsilon\left(\mathbb{G}\hat{\mathcal{H}} - \hat{\mathcal{H}}\mathbb{G}\right)$$
$$= \hat{\mathcal{H}} + i\varepsilon\left[\mathbb{G}, \hat{\mathcal{H}}\right]$$

For a symmetry, $\hat{\mathcal{H}}' = \hat{\mathcal{H}} \Rightarrow \left[\mathbb{G}, \hat{\mathcal{H}}\right] = 0$

Heisenberg equation of motion:

$$\frac{d}{dt}\mathbb{G} = \frac{i}{\hbar}\left[\mathbb{G}, \hat{\mathcal{H}}\right] = 0$$

Eigenvalues of $\mathbb{G}$ are conserved quantum numbers…

For a multivariate Lie group (dimension n)

$$\mathbb{T}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = \mathbb{1} + i\varepsilon_1 \mathbb{G}_1 + i\varepsilon_2 \mathbb{G}_2 + \cdots + i\varepsilon_n \mathbb{G}_n = \mathbb{1} + i\vec{\varepsilon} \cdot \vec{\mathbb{G}}$$

$$\mathbb{T}(\vec{\theta}) = \lim_{N\to\infty} \left(\mathbb{1} + \frac{i}{N}\vec{\theta} \cdot \vec{\mathbb{G}}\right)^N = e^{i\vec{\theta}\cdot\vec{\mathbb{G}}}$$

For a symmetry,

$$\left[i\vec{\varepsilon} \cdot \vec{\mathbb{G}}, \widehat{\mathcal{H}}\right] = 0 \quad \Rightarrow \quad \left[\mathbb{G}_a, \widehat{\mathcal{H}}\right] \text{ for } a = 1,2, \dots, n$$

i.e. all the generators will have conserved eigenvalues

…but it is not guaranteed they will be conserved simultaneously…!

Group closure property:

$$\mathbb{T}(\vec{\alpha})\mathbb{T}(\vec{\beta}) = \mathbb{T}(\vec{\gamma}) \qquad \Rightarrow \qquad e^{i\vec{\alpha}\cdot\vec{\mathbb{G}}}\, e^{i\vec{\beta}\cdot\vec{\mathbb{G}}} = e^{i\vec{\gamma}\cdot\vec{\mathbb{G}}}$$

$$e^{i\vec{\alpha}\cdot\vec{\mathbb{G}}}\, e^{i\vec{\beta}\cdot\vec{\mathbb{G}}} = e^{i\vec{\alpha}\cdot\vec{\mathbb{G}} + i\vec{\beta}\cdot\vec{\mathbb{G}} + \frac{1}{2}\left[i\vec{\alpha}\cdot\vec{\mathbb{G}}, i\vec{\beta}\cdot\vec{\mathbb{G}}\right] + \cdots} = e^{i\vec{\gamma}\cdot\vec{\mathbb{G}}}$$

Must have $\left[i\vec{\alpha} \cdot \vec{\mathbb{G}}, i\vec{\beta} \cdot \vec{\mathbb{G}}\right] \propto G_a$'s $\Rightarrow [\mathbb{G}_a, \mathbb{G}_b] = i f_{abc} \mathbb{G}_c$

Lie Algebra

Generators of a Lie Group obey

$$[\mathbb{G}_a, \mathbb{G}_b] = i f_{abc} \mathbb{G}_c$$

The f_{abc} are called structure constants (real numbers).
Only if all the f_{abc} for a given a, b vanish, can the generators $\mathbb{G}_a, \mathbb{G}_b$ have simultaneous eigenvalues.

Example:

Set of 3D rotations $\{\mathbb{R}(\theta_x, \theta_y, \theta_z) \mid 0 \leq \theta_x, \theta_y, \theta_z < 2\pi\}$ is a Lie group
Generators are $\mathbb{J}_x, \mathbb{J}_y, \mathbb{J}_z$ i.e. the angular momenta, i.e. we can write

$$\mathbb{R}(\theta_x, \theta_y, \theta_z) = e^{i\vec{\theta}\cdot\vec{\mathbb{J}}}$$

Lie algebra is

$$[\mathbb{J}_a, \mathbb{J}_b] = i\epsilon_{abc}\mathbb{J}_c$$

Structure constants ϵ_{abc} i.e. only one of the $\mathbb{J}'s$ has conserved e'value

Quantum Field Theory

Every field $\psi(x)$ is an operator on the Fock space of states

$$\hat{\psi}(x)\,|n_1, n_2, \dots\dots\rangle$$

$|n_1, n_2, \dots\dots\rangle$ occupation number representation

$\hat{\psi}(x) = \hat{a} \oplus \hat{b}^\dagger$: annihilates particle, creates antiparticle

$\hat{\psi}^\dagger(x) = \hat{a}^\dagger \oplus \hat{b}$: creates particle, annihilates antiparticle

