Stability and causality in first-order relativistic hydrodynamics in a general frame for arbitrary interactions

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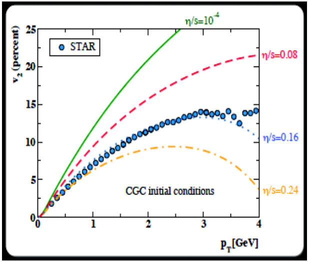
1. R Biswas, S Mitra, V Roy, Phys.Rev.D 106 (2022) 1, L011501
2. R Biswas, S Mitra, V Roy, Phys.Lett.B 838 (2023) 137725

ET-HCVM February 4, 2023

Hydrodynamics - A long-wavelength effective theory of fluids for the evolution of conserved quantities

- 1940-1950 : First order theories developed by Eckart and Landau-Lifshitz.
- 1970s : Second order theories developed by Israel and Stewart.
- 1980s : Hiscock and Lindblom's analysis of stability and causality in favour of second order theory at relativistic situation.



M. Luzum and P. Romatschke, PRC 78, 034915 (2008)

- 1982: Analytic fluid modelling of heavy-ion collision by Bjorken.
- 1990s/early 2000s: Relativistic ideal fluid modelling of heavy-ion collisions - non-viscous hydro simulations.
- 2005: Break-through in string theory, estimating shear viscosity for strongly coupled system.
- 2001-2003: Work on relativistic evolution for viscous fluids by Muronga, Teaney etc.
- 2008: The viscous hydrodynamics using small but finite $\eta/s(= 2 \times (1/4\pi))$ explains quantitatively STAR data.

Field redefinition of relativistic hydrodynamic theory

Motivation behind ···

- **Out of equilibrium thermodynamic fields have no first-principles microscopic definition.**
- **They are subjected to include dissipative effects from the medium.**

Key features ···

- The out of equilibrium fields are expressed in terms of thermodynamic quantities and their gradients - Constitutive relations.
- These constitutive relations are not unique and different out of equilibrium field choices can be adopted.
- Different redefinition schemes must always agree in equilibrium.
- Commonly the fields are set to their equilibrium values even in the dissipative medium by imposing certain matching or fitting conditions or hydrodynamic frame choices.
- A general frame choice can lead to a stable-causal first order theory.
  Ref: Bemfica, Disconzi and Noronha, PRD 98 (2018) no.10, 104064, PRD 100 (2019) no.10, 104020.
  Kovtun, JHEP 10 (2019), 034, JHEP 06 (2020), 067.

Formal framework - Hydrodynamics field redefinition

Identification of thermodynamic fields

Particle four flow: $N^{\mu}(x) = \int \frac{d^3p}{(2\pi)^3 p^0} p^{\mu} f = n u^{\mu} + V^{\mu},$

Energy-momentum tensor: $T^{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3 p^0} p^{\mu} p^{\nu} f = \epsilon u^{\mu} u^{\nu} - P\Delta^{\mu\nu} + \{W^{\mu} u^{\nu} + W^{\nu} u^{\mu}\} + \pi^{\langle\mu\nu\rangle}$

1. **Particle number density:** $n(x) = N^{\mu} u_{\mu} = \int \frac{d^3p}{(2\pi)^3 p^0} (p \cdot u) f$
2. **Energy density:** $\epsilon = u_{\mu} u_{\nu} T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 p^0} (p \cdot u)^2 f$
3. **Pressure:** $P(x) = -\frac{1}{3} T^{\mu\nu} \Delta_{\mu\nu} = -\frac{1}{3} \int \frac{d^3p}{(2\pi)^3 p^0} |\vec{p}|^2 f$
4. **Corrections to mean-particle velocity :** $V^{\mu} = \Delta^{\mu\nu} \delta N_{\nu} = \int \frac{d^3p}{(2\pi)^3 p^0} p^{\langle\mu\rangle} f$
5. **Corrections to energy flow or momentum density :** $W^{\alpha} = \Delta^{\alpha}_{\mu} u_{\nu} \delta T_{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 p^0} (p \cdot u) p^{\langle\alpha\rangle} f$
6. **Shear pressure tensor :** $\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \delta T^{\alpha\beta} = \int \frac{d^3p}{(2\pi)^3 p^0} p^{\langle\mu} p^{\nu\rangle} f$

