Fluid dynamics from a ‘least-biased’ truncation of the Boltzmann equation

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Emergent Topics In Relativistic Hydrodynamics, Chirality, Vorticity and Magnetic field

Toshali Sands (February 4, 2023)

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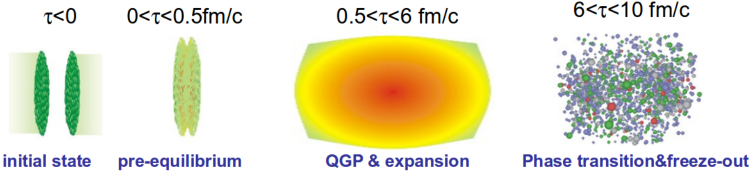
Introduction



(Figure by Steffen A. Bass)

- Traditional hydro: Description using macroscopic variables (T, μ, u^μ) and their gradients accompanied by transport coefficients (η, ζ, σ). Should be distinguished from Israel-Stewart type hydro (Jaiswal's talk yesterday).
- IS-type hydro [Muller '67, Israel, Stewart '76] is remarkably successful in describing intermediate stages of heavy-ion collisions [Heinz et al., Romatschke et al., Dusling and Teaney, Song et al., and several others].
- IS-type hydro derived from kinetic theory works even far-from-equilibrium [Heller et al., Romatschke, Strickland, Noronha, and others]. However, applicability sensitive to truncation scheme of moment-equations. How to choose an appropriate truncation procedure?

IS-type hydrodynamics from kinetic theory

- Consider a system of weakly interacting classical particles; description via kinetic theory using phase-space distribution function, $f(x,p)$.
- Evolution of $f(x,p)$ is governed by Boltzmann equation,

$$p^\mu \partial_\mu f = C[f],$$

where the collisional kernel $C[f]$ denotes interactions.

- Conserved curents $(T^{\mu\nu}, N^\mu)$ appearing in hydro are moments of $f(x,p)$. For example,

$$T^{\mu\nu}(x) \equiv \int dP\, p^\mu\, p^\nu\, f(x,p) = e\, u^\mu u^\nu - (P + \Pi)\, \Delta^{\mu\nu} + \pi^{\mu\nu};$$

here $dP \equiv d^3p/[(2\pi)^3 E_p]$ and $\Delta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$.

- If system is in perfect local equilibrium, $f \to f_{eq}$ (with say $f_{eq} = \exp(-(u\cdot p)/T)$), then $T^{\mu\nu} \to T^{\mu\nu}_{ideal}$, $N^\mu \to N^\mu_{ideal}$.
- Off-equilibrium parts of conserved currents stem from $\delta f \equiv f - f_{eq}$.

IS-type hydrodynamics from kinetic theory

- The bulk viscous pressure and shear stress tensor are:

$$\Pi = -\frac{1}{3}\,\Delta_{\mu\nu} \int dP\, p^{\mu}\, p^{\nu}\, \delta f, \tag{1}$$

$$\pi^{\mu\nu} = \int dP\, p^{\langle\alpha}\, p^{\beta\rangle}\, \delta f, \tag{2}$$

where $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta}\, A^{\alpha\beta}$ with the double-symmetric, traceless, and orthogonal (to u^{μ}) projector defined as,

$$\Delta^{\mu\nu}_{\alpha\beta} = (\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta}\Delta^{\nu}_{\alpha})/2 - \Delta^{\mu\nu}\Delta_{\alpha\beta}/3$$

- Applying co-moving time derivative operator $(u^{\mu}\partial_{\mu})$ on both sides of Eqs. (1, 2) and using the Boltzmann equation,

$$p^{\mu}\partial_{\mu} f = C[f],$$

one gets evolution equations for bulk and shear stresses.

IS-type hydrodynamics from kinetic theory

- For example, consider a massive Boltzmann gas. Also, take a simplistic collisional kernel given by the relaxation-time approximation (RTA) [Andersen & Witting '74],

$$C[f] \approx -\frac{u \cdot p}{\tau_R}\,(f - f_{eq})\,;$$

here τ_R is the time scale for relaxation to local equilibrium.

- One then obtains a relaxation-type evolution of Π [Denicol et al. '12]:

$$\dot{\Pi} + \frac{\Pi}{\tau_R} = -\alpha_1\,\theta + \alpha_2\,\Pi\,\theta + \alpha_3\,\pi^{\mu\nu}\,\sigma_{\mu\nu} + \frac{m^2}{3}\,\rho^{\mu\nu}_{(-2)}\,\sigma_{\mu\nu} + \frac{m^2}{3}\nabla_\mu\,\rho^{\mu}_{(-1)} + \frac{m^4}{9}\,\rho_{(-2)}\,\theta,$$

where $\alpha_i = \alpha_i(T, m)$.

- Standard definitions: $\dot{\Pi} = u^\mu \partial_\mu \Pi$ (time-derivative), $\theta = \partial_\mu u^\mu$ (expansion rate), $\nabla^\mu = \Delta^{\mu\nu}\,\partial_\nu$ (space-like derivative), velocity stress tensor $\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha u^\beta$.

IS-type hydrodynamics from kinetic theory

- However, the equation,

$$\dot{\Pi} + \frac{\Pi}{\tau_R} = -\alpha_1\,\theta + \alpha_2\,\Pi\,\theta + \alpha_3\,\pi^{\mu\nu}\,\sigma_{\mu\nu} + \frac{m^2}{3}\rho^{\mu\nu}_{(-2)}\,\sigma_{\mu\nu} + \frac{m^2}{3}\nabla_\mu\,\rho^{\mu}_{(-1)} + \frac{m^4}{9}\rho_{(-2)}\,\theta,$$

  is **not closed** due to couplings to $\rho-$ tensors.