Lie group of transformations

$$e^{i\vec{\theta}\cdot\vec{\mathbb{G}}}|n_1, n_2, \dots\dots\rangle \qquad\qquad e^{i\vec{\theta}\cdot\vec{\mathbb{G}}}\hat{\psi}(x)e^{-i\vec{\theta}\cdot\vec{\mathbb{G}}}$$

For an infinitesimal transformation

$$\hat{\psi}(x) \xrightarrow{T} \hat{\psi}'(x) = \hat{\psi}(x) + \left[i\vec{\varepsilon}\cdot\vec{\mathbb{G}}, \hat{\psi}(x)\right]$$

i.e. $$\delta_{\vec{\varepsilon}}\hat{\psi}(x) = \hat{\psi}'(x) - \hat{\psi}(x) = \left[i\vec{\varepsilon}\cdot\vec{\mathbb{G}}, \hat{\psi}(x)\right]$$

Successive transformations

$$\delta_{\vec{\alpha}}\delta_{\vec{\beta}}\hat{\psi}(x) = \left[i\vec{\alpha}\cdot\vec{\mathbb{G}},\left[i\vec{\beta}\cdot\vec{\mathbb{G}},\hat{\psi}(x)\right]\right]$$

$$\delta_{\vec{\beta}}\delta_{\vec{\alpha}}\hat{\psi}(x) = \left[i\vec{\beta}\cdot\vec{\mathbb{G}},\left[i\vec{\alpha}\cdot\vec{\mathbb{G}},\hat{\psi}(x)\right]\right]$$

Subtracting

$$\left[\delta_{\vec{\alpha}}\delta_{\vec{\beta}} - \delta_{\vec{\beta}}\delta_{\vec{\alpha}}\right]\hat{\psi}(x) = \left[i\vec{\alpha}\cdot\vec{\mathbb{G}},\left[i\vec{\beta}\cdot\vec{\mathbb{G}},\hat{\psi}(x)\right]\right] + \left[i\vec{\beta}\cdot\vec{\mathbb{G}},\left[\hat{\psi}(x),i\vec{\alpha}\cdot\vec{\mathbb{G}}\right]\right]$$

$$= -\left[\hat{\psi}(x)\left[i\vec{\alpha}\cdot\vec{\mathbb{G}},i\vec{\beta}\cdot\vec{\mathbb{G}}\right]\right]$$

(Jacobi identity $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$)

Lie Algebra of generators : $\left[i\vec{\alpha}\cdot\vec{\mathbb{G}},i\vec{\beta}\cdot\vec{\mathbb{G}}\right] = -i\vec{\omega}\cdot\vec{\mathbb{G}}$

$$\left[\delta_{\vec{\alpha}},\delta_{\vec{\beta}}\right]\hat{\psi}(x) = \delta_{\vec{\omega}}\,\hat{\psi}(x)$$

closure condition on fields

Isomorphism

Two groups are isomorphic if there is a one-one map $a \to a'$

s.t. if $ab = c$, then $a'b' = c'$.

Isomorphic groups are mathematically equivalent as they have the same multiplication table…

a	b	c	…	x
b	d	a		p
c	b	q		b
$\vdots$			$\ddots$	
x	c	a		f

a'	b'	c'	…	x'
b'	d'	a'		p'
c'	b'	q'		b'
$\vdots$			$\ddots$	
x'	c'	a'		f'

Lie groups will be isomorphic if they have the same Lie Algebra, i.e.

if $[\mathbb{G}_a, \mathbb{G}_b] = if_{abc}\mathbb{G}_c$ then $[\mathbb{G}'_a, \mathbb{G}'_b] = if_{abc}\mathbb{G}'_c$

If any Lie group of operators is isomorphic to a Lie group of square $n \times n$ matrices, the matrices are said to form a representation

Unitary Groups

Set of $n \times n$ unitary matrices forms a Lie group… isomorphic to $U(n)$

No of real elements: $\nu = 2n^2 - n^2 - {}^{n}C_2 = n^2$

Set of $n \times n$ unitary unimodular (determinant = 1) matrices also forms a Lie group… isomorphic to $SU(n)$

No of real elements: $\nu = 2n^2 - n^2 - {}^{n}C_2 - 1 = n^2 - 1$

- $n = 1$

$$\mathbb{M}^*\mathbb{M} = 1 \qquad \Rightarrow \quad \mathbb{M} = e^{i\theta\mathbb{1}}$$

i.e. a phase transformation is isomorphic to $U(1)$; generator $\mathbb{1}$

- $n = 2$

$$\mathbb{M} = e^{i\mathbb{h}}$$

where $\mathbb{h}$ is Hermitian

generic 2×2 Hermitian matrix (real parameters $\alpha, \beta, \gamma, \delta$)

$$\mathbb{h} = \frac{1}{2}\begin{pmatrix} \alpha+\delta & \beta-i\gamma \\ \beta+i\gamma & \alpha-\delta \end{pmatrix}$$

$$= \frac{1}{2}\alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\gamma\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{2}\delta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \tfrac{1}{2}\alpha\mathbb{1} + \tfrac{1}{2}\beta\sigma_1 + \tfrac{1}{2}\gamma\sigma_2 + \tfrac{1}{2}\delta\sigma_3$$

