Spin Hydrodynamics from the Dirac Equation

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Introduction

Hydrodynamics from the Dirac equartion

Dirac equation and Hydrodynamics of ”micro-variables”

It is known that the Dirac equation can be expressed in terms of the covariant bilinear ”classical” variables (Takabayasi,1957). Casting the Dirac equation into fluid form can provide us, I believe good insight into incorporating spin spin into a hydrodynamic description. Here is the outline:

- Dirac equation:

$$i\gamma^{\mu}\partial_{\mu}\psi - e\gamma^{\mu}A_{\mu}\psi - m\psi = 0, \quad (1)$$

$$i\partial_{\mu}\bar{\psi}\gamma^{\mu} + e\bar{\psi}\gamma^{\mu}A_{\mu} - m\bar{\psi} = 0, \quad (2)$$

  where, $\bar{\psi} = \psi^{\dagger}\gamma^{0}$.

- Let γ^{A} be any one of the following 16-combinations

$$\hat{1},\, \gamma^{5},\, \gamma^{\mu},\, \gamma^{\mu}\gamma^{5}\, \&\, \sigma^{\mu\nu} = -1/2\,[\gamma^{\mu},\, \gamma^{\nu}]$$

- Define bilinear variables: Scalar $\Omega = \bar{\psi}\hat{1}\psi$, pseudo-scalar $\bar{\Omega} = i\bar{\psi}\gamma_{5}\psi$, vector $\bar{S}_{\mu} = \bar{\psi}\gamma_{5}\gamma_{\mu}\psi$, tensor $M^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi$ and pseudo-tensor $\bar{M}^{\mu\nu} = i\bar{\psi}\gamma_{5}\sigma^{\mu\nu}\psi$.

Fluidization of the Dirac equation ....

Couple of things: a) For dynamics of the new variables: Multiply Eq.(1) by $\bar{\psi}\gamma^A$ from left & Eq.(2) by $\gamma^A\psi$ from right and subtract. b) Step a) gives new higher rank tensors up to rank-3 .

- New variables:
  $J_\mu = \frac{1}{2m}\delta^*_\mu\Omega,\ \bar{J}_\mu = \frac{1}{2m}\delta^*_\mu\bar{\Omega},\ T^\mu_\nu = \frac{1}{2m}\delta^*_\nu S^\mu,\ \bar{T}^\mu_\nu = \frac{1}{2m}\delta^*_\nu\bar{S}^\mu$ and $N^{\mu\nu}_\alpha = \frac{1}{2m}\delta^*_\alpha M^{\mu\nu}$ & $\bar{N}^{\mu\nu}_\alpha = \frac{1}{2m}\delta^*_\alpha\bar{M}^{\mu\nu}$, where,

$$\delta^*\mu = i(\bar{\psi}\gamma^a\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^A\psi - eA_\mu\bar{\psi}\gamma^A\psi). \quad (3)$$

  $T^{\mu\nu}$ and $\bar{T}^{\mu\nu}$ satisfy

$$T^{\mu\nu} - T^{\nu\mu} = \frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha\bar{S}_\beta$$

$$\epsilon^{\mu\nu\alpha\beta}\bar{T}_{\alpha\beta} = M^{\mu\nu} - \frac{1}{2m}\left(\partial^\mu S^\nu - \partial^\nu S^\mu\right)$$

- Next, we introduce "fluid" like micro-variables: fluid-velocity $v_\mu = \frac{1}{\rho}S_\mu$, spin $w_\mu = \frac{1}{\rho}\bar{S}_\mu$ & fluid-momentum defined below: $k^\mu = \frac{1}{\rho^2}\left(\Omega J^\mu + \bar{\Omega}\bar{J}^\nu\right)$

Momentum k^μ satisfies :

$$\partial_\mu k_\nu - \partial_\nu k_\mu = -\frac{i}{2m}\epsilon^{\alpha\beta\gamma\delta} v_\alpha W_\beta (\partial_\mu v_\gamma \partial_\nu v_\delta - \partial_\mu w_\gamma \partial_\nu w_\delta) - \frac{e}{m} F_{\mu\nu} \quad (4)$$

