Hydrodynamics and the early thermalization (?) of quark-gluon plasma

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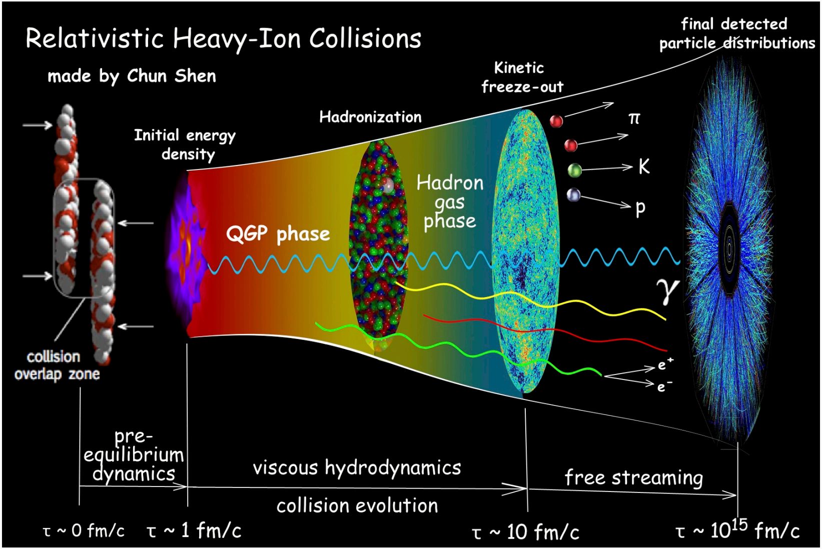
Heavy-ion collision



Hydrodynamic simulation of HIC

$\sim$ 15-20 years ago:

Discovery of the "unreasonable effectiveness of hydrodynamics" in describing ultrarelativistic heavy-ion collision dynamics. hundreds of papers..

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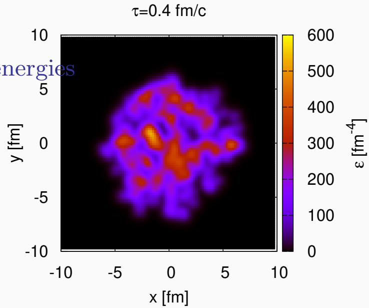
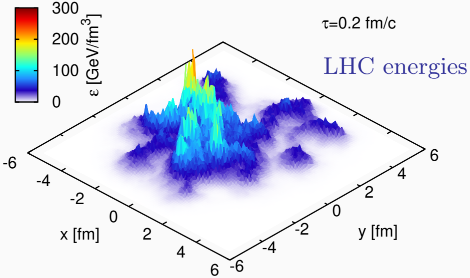


Schenke, Jeon, Gale, PRL **106** (2011), 042301

LHC energies



Heinz, Snellings, Ann. Rev. Nucl. Part. Sci. 63 (2013) 123

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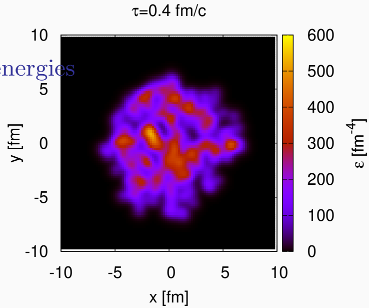

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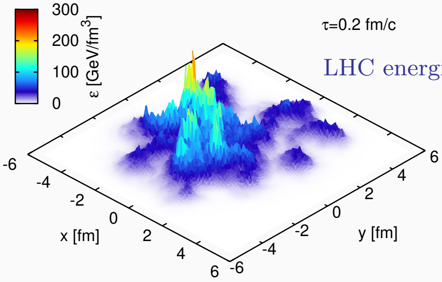

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Hydrodynamics is applied in regime of large gradients...