Formal framework - Hydrodynamics from kinetic theory

Relativistic Transport Equation of Single particle Distribution Function

Homogeneous solution	**Inhomogeneous solution**
Fully arbitrary	Controlled by system interaction
Decided by the matching condition owing to the frame choice	Extracted solely from the underlying microscopic theory

↓

Macroscopic Hydrodynamic Equation of Conserved Quantities

Special cases :

- Landau-Lifshitz and Eckart frames both set the energy and particle number density to their equilibrium values - Energy density and particle number density correction is zero.
- Landau-Lifshitz frame sets the hydrodynamic velocity as the eigen vector of energy momentum tensor - Energy flux or momentum density is zero.
- Eckart frame expresses the hydrodynamic velocity as the particle-4-flow - Particle current is zero.

Formal framework - Hydrodynamics from kinetic theory

Relativistic transport equation

$$p^\mu \partial_\mu f(x,p) = C[f] = -\mathcal{L}[\phi]$$

$f = f^{(0)} + f^{(0)}(1 \pm f^{(0)})\phi$

collision term

Linearized collision term and its properties:

$$\mathcal{L}[\phi] = \int \frac{d^3p_1}{(2\pi)^3 p_1^0} \frac{d^3p'}{(2\pi)^3 p'^0} \frac{d^3p_1'}{(2\pi)^3 p_1'^0} f^{(0)} f_1^{(0)} (1 \pm f'^{(0)})(1 \pm f_1'^{(0)}) \{\phi + \phi_1 - \phi' - \phi_1'\} W(pp_1 \mid p'p_1')$$

- $\mathcal{L}[p^\mu] = 0, \;\; \mathcal{L}[1] = 0$: Energy-momentum and particle number conservation.
- $\int d\Gamma_p \psi \mathcal{L}[\phi] = \int d\Gamma_p \phi \mathcal{L}[\psi]$ with $\psi = \psi(x, p^\mu)$: Self adjoint properties.
- $\int d\Gamma_p \mathcal{L}[\phi] = 0$ and $\int d\Gamma_p p^\mu \mathcal{L}[\phi] = 0$: Summational invariant property.
- $\int d\Gamma_p \phi \mathcal{L}[\phi] \geq 0$: Non-negative entropy production rate.

$\phi \sim 1, p^\mu \longrightarrow$ Homogeneous solution - Fully arbitrary

$\phi \sim p^\mu p^\nu$ or beyond $\longrightarrow$ Inhomogeneous solution - Controlled by system interaction

Formal framework : Hydrodynamics from kinetic theory

r^{th}order of gradient expansion : $\sum_l Q_l^{(r)} X_l^{(r)} + \sum_m R_m^{(r)\mu} Y_{m\mu}^{(r)} + \sum_n S_n^{(r)\mu\nu} Z_{n\mu\nu}^{(r)} = -\mathcal{L}[\phi^{(r)}]$

$X_l^{(r)}, Y_{m\mu}^{(r)}$ and $Z_{n\mu\nu}^{(r)} \rightarrow$ scalar, vector and rank-2 tensor thermodynamic forces of gradient expansion order r

General solution for ϕ^r as a linear combination of the thermodynamic forces

$$\phi^{(r)} = \sum_l A_l^r X_l^{(r)} + \sum_m B_m^{r\mu} Y_{m\mu}^{(r)} + \sum_n C_n^{r\mu\nu} Z_{n\mu\nu}^{(r)}$$

$A_l^r, B_m^{r\mu}$ and $C_n^{r\mu\nu}$ are unknown coefficients needed to be estimated from the transport equation