- further we have introduce total density ρ

$$\rho = \sqrt{\Omega^2 + \bar{\Omega}^2} \quad (5)$$

  and the pseudo-scalar parameter $\theta = tan^{-1}\left(\frac{\bar{\Omega}}{\Omega}\right)$.
- Non-relativistic limit: v^μ does not reproduce non-relativistic limit, but k^μ does. For the Dirac Hamiltonian α matrix represent the "velocity" which has eigen-values ± 1. This could be related with particle & anti-particle mixing.
- To get the macroscopic description one need to express N-body wave function $\Psi(r_1, r_2 ... r_N)$ as $4^N \times 4^N$ of Slater determinant of N one-particle state and it can be written as $4^N \times 4^N$ of Slater determinant of N one-particle states,

$$\Psi(r_1, r_2, ..., r_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{a_1}(r_1) & \psi_{a_1}(r_2) & ... & \psi_{a_1}(r_N) \\ \psi_{a_2}(r_1) & \psi_{a_2}(r_2) & ... & \psi_{a_2}(r_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{a_N}(r_1) & \psi_{a_N}(r_2) & ... & \psi_{a_N}(r_N) \end{vmatrix}$$

It is to be noted that the wave function $\Psi(r_1, r_2 ... r_N)$ satisfy the generalized Dirac equation and the N-body spinor requires the Dirac matrices generalization to the suitable dimensions (e.g. P. Strange, Rel. Q.M., CUP, 1998).
The fluid equations are obtained by averaging over these N-particle states(Asenjo et,al 2011, also Marklund & Brodin, 2007). If p_α is the probability of finding particle in the state α, then the fluid density n in the rest frame $n = \Sigma_\alpha\, p_\alpha \rho_\alpha$

Spin Hydrodynamics

Average of quantity x_α^μ one may write $X^\mu = \langle x_\alpha^\mu \rangle = \frac{1}{n}\Sigma_\alpha p_\alpha \rho_\alpha x_\alpha^\mu$

Thus we write the fluid velocity $U^\mu = \langle v^\mu \rangle$ and the spin $W^\mu = \langle w_\alpha^\mu \rangle$.

$$\partial_\mu \left(nU^\mu\right) = 0 \tag{6}$$

$$\partial_\mu \left(nW^\mu\right) = -2mn\langle \sin\theta \rangle \tag{7}$$

$$\begin{aligned} u^\nu \partial_\nu W^\nu &= W_\nu \left(\partial^\nu U^\mu - \partial^\mu U^\nu\right) + ig^{\mu\beta}\epsilon_{\beta\nu\kappa\lambda} U^\nu W^\kappa \langle \partial^\lambda \theta \rangle \\ &- \langle v^\nu \partial_\nu s^\mu \rangle + \langle w_\nu \left(\partial^\nu z^\mu - \partial^\mu z^\nu\right) \rangle \\ &+ ig^{\mu\beta}\epsilon_{\beta\nu\kappa\lambda} \left(\langle v^\nu s^\kappa \partial^\lambda \theta \rangle + \langle z^\nu \partial^\lambda \theta \rangle W^\kappa \right) \end{aligned} \tag{8}$$

$$\begin{aligned} \langle \cos\theta \rangle U^\mu \partial_\mu W^\nu &= \frac{e}{m} W_\mu F^{\nu\mu} - \left\langle \frac{i}{2m\rho} g^{\nu\lambda} \epsilon_{\lambda\alpha\beta\gamma} \partial_\mu \left(\rho v^\alpha w^\beta \partial^\mu w^\gamma\right) \right\rangle \\ &+ \left\langle \frac{1}{2m\rho} \partial_\mu \left(\rho \partial^\mu \theta v^\nu + \rho w^\nu \partial_\alpha \theta \left(v^\alpha w^\mu - v^\mu w^\alpha\right)\right) \right\rangle \\ &+ \left\langle \partial_\mu \left(i w^\nu \epsilon^{\mu\alpha\kappa\lambda} \partial_\alpha \left(\rho v_\kappa w_\lambda\right)\right) \right\rangle - \langle \cos\theta z^\mu \partial_\mu W^\nu \rangle \\ &- U^\mu \langle \cos\theta \partial_\mu s^\nu \rangle + \langle \sin\theta w^\nu v^\mu \partial_\mu \theta \rangle \end{aligned} \tag{9}$$