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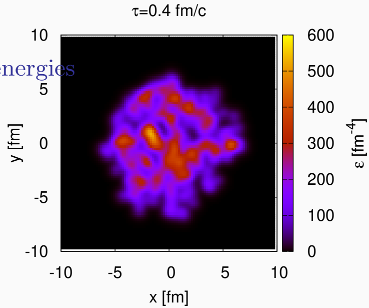

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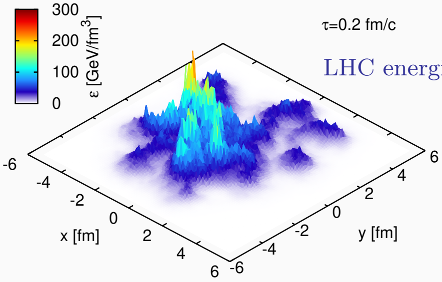

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Simulations like these explains data → "Early thermalization puzzle"

What is the domain of hydrodynamics?

- Textbooks: Close to local equilibrium, $\lambda_{\rm mfp} \ll L$ or $|\nabla^\mu u^\nu|/T \ll 1$.
  Hence, hydrodynamics is formulated as an expansion in velocity gradients.
  Eckart, Phys. Rev.58 (1940), Landau and Lifshitz, "Fluid mechanics" (1987)

- Viscous hydrodynamics: Add out-of-equilibrium corrections to $T^{\mu\nu}_{\rm ideal}$:

$$T^{\mu\nu} = T^{\mu\nu}_{\rm ideal} + \Pi^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu} + \pi^{\mu\nu} + \Pi\Delta^{\mu\nu}$$

  - The operator $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ projects on the space orthogonal to u^μ.
  - Landau frame chosen: $T^{\mu\nu}u_\nu = \epsilon u^\mu, \quad \epsilon = \epsilon_{\rm eq}$.
  - Vanishing chemical potential – no net conserved charge.

- $1^{\rm st}$ order hydrodynamics: Navier-Stokes:
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$$\pi^{\mu\nu} = \eta\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha\right) = 2\,\eta\,\sigma^{\mu\nu}, \qquad \Pi = -\zeta\,\partial_\mu u^\mu.$$

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Müller-Israel-Stewart theory : 1

Müller, Z. Phys, 198, 329 (1967), Israel and Stewart, Ann. Phys. 100, 310 (1976)

- Starting point:

$$S^\mu \equiv S^\mu\left(T, u^\mu, \mu, N^\mu, T^{\mu\nu}\right) \equiv S^\mu\left(T, u^\mu, \mu, \Pi, \pi^{\mu\nu}, V^\mu\right)$$

Here, N^μ is conserved current, V^μ is particle diffusion.

- Expand S^μ in powers of the dissipative currents around a fictitious equilibrium state

$$S^\mu = \frac{P}{T}u^\mu + \frac{1}{T}u_\nu T^{\mu\nu} - \frac{\mu}{T}N^\mu - X^\mu\left(\delta N^\mu, \delta T^{\mu\nu}\right)$$

- Expanding X^μ to second-order

$$S^\mu = s u^\mu - \frac{\mu}{T}V^\mu - \frac{u^\mu}{2}\left(\delta_0 \Pi^2 - \delta_1 V_\alpha V^\alpha + \delta_2 \pi_{\alpha\beta}\pi^{\alpha\beta}\right) - \gamma_0 \Pi V^\mu - \gamma_1 \pi^\mu_\nu V^\nu + \mathcal{O}(\delta^3)$$

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Müller-Israel-Stewart theory : 2

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- Demand entropy divergence is positive

$$\partial_\mu S^\mu = \frac{\Pi}{T}\underbrace{\left(-\theta - T\delta_0\dot{\Pi} - \frac{T}{2}\Pi\dot{\delta}_0 - \frac{T}{2}\delta_0\Pi\theta - T\gamma_0\partial_\mu V^\mu - T(1-r)V^\mu\nabla_\mu\gamma_0\right)}_{\Omega_\Pi\Pi}$$

$$+V_\mu\underbrace{\left(-\nabla^\mu\left(\frac{\mu}{T}\right) + \delta_1\dot{V}^{\langle\mu\rangle} + \frac{V^\mu}{2}\dot{\delta}_1 + \frac{\delta_1}{2}V^\mu\theta - \gamma_0\nabla^\mu\Pi - r\Pi\nabla^\mu\gamma_0 - \gamma_1\partial_\nu\pi^{\mu\nu} - y\pi^{\mu\nu}\nabla_\nu\gamma_1\right)}_{-\Omega_V V^\mu}$$