Solution from relativistic transport equation

$$Q_l^{(r)} = -\mathcal{L}[A_l^r] \,, R_l^{(r)\mu} = -\mathcal{L}[B_l^{r\mu}] \,, S_l^{(r)\mu\nu} = -\mathcal{L}[C_l^{r\mu\nu}]$$

$A_l^{(r)} = \sum_{s=0}^{\infty} A_l^{r,s}(z,x) P_s^{(0)}$

$B_m^{(r)\mu} = \sum_{s=0}^{\infty} B_m^{r,s}(z,x) P_s^{(1)} \tilde{p}^{\langle\mu\rangle}$

$C_n^{(r)\mu\nu} = \sum_{s=0}^{\infty} C_n^{r,s}(z,x) P_s^{(2)} \tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}$

Semi-orthogonal monic polynomial : $P_0^{(0)}, P_0^{(1)}, P_0^{(2)} = 1, \quad P_1^{(0)} = \tilde{E}_p$

$$\int dF_p (\tilde{E}_p/\tau_R)(\Delta_{\mu\nu}p^\mu p^\nu)^n P_s^{(n)} P_r^{(n)} \sim \delta_{s,r}$$

Formal framework : Hydrodynamics from kinetic theory

Interaction solutions $\rightarrow (A_l^r)^2, (B_l^r)^1, (C_l^r)^0$ **or onwards :**

Can be extracted from transport equation with the form $\sim 1/[\phi,\phi]$

Bracket quantity $\rightarrow [\phi,\phi] = \int d\Gamma_p \phi \mathcal{L}[\phi] \rightarrow$ always non-negative, contains interaction

Homogeneous solutions $\rightarrow (A_l^r)^0, (A_l^r)^1$**,** $(B_l^r)^0$ **:**

Can not be extracted from transport equation, needed to be fixed from matching conditions

Matching condition

Constraints that set the thermodynamic fields to their equilibrium values even in the presence of dissipation

$$\int dF_p \tilde{E}_p^i \phi = 0 \qquad \int dF_p \tilde{E}_p^j \phi = 0 \qquad \int dF_p \tilde{E}_p^k \tilde{p}^{\langle\mu\rangle} \phi = 0$$

$(i,j,k) = (1,2,1) \rightarrow$ Landau-Lifshitz frame $\qquad (i,j,k) = (1,2,0) \rightarrow$ Eckart frame

Out of equilibrium distribution function

$$\phi^{(r)} = \phi_{\rm int}^{(r)} - \tilde{E}_p \left[\frac{I_j}{\mathcal{D}_{i,j}^{1,0}} \int dF_p \tilde{E}_p^i \phi_{\rm int}^{(r)} + (i \leftrightarrow j)\right] - \left[\frac{I_{j+1}}{\mathcal{D}_{i,j}^{0,1}} \int dF_p \tilde{E}_p^i \phi_{\rm int}^{(r)} + (i \leftrightarrow j)\right] - \frac{\tilde{p}_{\langle\nu\rangle}}{J_k} \int dF_p \tilde{E}_p^k \tilde{p}^{\langle\nu\rangle} \phi_{\rm int}^{(r)}$$

Formal framework : Hydrodynamics from kinetic theory

Momentum-dependent relaxation time approximation (MDRTA)

$$\tau_R = \tau_R^0(x)\,(p\cdot u/T)^{\Lambda}$$

$$\mathcal{L}_{\rm MDRTA}[\phi] = \frac{(p\cdot u)}{\tau_R} f^{(0)}(1\pm f^{(0)})\Bigg[\phi - \frac{\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p^2\rangle\langle\frac{\tilde{E}_p}{\tau_R}\phi\rangle - \langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\rangle\langle\frac{\tilde{E}_p}{\tau_R}\phi\tilde{E}_p\rangle}{\langle\frac{\tilde{E}_p}{\tau_R}\rangle\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p^2\rangle - \langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\rangle^2}$$
$$- \tilde{E}_p\frac{\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\rangle\langle\frac{\tilde{E}_p}{\tau_R}\phi\rangle - \langle\frac{\tilde{E}_p}{\tau_R}\rangle\langle\frac{\tilde{E}_p}{\tau_R}\phi\tilde{E}_p\rangle}{\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\rangle^2 - \langle\frac{\tilde{E}_p}{\tau_R}\rangle\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p^2\rangle} - \tilde{p}_{\langle\nu\rangle}\frac{\langle\frac{\tilde{E}_p}{\tau_R}\phi\tilde{p}^{\langle\nu\rangle}\rangle}{\frac{1}{3}\langle\frac{\tilde{E}_p}{\tau_R}\tilde{p}^{\langle\mu\rangle}\tilde{p}_{\langle\mu\rangle}\rangle}\Bigg]$$