Thus the generators of $U(2)$ are represented by ($\nu = 4$)

$$\mathbb{G}_0 = \frac{1}{2}\mathbb{1}, \qquad \mathbb{G}_1 = \frac{1}{2}\sigma_1, \qquad \mathbb{G}_2 = \frac{1}{2}\sigma_2, \qquad \mathbb{G}_3 = \frac{1}{2}\sigma_3$$

(fundamental representation)

If $\det \mathbb{M} = 1$, then $\mathrm{Tr}\,\mathbb{h} = 0 \;\Rightarrow\; \alpha = 0$

Thus the generators of $SU(2)$ are represented by ($\nu = 3$)

$$\mathbb{G}_1 = \frac{1}{2}\sigma_1, \qquad \mathbb{G}_2 = \frac{1}{2}\sigma_2, \qquad \mathbb{G}_3 = \frac{1}{2}\sigma_3$$

Since the Pauli matrices obey the commutation relations

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$$

the $\mathbb{G}_a = \frac{1}{2}\sigma_a$ must obey

$$[\mathbb{G}_a, \mathbb{G}_b] = i\epsilon_{abc}\mathbb{G}_c$$

which is the same as the rotation generators (angular momenta).

Hence, every $SU(2)$ must be isomorphic to a set of rotations

Actually, to 2 sets of rotations, proper and improper

$\Rightarrow$ homomorphism rather than isomorphism)

To see this, construct the matrix

$$\mathbb{X} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$$

so that $\det \mathbb{X} = -x_3^2 - (x_1^2 + x_2^2) = -r^2$

Under rotations, r^2 is invariant, so we must have

$$\mathbb{X} \to \mathbb{X}' = \mathbb{S}\mathbb{X}\mathbb{S}^\dagger$$

such that

$$\det \mathbb{X}' = \det \mathbb{X}\ |\det \mathbb{S}|^2$$

We will have $\det \mathbb{X}' = \det \mathbb{X}$ only if $|\det \mathbb{S}|^2 = 1$, i.e. $\mathbb{S}$ is an $SU(2)$ transformation. But note that both $\mathbb{S}$ and $-\mathbb{S}$ leave the determinant unchanged, so we have to include both proper and improper rotations.

The $SU(2)$ matrix $\mathbb{S}$ will act on some 2-component objects which transform as

$$\mathfrak{u} \to \mathfrak{u}' = \mathbb{S}\mathfrak{u}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

These are called Pauli spinors, or spinors of $SU(2)$.

- $n = 3$

$$\mathbb{M} = e^{i\mathbb{h}}$$

where $\mathbb{h}$ is Hermitian, but now ($\nu = 8$)

$$\mathbb{h} = \sum_{k=1}^{8} \alpha_k \, \mathbb{G}_k = \sum_{k=1}^{8} \frac{1}{2} \alpha_k \, \lambda_k$$

where the λ_k are the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\mathbb{G}_k = \frac{1}{2}\lambda_k$ are the generators of $SU(3)$ ($\nu = 8$) or $U(3)$ ($\nu = 9$)

Adjoint Representation

The generators obey the Jacobi identity

$$\left[[\mathbb{G}_a, \mathbb{G}_b], \mathbb{G}_c\right] + \left[[\mathbb{G}_b, \mathbb{G}_c], \mathbb{G}_a\right] + \left[[\mathbb{G}_c, \mathbb{G}_a], \mathbb{G}_b\right] = 0$$

reduces to

$$f_{abd} f_{cde} + f_{bcd} f_{ade} + f_{cad} f_{bde} = 0$$

Defining a set of $\nu \times \nu$ matrices

$$[\mathbb{D}_a]_{bc} = -i f_{abc}$$

the above equation can be rewritten

$$[\mathbb{D}_a, \mathbb{D}_b] = i f_{abc} \mathbb{D}_c$$

The $\mathbb{D}$ matrices form the adjoint representation of the Lie group.

If $\mathfrak{u}$ and $\mathbb{G}_a$ are written in the fundamental representation of the Lie group, then the set of ν objects $\mathcal{J}_a = \mathfrak{u}^\dagger \mathbb{G}_a \mathfrak{u}$ forms a multiplet of the adjoint representation, i.e. the $\mathbb{D}$ act on them just as the $\mathbb{G}$ on spinors.