$$+\frac{\pi^{\mu\nu}}{T}\underbrace{\left(\sigma^{\mu\nu} - T\delta_2\dot{\pi}^{\langle\mu\nu\rangle} - \frac{T}{2}\pi^{\mu\nu}\dot{\delta}_2 - \frac{T}{2}\delta_2\pi^{\mu\nu}\theta - T\gamma_1\nabla^{\mu\langle}V^{\rangle\nu} - T(1-y)V^{\langle\mu}\nabla^{\nu\rangle}\gamma_1\right)}_{\Omega_\pi\pi_{\mu\nu}}$$

Here, Ω_Π , Ω_V , $\Omega_\pi \geq 0$. Co-moving derivative $\dot{A} \equiv u^\mu\partial_\mu A$.

- Relaxation type equations for dissipative stresses

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\Omega_\pi}{T\delta_2}\pi^{\mu\nu} = \frac{1}{T\delta_2}\sigma^{\mu\nu} + \cdots$$

- Causal and stable phenomenological theory. Π, $\pi^{\mu\nu}$, V^μ promoted to independent dynamical variables.

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Second-order hydrodynamics

- Gradient expansion till second order (for conformal systems):
  R. Baier et al., JHEP 04, 100 (2008) S. Bhattacharyya et al., JHEP 02, 045 (2008)

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - 2\eta\tau_\pi \left[\dot{\sigma}^{\langle\mu\nu\rangle} + \frac{1}{3}\sigma^{\mu\nu}\theta\right] + \lambda_1\, \sigma_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} + \lambda_2\, \sigma_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} + \lambda_3\, \omega_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}$$

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  - Restores causality and stability. Israel-Stewart-like equation.
    Many variants A. Muronga '02; Denicol, Niemi, Molnár, Rischke '12; A. Jaiswal '13; . . .
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These ISL theories are used in hydrodynamic simulations of heavy-ion.

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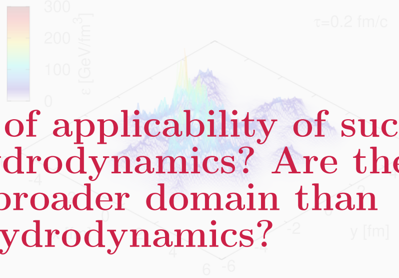


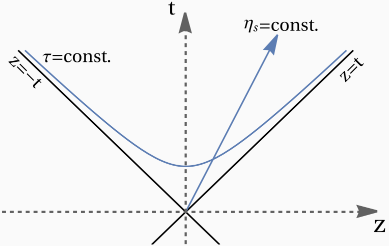
What is the domain of applicability of such Israel-Stewart-like hydrodynamics? Are they applicable over a broader domain than traditional hydrodynamics?

Schenke, Jeon, Gale, PRL **106** (2011), 042301   Heinz, Snellings, Ann. Rev. Nucl. Part. Sci. 63 (2013) 123

Simulations like these explains data, however, hydrodynamics is applied in regime of large gradients. Does it even make sense?

Bjorken flow J. D. Bjorken, PRD 27, 140 (1983)

- Bjorken symmetries: homogeneity in the transverse (x, y) plane, boost invariance along the z (beam) direction, and reflection symmetry $z \to -z$.
- Appropriate description of early-time dynamics of matter formed in ultra-relativistic heavy-ion collisions.



Milne coordinate system $(\tau, x_\perp, \phi, \eta_s)$.

Proper time: $\tau = \sqrt{t^2 - z^2}$.

Space-time rapidity: $\eta_s = \tanh^{-1}(z/t)$.