1. $\mathcal{L}_{\rm MDRTA}[\phi] = 0$ if $\phi = a + b(p\cdot u) + c^{\mu}p_{\langle\mu\rangle}$
2. Satisfies the self adjoint property as well, $\int d\Gamma_p \psi \mathcal{L}_{\rm MDRTA}[\phi] = \int d\Gamma_p \phi \mathcal{L}_{\rm MDRTA}[\psi]$
3. Satisfies summation invariant property $\int d\Gamma_p \psi \mathcal{L}_{\rm MDRTA}[\phi] = 0$ for $\psi = a + b(p\cdot u) + c^{\mu}p_{\langle\mu\rangle}$
4. Gives conservation laws $\partial_\mu N^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$ microscopically
5. The conservation laws are not needed to be estimated order by order and are treated non-perturbatively
6. The conservation laws are irrespective of the frame indices or particular momentum dependence of τ_R

First-order thermodynamic field corrections

- **General frame**
- **Arbitrary interaction via MDRTA**

$$\frac{\phi_{\text{int}}^{(1)}}{\tau_R^0} = -\tilde{E}_p^{\Lambda-1}\left[\tilde{E}_p^2\frac{DT}{T} + \tilde{E}_p D\tilde{\mu} + \left(\frac{\tilde{E}_p^2}{3} - \frac{z^2}{3}\right)(\partial\cdot u) + \tilde{E}_p\tilde{p}^{\langle\mu\rangle}\left(\frac{\nabla_\mu T}{T} - Du_\mu\right) + \tilde{p}^{\langle\mu\rangle}\nabla_\mu\tilde{\mu} - \tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}\sigma_{\mu\nu}\right]$$

Scalar field correction

$$\begin{bmatrix}\delta n^{(1)}\\ \delta\epsilon^{(1)}\\ \delta P^{(1)}\end{bmatrix} = \begin{bmatrix}\nu_1\\ \varepsilon_1\\ \pi_1\end{bmatrix}\frac{DT}{T} + \begin{bmatrix}\nu_2\\ \varepsilon_2\\ \pi_2\end{bmatrix}(\partial\cdot u) + \begin{bmatrix}\nu_3\\ \varepsilon_3\\ \pi_3\end{bmatrix}D\tilde{\mu}\ ,$$

Vector field correction

$$\begin{bmatrix}W^{(1)\mu}\\ V^{(1)\mu}\end{bmatrix} = \begin{bmatrix}\theta_1\\ \gamma_1\end{bmatrix}\left(\frac{\nabla^\mu T}{T} - Du^\mu\right) + \begin{bmatrix}\theta_3\\ \gamma_3\end{bmatrix}\nabla^\mu\tilde{\mu}$$

The most general field expressions including corrections

$$N^\mu = (n_0 + \delta n)u^\mu + V^\mu$$
$$T^{\mu\nu} = (\epsilon_0 + \delta\epsilon)u^\mu u^\nu - (P_0 + \delta P)\Delta^{\mu\nu} + (W^\mu u^\nu + W^\nu u^\mu) + \pi^{\mu\nu}$$

First-order thermodynamic field corrections

14 First order field correction coefficients - $\nu_1, \nu_2, \nu_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \pi_1, \pi_2, \pi_3, \theta_1, \theta_3, \gamma_1, \gamma_3, \eta$

- Not all transport coefficients but their certain combinations remain invariant under field redefinition due to hydrodynamic frame choice.
- Homogeneous part of correction, i,e frame information exactly cancels to retain only the interaction part in flux transport at any order.