Spin Hydrodynamics

The force equation:

$$\begin{aligned}\langle\cos\theta\rangle U^\nu\partial_\nu U_\mu &= -\frac{e}{m}F_{\mu\nu}U^\nu + \frac{1}{2m}\langle v^\nu\partial_\nu\left(\partial_\alpha\theta\left(v^\alpha w_\mu - v_\mu w^\alpha\right)\right)\rangle \\ &+ \left\langle\frac{1}{m\rho}\partial^\nu\left(\rho\partial_\mu\theta w_\nu - i\rho\epsilon_{\nu\alpha\beta\lambda}v^\alpha w^\beta\partial_\mu v^\lambda\right)\right\rangle \\ &+ \frac{i}{2m}\left\langle v^\nu\partial_\nu\left(\frac{1}{\rho}g_{\mu\beta}\epsilon^{\beta\alpha\kappa\lambda}\right)\right\rangle - \langle\cos\theta z^\nu\partial_\nu v_\mu\rangle \\ &- U^\nu\langle\cos\theta\partial_\nu z_\mu\rangle + \langle\sin\theta v_\mu v^\nu\partial_\nu\theta\rangle \end{aligned} \tag{10}$$

& A constraint:

$$\begin{aligned}\epsilon_{\kappa\lambda\mu\nu}\langle v^\kappa w^\lambda\partial^\mu k^\nu\rangle &= -\frac{i}{4m}\epsilon_{\kappa\lambda\mu\nu}\epsilon^{\alpha\beta\gamma\delta}\langle v^\kappa w^\lambda v_\alpha w_\beta\left(\partial^\mu v_\gamma\partial^\nu - \partial^\mu w_\gamma\partial^\nu w_\delta\right)\rangle \\ &- \frac{e}{2m}\epsilon_{\kappa\lambda\mu\nu}\langle v^\kappa w^\lambda\rangle F^{\mu\nu}\end{aligned} \tag{11}$$

- Label α on the variable inside the angular bracket has been dropped for the simplicity

- Nonlinear and quantities like $\langle X \rangle$ make it more difficult to solve.
- These equations applied to study relativistic electromagnetic plasmas (Asenjo et.al 2011), in studying neutrino-induced vorticity and magnetization in astrophysical scenario (Bhattt & George (2017))

Assumptions: Initial state have no flow-velocity & spin-polarization i.e. $U_o^\mu = \left(U_o^0, 0, 0, 0\right)$ and $W_o^\mu = (0, 0, 0, 0)$, subscript o denote background quantities & they are constants in space and time.

- Eq.(7): $-2mn_o\langle\sin\theta_o\rangle \Longrightarrow \langle\sin\theta_o\rangle = 0$ & thus $\cos\langle\cos\theta_o\rangle = 1$.
- Perturbed quantities are functions of x & t . These include macro or fluid variables: $\delta U^\mu = \left(0, U_o^0\delta v^i\right)$, $\delta W^\mu, \delta n$ and micro-variables $\delta z^\mu, \delta s^\mu$& $\delta\theta$ whose average yet to be found.
- z^μ is velocity in the fluid rest-frame and it is related with thermal speed of fluid particles. Spatial components of Eq.(10) shown to contain a term $\partial_j\langle\cos\theta z^i z^j\rangle$ which is equivalent to the pressure gradient term in the Euler equation (for a normal fluid $\cos\theta = 1$).