- Fluid appears static, $u^\mu = (1, 0, 0, 0)$. Finite expansion rate, $\partial_\mu u^\mu = 1/\tau$.
- All scalars depends only on proper time τ. Shear tensor is diagonal,

$$\pi^{\mu\nu} = \mathrm{diag}\left(0, \frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{\tau^2}\right). \qquad T^{\mu\nu} = \mathrm{diag}\left(\epsilon, P + \Pi + \frac{\pi}{2}, P + \Pi + \frac{\pi}{2}, P + \Pi - \pi\right).$$

Set of special moments of distribution function

- Boltzmann equation in RTA approximation undergoing Bjorken expansion:

$$\left(\frac{\partial}{\partial \tau} - \frac{p_z}{\tau}\frac{\partial}{\partial p_z}\right) f(\tau, p) = -\frac{f(\tau,p) - f_{\rm eq}(p_0/T)}{\tau_R(\tau)}$$

- Consider the moments:

$$\mathcal{L}_n \equiv \int_p p_0^2 \; P_{2n}(p_z/p_0) \; f(\tau,p), \qquad \mathcal{M}_n \equiv m^2 \int_p P_{2n}(p_z/p_0) \; f(\tau,p)$$

where $\int_p \equiv \frac{d^3p}{(2\pi)^3 \, p_0}$ and P_{2n} is the Legendre polynomial of order $2n$.

Blaizot and Yan, PLB **780** (2018) SJ, Blaizot, Bhalerao, Chen, Jaiswal, Yan; PRC **106**, 044912 (2022)

- Boltzmann equation can be recast as:

$$\frac{\partial \mathcal{L}_n}{\partial \tau} = -\frac{1}{\tau}\left(a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}\right) - (1 - \delta_{n,0})\frac{(\mathcal{L}_n - \mathcal{L}_n^{\rm eq})}{\tau_R}$$

$$\frac{\partial \mathcal{M}_n}{\partial \tau} = -\frac{1}{\tau}\left(a'_n \mathcal{M}_n + b'_n \mathcal{M}_{n-1} + c'_n \mathcal{M}_{n+1}\right) - \frac{(\mathcal{M}_n - \mathcal{M}_n^{\rm eq})}{\tau_R}$$

The coefficients $a_n, b_n, c_n, a'_n, b'_n, c'_n$ are pure numbers. Depends on expansion geometry.

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Fixed point structure

- Equation of $\mathcal{L}_n$ moments are decoupled from $\mathcal{M}_n$ moments $\implies$ evolution of energy density ($\mathcal{L}_0$) does not depend on $\mathcal{M}_n$ evolution.

- Consider the quantity: $g_0 \equiv \frac{\tau}{\mathcal{L}_0}\frac{\partial \mathcal{L}_0}{\partial \tau}$. In the regimes where the energy density behave as power law, g_0 is the exponent in that power law.

- Define $\beta(g_0, w) \equiv w\frac{\mathrm{d}g_0}{\mathrm{d}w}$ where $w = \tau/\tau_R$. Equation for $\mathcal{L}_n$ becomes:

$$-\beta(g_0, w) = g_0^2 + g_0\left(a_0 + a_1 + w\right) + a_0 a_1 - c_0 b_1 + a_0 w - c_0 c_1 \frac{\mathcal{L}_2}{\mathcal{L}_0} - \frac{c_0}{2} w \left(1 - 3\frac{P}{\epsilon}\right)$$

- Zeros of $\beta(g_0, w)$ gives fixed points.

- Free-streaming fixed points ($w \ll 1$):
  - Exact fixed point: $g_0 = -1$ (stable) and $g_0 = -2$ (unstable).

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- Hydrodynamic fixed point ($w \gg 1$) : $g_* = -1 - P/\epsilon$ (governed by EoS).

Three-moment truncation

- Equation of three moments:

$$\frac{\partial \mathcal{L}_0}{\partial \tau} = -\frac{1}{\tau}\left(a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1\right), \qquad \frac{\partial \mathcal{L}_1}{\partial \tau} = -\frac{1}{\tau}\left(a_1 \mathcal{L}_1 + b_1 \mathcal{L}_0 + c_1 \mathcal{L}_2\right) - \frac{\left(\mathcal{L}_1 - \mathcal{L}_1^{\mathrm{eq}}\right)}{\tau_R},$$

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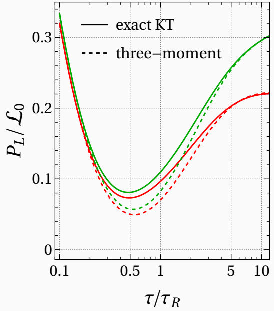
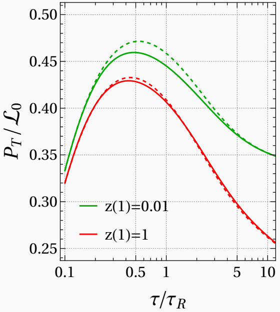
- Considering three lowest moments ($\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{M}_0$) is enough to approximately capture the exact evolution.