Scalar coefficients

$$f_i = \pi_i - \varepsilon_i \left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{n_0} - \nu_i \left(\frac{\partial P_0}{\partial n_0}\right)_{\epsilon_0}$$

Bulk viscosity : $\zeta = -f_2 + \left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{n_0} f_1 + \frac{1}{T}\left(\frac{\partial P_0}{\partial n_0}\right)_{\epsilon_0} f_3$

Vector coefficients

$$l_i = \gamma_i - \frac{n_0}{\epsilon_0 + P_0}\theta_i$$

Charge conductivity : $k_n = l_3 - \frac{n_0 T}{(\epsilon_0 + P_0)} l_1$

Stability and causality analysis

Rising questions $\cdots$

- Is general frame is enough for producing a stable-causal first order theory ?
- Does arbitrary interaction from the coarse-grained theory control stability-causality by any manner ?
- Is stability a Lorentz invariant property ?
- What is the connection between causality and stability is a dissipative system ?

Linear stability and causality analysis in Lorentz rest frame

$$\epsilon(t,x) = \epsilon_0 + \delta\epsilon(t,x) \qquad n = n_0 + \delta n(t,x) \qquad P(t,x) = P_0 + \delta P(t,x)$$
$$u^\mu(t,x) = (1,\vec{0}) + \delta u^\mu(t,x)$$

Fluctuations are expressed in plane wave solutions via a Fourier transformation

$$\delta\psi(t,x) \to e^{i(\omega t - kx)}\delta\psi(\omega,k), \qquad k^\mu = (\omega,k,0,0)$$

Stability and causality analysis

Shear channel stability and causality situation

Small k limit

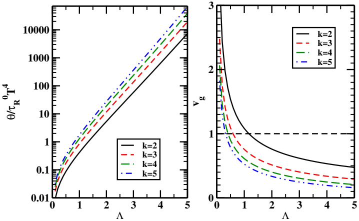
$$\omega_1^T = i\frac{\eta}{(\epsilon_0+P_0)}k^2 + \mathcal{O}(k^4)$$

$$\omega_2^T = i\frac{(\epsilon_0+P_0)}{\theta} + \mathcal{O}(k^2)$$

Large k limit

$$\omega_{1,2}^T = \pm\sqrt{\eta/\theta}\ k + i\frac{(\epsilon_0+P_0)}{2\theta} + \mathcal{O}(\tfrac{1}{k})$$

Asymptotic group velocity : $v_g = \lim_{k\to\infty}\left|\frac{\partial\ \mathrm{Re}(\omega)}{\partial k}\right| = \sqrt{\eta/\theta}$



- Condition for stability : $\theta\ (-\theta_1) > 0$
- Condition for causality : $\theta > \eta$
- $\theta_1 = -\tau_R^0 T^2\left(J_{\Lambda+1} + \frac{\epsilon_0+P_0}{T^2}\frac{J_{k+\Lambda}}{J_k}\right)$
  $\theta_1 = 0$ at $k = 1$ (LL frame)
  $\Lambda = 0$ (momentum independent RTA)

Stability and causality analysis

Sound channel at small k limit

- **Hydrodynamic modes :**

$$\omega_6^L = i\hat{h}^2 \frac{k_n T}{(\epsilon_0 + P_0)} k^2 \qquad \hat{h} = (\epsilon_0 + P_0)/n_0 T$$

$$\omega_{4,5}^L = \pm c_s k + i\frac{\Gamma_s}{2} k^2 + \mathcal{O}(k^3) \qquad \Gamma_s = \left[\frac{4}{3}\eta + \zeta + \frac{k_n T}{c_s^2}\left(\frac{1}{T}\frac{\partial P_0}{\partial n_0}\right)^2_{\epsilon_0}\right] / (\epsilon_0 + P_0)$$

- **Non-hydrodynamic modes :**

$$(i\omega^L)^3 A_6 + (i\omega^L)^2 A_5 + (i\omega^L) A_4^0 + A_3^0 = 0$$

Routh-Hurwitz criteria for stability of non-hydro modes

$$A_6 > 0\ ,\ A_5 > 0\ ,\ A_3^0 > 0\ ,\ B_2 = (A_4^0 A_5 - A_3^0 A_6)/A_5 > 0$$

- A's are explicit functions of frame indices (i, j, k) and MDRTA parameter Λ
- Stability critically depends upon frame indices and medium interaction.