A Linear Mode Analysis

- Average value of δ_s^μ one can estimate using Li, Stephanov & Yee, PRL127(2021) , which gives the expression for the spatial components $\delta s^i = \frac{1}{12\beta^2}\epsilon^{ijk}\delta\omega_{jk}$, here, ω^{ij} is vorticity and $\beta = 1/T$. δs^0 can be constructed by taking a scalar product of velocity to get $s^\mu = \left(0, \delta s^i\right)$.
- Further assumptions are necessary to evaluate the statistical averages in the equations of spin-hydrodynamics. We assume that $z^\mu, s^\mu\, \theta$ are not correlated (this can be questioned, but it is assumed for simplicity) and $\langle \delta s^\mu s_o^\nu \rangle = 0$ & $\langle \delta (z^\mu s^\nu) \rangle = 0$.
- Under these assumptions, all the terms with $\epsilon_{\alpha\beta,\gamma\lambda}$ in Eq.(8) (spin dynamics) are zero and one gets $\delta W^i = -\frac{1}{12\beta_0^2}\epsilon^{ijk}\delta\omega_{jk}$ and $\delta W^0 = 0$. Exactly the same relation for δW^i from Eq.(9).
- From "spin-current" (Eq.(7)) one gets $\langle \delta\theta \rangle = 0$ as $\partial^i \epsilon^{ijk}\delta\omega_{jk} = 0$.
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Linear Mode Analysis ...

- Finally setting $\theta_o = 0 \& \delta\theta = 0$ in the "Euler" equation, we get the following set of equations from the original spin hydrodynamics:

$$\delta W^i = -\frac{1}{12\beta_0^2}\epsilon^{ijk}\delta\omega_{jk}, \tag{12}$$

$$\partial_0 \delta n = -n_0 \partial_i \delta V^i, \tag{13}$$

$$\partial_0 \delta V^i = -\frac{1}{\epsilon_o + p_o}\nabla^i \delta p \tag{14}$$

- If one takes time derivative of Eq.(13) and substitute for $\partial_0 \delta V^i$, we get equation for sound-waves which is similar to reported by us in G. Sarwar et.all [arXiv:2209.08652]
- If the fluid-velocity perturbation has non-zero vorticity, then it can drive spin-polarization $\delta W^i \neq 0$ (Eq.(12)).

Conclusions

- We have outlined the procedure of obtaining fluid description from the Dirac equation. The equations of spin-hydrodynamics seems to have very different structure from the other models used in QGP.
- Normal mode analysis: Using some simplifying assumptions it is shown that the spin-hydrodynamic model support sound waves and perturbations remain stable. The model presented here does not contain the effect of dissipation.
- It would be interesting investigate this model in the presence of finite vorticity and magnetic field in the initial state.

THANK YOU

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\begin{aligned}
\partial_\mu\left(\rho v^\mu\right) &= 0, \\
\partial_\mu\left(\rho w_\mu\right) &= -2m\rho sin\theta, \\
v^\nu\partial_\nu w^\mu &= w_\nu\left(\partial^\nu v^\mu - \partial^\mu v^\nu\right) + ig^{\mu\beta}\epsilon_{\beta\nu\kappa\lambda}v^\nu w^\kappa\partial^\lambda\theta, \\
\partial_\mu\left(\rho k^\mu w^\nu\right) &= -\frac{e}{m}\rho w_\mu F^{\mu\nu} + \frac{1}{2m}\partial_\mu(\rho\partial^\mu\theta v^\nu) \\
&\quad - \frac{i}{2m}g^{\nu\lambda}\epsilon_{\lambda\alpha\beta\gamma}\partial_\mu\left(\rho v^\alpha w^\beta\partial^\nu w^\lambda\right) \\
\rho v^\nu\partial_\nu k_\mu &= -\frac{e\rho}{m}F_{\mu\nu}v^\nu + \frac{1}{2}\partial^\nu\left(\rho\partial_\mu\theta w_\nu - i\rho\epsilon_{\nu\alpha\beta\lambda}v^\alpha w^\beta\partial_\mu v^\lambda\right)\epsilon_{\beta\nu\kappa\lambda}v^\nu w^\kappa\partial^\lambda\theta \\
\rho v^\nu\partial_\nu k_\mu &= -\frac{e\rho}{m}F_{\mu\nu}v^\nu + \frac{1}{2m}\partial^\nu\left(\rho\partial_\mu\theta w_\nu - i\epsilon_{\nu\alpha\beta\lambda}v^\alpha w^\beta\partial_\mu v^\lambda\right)
\end{aligned}
$$