$z = m/T$

Isotropic IC

Constant τ_R

Three-moment truncation: with interpolation for $\mathcal{L}_2$ and $\mathcal{M}_1$

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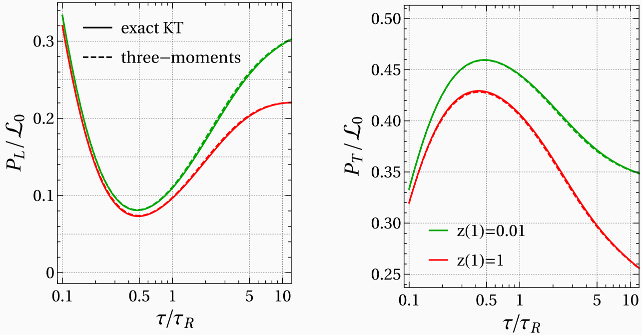
- Using a simple interpolation for $\mathcal{L}_2$ and $\mathcal{M}_1$ $\rightarrow$



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- ISL hydro equations can be obtained from truncation of $\mathcal{L}_2$ and $\mathcal{M}_1$. However, there is an inherent ambiguity in the definition of some second-order transport coefficients.
SJ, Blaizot, Bhalerao, Chen, Jaiswal, Yan; PRC **106**, 044912 (2022)

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- Relaxation-type structure inherent in moments equations – necessary for extending domain in free-streaming regime.

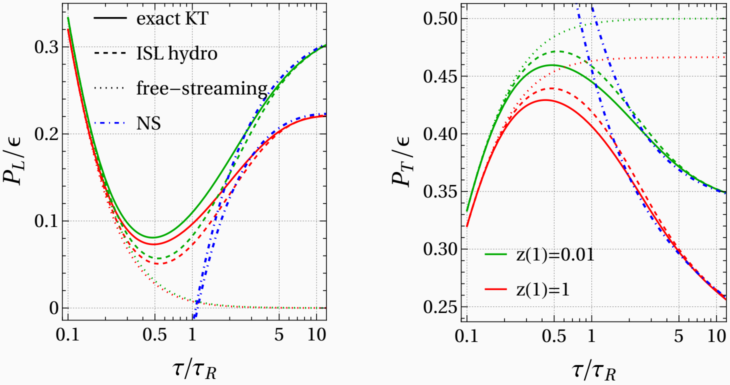
- Time derivative of $\mathcal{L}_1$ and $\mathcal{M}_0$, and correspondingly, $\pi \equiv -\frac{2}{3}\left(\mathcal{L}_1 + \frac{\mathcal{M}_0}{2}\right)$ and $\Pi \equiv \left(\mathcal{L}_0 - 3P - \mathcal{M}_0\right)/3$ in ISL hydro, captures approximately the features of the collisionless regime of the expanding system.

Illustration ⇒

ISL hydrodynamics captures free-streaming!

SJ, Blaizot, Bhalerao, Chen, Jaiswal, Yan; PRC **106**, 044912 (2022)

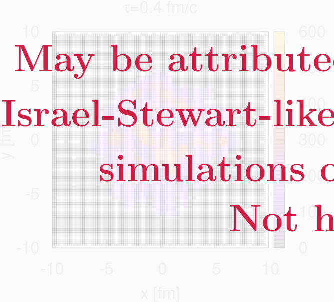
Isotropic initial conditions.