Ref : Phys.Rev.D 106 (2022) 1, L011501

Stability and causality analysis

Sound channel stability criterion

$$A_6 > 0\ ,\ A_5 > 0\ ,\ A_3^0 > 0\ ,\ B_2 = (A_4^0 A_5 - A_3^0 A_6)/A_5 > 0$$

$$\begin{aligned}
A_3^0 &= n_0(\epsilon_0 + P_0) > 0 \\
A_6 &= \frac{\theta_1}{(\epsilon_0 + P_0)} \hat{h} c_s^2 (\nu_1 \epsilon_3 - \nu_3 \epsilon_1) \\
A_5 &= \hat{h} c_s^2 (\nu_3 \epsilon_1 - \nu_1 \epsilon_3) - \theta_1 \left[(\nu_1 f + \nu_3 c) + \frac{1}{\hat{h} T} (\epsilon_1 g + \epsilon_3 d) \right] \\
A_4^0 &= (\epsilon_0 + P_0)(\nu_1 f + \nu_3 c) + n_0(\epsilon_1 c + \epsilon_3 d - \theta_1)
\end{aligned}$$

Field correction coefficients

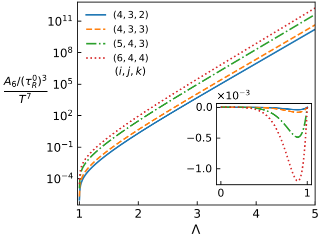
$$\varepsilon_1 = \tau_R^0 \left[\frac{\partial \epsilon_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda+1,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial \epsilon_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda+1,0}}{\mathcal{D}_{i,j}^{1,0}} - T^2 I_{\Lambda+3} \right] \qquad \varepsilon_3 = \tau_R^0 \left[\frac{\partial \epsilon_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial \epsilon_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda,0}}{\mathcal{D}_{i,j}^{1,0}} - T^2 I_{\Lambda+2} \right]$$

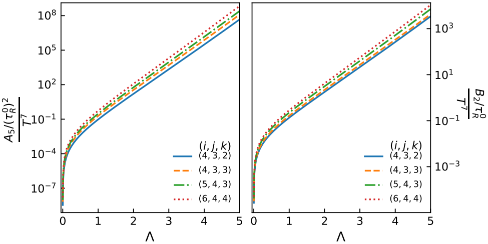
$$\nu_1 = \tau_R^0 \left[\frac{\partial n_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda+1,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial n_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda+1,0}}{\mathcal{D}_{i,j}^{1,0}} - T I_{\Lambda+2} \right] \qquad \nu_3 = \tau_R^0 \left[\frac{\partial n_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial n_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda,0}}{\mathcal{D}_{i,j}^{1,0}} - T I_{\Lambda+1} \right]$$

$$\theta_1 = -\tau_R^0 T^2 \left(J_{\Lambda+1} + \frac{\epsilon_0 + P_0}{T^2} \frac{J_{k+\Lambda}}{J_k} \right)$$

Stability and causality analysis

Sound channel stability situation





- The coefficients ν_1, ε_1 and hence A_5, A_4^0, A_6 vanish both for LL+Eckart $(i = 1, j = 2)$ $\forall\Lambda$.
- The coefficients make A_5 and A_4^0 vanish for $\Lambda = 0$ and A_6 vanish for both $\Lambda = 0$ and 1 at any frame.
- A_6 is negative for $0 < \Lambda < 1$, with higher values of the frame indices making the situation worse with larger negative values $\rightarrow$ increased instability.