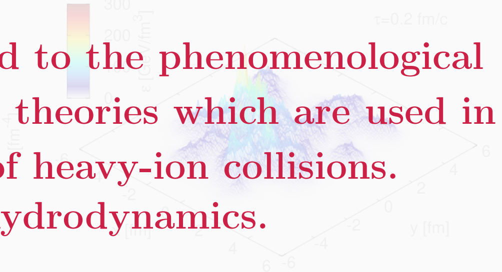


Short free-streaming regime (dotted curves) seen in both the kinetic theory and ISL hydrodynamic. There is nothing typically "hydrodynamic" here; hydrodynamics becomes a valid description only for times $\tau \gtrsim \tau_R$.

$\sim$ 15-20 years ago:

Discovery of the **"unreasonable effectiveness of hydrodynamics"** in describing ultrarelativistic heavy-ion collision dynamics.





May be attributed to the phenomenological Israel-Stewart-like theories which are used in simulations of heavy-ion collisions. Not hydrodynamics.

Schenke, Jeon, Gale, PRL **106** (2011), 042301 Heinz, Snellings, Ann. Rev. Nucl. Part. Sci. 63 (2013) 123

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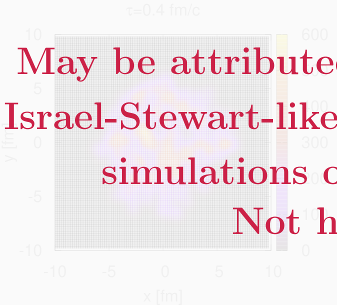

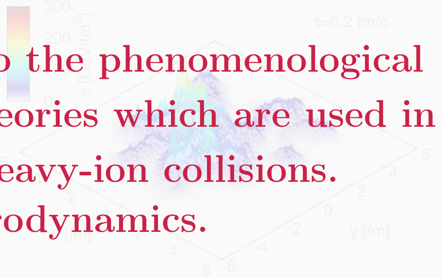

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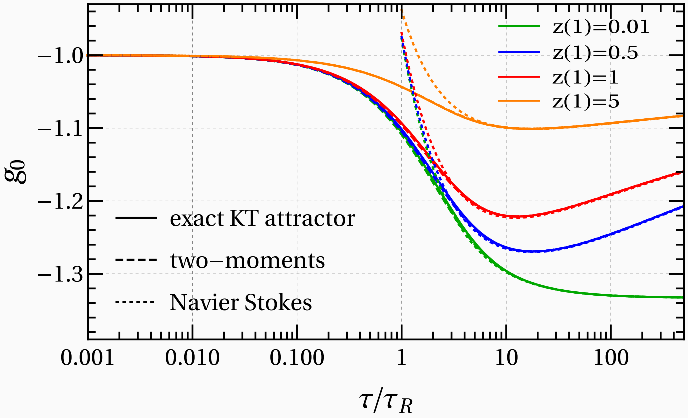
Thank You!

Extras

Evolution of g_0 SJ, Blaizot, Bhalerao, Chen, Jaiswal, Yan; PRC **106**, 044912 (2022)

Solving the truncated three-moment eqs. using a simple interpolation for $\mathcal{L}_2$ and $\mathcal{M}_1$.



- Attractor initial condition. $z = m/T$

Ambiguity of second-order transport coefficients

SJ, Blaizot, Bhalerao, Chen, Jaiswal, Yan; PRC **106**, 044912 (2022)

- Equation of $\mathcal{L}_n$ moments are decoupled from $\mathcal{M}_n$ moments $\Longrightarrow$ evolution of energy density ($\mathcal{L}_0$) does not depend on $\mathcal{M}_n$ evolution.
- Since only $\Pi - \pi = c_0(\mathcal{L}_1 - \mathcal{L}_1^{\rm eq})$ enters in evolution of ϵ, similar decoupling in the hydrodynamic equations expected. Such decoupling holds in the ISL hydro iff

$$\delta_{\Pi\Pi} + \frac{2}{3}\lambda_{\pi\Pi} = \lambda_{\Pi\pi} + \frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi}$$

  Not satisfied by transport coefficients derived in A. Jaiswal et. al., PRC 90 (2014) 044908
- New transport coefficients derived following a different truncation for $\mathcal{L}_2$ and $\mathcal{M}_1$ appearing in the equation for $\mathcal{L}_1$ and $\mathcal{M}_0$. Coefficients of the gradient series of Π and π unchanged.