A general frame and the nature of underlying interactions are both crucial for the stability and causality of a first-order theory

Stability and causality analysis

Big questions $\cdots$

1. **The stability of a relativistic dissipative system is a Lorentz-invariant property or not !**

   • Stability analysis in local rest frame is often inadequate even sometimes misleading

   • With a boosted background the stability situation can alter drastically

2. **What is the connection between the stability and the speed of signal propagation ?**

   • Correlation studied for **Mueller-Israel-Stewarts (MIS)** - Hiscock et al, Annals Phys.151, 466 (1983), Olson, Annals Phys. 199, 18 (1990), Pu et al, Phys. Rev. D 81, 114039 (2010)

   • Correlation studied for first-order stable-causal **BDNK** theory - Bemfica et al, Phys. Rev.X 12, no.2, 021044 (2022)

3. **Gavassino, Phys. Rev.X 12, no.4, 041001 (2022) provide an intuitive understanding of this puzzle.**

Stability and causality analysis

Linear stability and causality analysis with a Lorentz boosted background

$$\epsilon(t,x)=\epsilon_0+\delta\epsilon(t,x) \qquad n=n_0+\delta n(t,x) \qquad P(t,x)=P_0+\delta P(t,x)$$
$$u^\mu(t,x)=(1,\vec{0})+\delta u^\mu(t,x)$$

Fluctuations are expressed in plane wave solutions via a Fourier transformation

$$\delta\psi(t,x)\rightarrow e^{i(\omega t-kx)}\delta\psi(\omega,k)$$

Background fluid is boosted along x-axis with a constant velocity **v**

$$u_0^\mu=\gamma(1,\mathbf{v},0,0) \qquad \gamma=1/\sqrt{1-\mathbf{v}^2}$$

Velocity fluctuation $\delta u^\mu=(\gamma\mathbf{v}\delta u^x,\gamma\delta u^x,\delta u^y,\delta u^z)\rightarrow u_0^\mu\delta u_\mu=0$ velocity normalization maintained

Transformations : $\omega\rightarrow\gamma(\omega-k\mathbf{v})$ and $k^2\rightarrow\gamma^2(\omega-k\mathbf{v})^2-\omega^2+k^2$ to the local rest frame $\rightarrow$ Dispersion relation in the boosted frame

Shear channel in boosted frame

Small k limit

$$\omega^{\perp}_{1,2} = 0,\ i\frac{(\epsilon_0 + P_0)}{\gamma(\theta - \eta\mathbf{v}^2)} + \mathcal{O}(k)$$

Large k limit

$$\omega^{\perp}_{1,2} = k\frac{\left(\mathbf{v} \pm \sqrt{\frac{\eta}{\theta}}\right)}{\left(1 \pm \mathbf{v}\sqrt{\frac{\eta}{\theta}}\right)} + i\frac{(\epsilon_0 + P_0)\left[1 \pm \sqrt{\frac{\eta\mathbf{v}^2}{\theta}}\right]}{2\gamma(\theta - \eta\mathbf{v}^2)} + \mathcal{O}(\frac{1}{k})$$

Condition for stability : $\theta > \eta\mathbf{v}^2$

The asymptotic causality condition for shear channel in Lorentz rest frame $\theta > \eta$ readily reproduces the stability condition for all $0 < \mathbf{v} < 1$.

With boosted background, the group velocity of the propagating shear mode

$$v_g^{\perp} = \lim_{k\to\infty}\left|\frac{\partial\ \mathrm{Re}(\omega^{\perp})}{\partial k}\right| = (\mathbf{v} + \sqrt{\eta/\theta})/(1 + \mathbf{v}\sqrt{\eta/\theta})$$

Subluminal as long as the asymptotic causality condition in local rest frame is satisfied.

Stability and causality analysis

Causality situation at sound channel

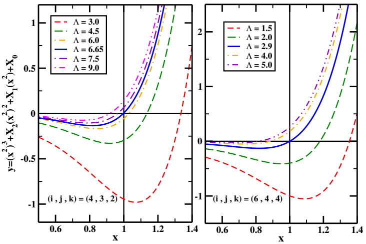
At asymptotic limit $k \to \infty$, the sound dispersion relation becomes,

$$H_6(\omega^{\|})^6 + H_5(\omega^{\|})^5 k + H_4(\omega^{\|})^4 k^2 + H_3(\omega^{\|})^3 k^3 + H_2(\omega^{\|})^2 k^4 + H_1(\omega^{\|}) k^5 + H_0 k^6 = 0$$

Large k expansion : $\omega^{\|} = v_g^{\|} k + \sum_{n=0}^{\infty} c_n k^{-n}$

Asymptotic group velocity values : $H_6(v_g^{\|})^6 + H_5(v_g^{\|})^5 + H_4(v_g^{\|})^4 + H_3(v_g^{\|})^3 + H_2(v_g^{\|})^2 + H_1 v_g^{\|} + H_0 = 0$

Estimating the value of $v_g^{\|}$ at local rest frame



- $A_6(\mathbf{x}^2)^3 - A_4^2(\mathbf{x}^2)^2 + A_2^4 X_1(\mathbf{x}^2) - A_0^6 = 0$
- A's are explicit functions of i, j, k and Λ.
- Increasing Λ corresponds to smaller $v_g^{\|}$ that eventually becomes subluminal.
- $\Lambda_c = 6.65$ for frame $(i, j, k) = (4, 3, 2)$
  $\Lambda_c = 2.9$ for frame $(i, j, k) = (6, 4, 4)$

Stability and causality analysis

Stability situation at sound channel

At large wavelength limit $k \to 0$, the sound dispersion relation becomes,

$$G_6(i\omega^{\|})^6 + G_5(i\omega^{\|})^5 + G_4(i\omega^{\|})^4 + G_3(i\omega^{\|})^3 + G_2(i\omega^{\|})^2(ik) + G_1(i\omega^{\|})k^2 + G_0 ik^3 = 0$$

Small k expansion : $\omega^{\|} = a_0 + a_1 k$

a_1 values : always real, propagating, hydro-mode : does not control stability

a_0 values : always imaginary, non-propagating, non-hydro mode

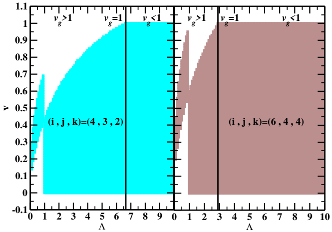
Non-hydro modes : $G_6(ia_0)^3 + iG_5(ia_0)^2 + G_4(ia_0) + G_3 = 0$

Routh-Hurwitz criteria for stability of non-hydro modes

$$G_6, G_5, G_3 > 0\ , \quad G_{B2} = (G_4G_5 - G_3G_6)/G_5 > 0$$

Stability and causality analysis

Island of stability for different boost velocities



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- The Λ values beyond which the asymptotic causality condition holds, the system is also stable for all $0 < \mathbf{v} < 1$.
- For $v_g > 1$ there is no Λ value for which the system is either stable or unstable for all values of $\mathbf{v}$.
- Beyond the $v_g = 1$ line, the stability situation agrees in all reference frames.

Stability is a Lorentz invariant property if and only if the signal propagation within the medium is causal

Gavassino, PRX 12, no.4, 041001 (2022) : Causality violation can chronologically reorder a perturbation in different reference frames by the relativity of simultaneity, so that in a dissipative medium, two observers can disagree on whether the perturbation is growing or decaying.

Conclusion and outlook

Summary· · ·

- **A first-order, stable and causal, relativistic hydrodynamic theory has been derived from a coarse-grained microscopic kinetic theory.**

- **In order to hold stability and causality at first-order theories, besides a general frame, the system interactions need to be carefully taken into account.**

- **The stability situation in different reference frames agrees with each other only as long as the signal propagation respects causality**

Final remarks· · ·

- **A full non-linear analysis along with the study of characteristics is needed to be explored. Work is in progress.**