- The $\rho-$ tensors are non-hydrodynamic moments of f. For example,

$$\rho^{\mu}_{(-1)} \equiv \Delta^{\mu}_{\alpha}\int dP\,(u\cdot p)^{-1}\;p^{\alpha}\,\delta f,$$

$$\rho^{\mu\nu}_{(-2)} \equiv \int dP\,(u\cdot p)^{-2}\;p^{\langle\mu}\,p^{\nu\rangle}\,\delta f,$$

- Similar feature exists for shear stress evolution equation.

- Needs truncation, i.e., to express δf in terms of quantities appearing in $T^{\mu\nu}$.

Standard truncation procedures

- Expanding $\delta f(x,p) \equiv f_{eq}\,\phi$ in powers of momenta. For example, Grad's '14-moment' expansion [Dusling, Teaney '08],

$$\phi(x,p) \approx \frac{p^\mu p^\nu}{2(e+P)T^2}\left(\pi_{\mu\nu} + \frac{2}{5}\Pi\Delta_{\mu\nu}\right).$$

- δf motivated by Chapman-Enskog (CE) like expansion of a simplistic Boltzmann collisonal kernel [Bhalerao, Jaiswal et al. '14]:

$$\phi(x,p) \approx -\frac{\beta}{3\beta_\Pi}\left(3c_s^2(u\cdot p)^2 + p_{\langle\mu\rangle}p^{\langle\mu\rangle}\right)\frac{\Pi}{u\cdot p} + \frac{\beta}{2\beta_\pi}\frac{p_\mu p_\nu \pi^{\mu\nu}}{u\cdot p},$$

where $p^{\langle\mu\rangle} = \Delta^\mu_\alpha\, p^\alpha$. The above two δf's are **linear** in viscous stresses.

- Approximating $f(x,p)$ by an anisotropic ansatz [Romatschke and Strickland '03]:

$$f(x,p) \approx f_{eq}\left(\frac{1}{\lambda}\sqrt{p_\mu\, p_\nu\, \Xi^{\mu\nu}}\right),\quad \Xi^{\mu\nu} = u^\mu u^\nu + \xi^{\mu\nu} - \Delta^{\mu\nu}\,\psi$$

Why a new truncation scheme?

- Grad assumes δf to be quadratic in momenta (ad-hoc); Chapman-Enskog δf should not be valid far-from-equilibrium. Both become negative (unphysical) at large momenta. Resulting hydrodynamics breaks down in certain flow profiles.
- The aHydro ansatz does not become negative and can handle large shear deformations at early stages of heavy-ion collisions.
  - But: its form is ad-hoc. Not possible to describe large negative bulk viscous pressures, especially, for small masses of particles.
  - May not be the most suitable distribution to model arbitrary flow profiles.
- We want to implement a truncation scheme that (i) leads to a framework which may work both near and far from local equilibrium and ii) does not invoke uncontrolled assumptions about the microscopic physics.

The 'least-biased' distribution [E. Jaynes, Phys. Rev. 106, 620 (1957)]

- We want δf to be expressed solely in terms of quantities appearing in $T^{\mu\nu}$.

- The '**least-biased**' distribution that uses **all of, and only** the information provided by $T^{\mu\nu}$ is one that maximizes the non-equilibrium entropy,

$$s[f] = -\int dP\,(u\cdot p)\,\Phi[f],\;\; \Phi[f] \equiv f\ln(f) - \frac{1+a\,f}{a}\ln(1+a\,f),$$

($a = (-1, 0, 1)$ for FD, MB, BE statistics),

- subject to constraints,

$$\int dP\,(u\cdot p)^2\; f = e,\;\; -\frac{1}{3}\int dP\,p_{\langle\mu\rangle}\,p^{\langle\mu\rangle}\; f = P + \Pi,$$

$$\int dP\,p^{\langle\mu}\,p^{\nu\rangle}\; f = \pi^{\mu\nu},$$

where $p^{\langle\mu}\,p^{\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta}\,p^{\alpha}\,p^{\beta}$.

The idea behind ‘least-biasedness’

- Consider a system in a macrostate specified by (E, V, N). The system can be in a variety of microstates consistent with the macrostate.
- One may, in general, assign any probability distribution to these microstates.
- But, the probability distribution where all such microstates are assumed to be equally probable is the ‘least-biased’ one.
- Such a distribution also maximizes the Shannon (or information) entropy, $S = -\sum_i p_i \ln(p_i)$.

Lagrange's method of undetermined multipliers

- Introduce Lagrange multipliers,

$$s[f] = -\int dP\ (u\cdot p)\ \Phi[f] + \Lambda\left[e - \int dP\,(u\cdot p)^2\, f\right] + \lambda_\Pi\left[P + \Pi + \frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha\, p^\beta\, f\right] + \gamma_{\langle\alpha\beta\rangle}\left[\pi^{\alpha\beta} - \int dP\, p^{\langle\alpha} p^{\beta\rangle} f\right].$$

- Functional derivative w.r.t. f: $\frac{\delta s[f]}{\delta f} = 0.$
- The solution for distribution function,

$$f_{\mathrm{ME}}(x,p) = \left[\exp\left(\Lambda(u\cdot p) - \frac{\lambda_\Pi}{u\cdot p} p_{\langle\alpha\rangle} p^{\langle\alpha\rangle} + \frac{\gamma_{\langle\alpha\beta\rangle}}{u\cdot p} p^{\langle\alpha} p^{\beta\rangle}\right) - a\right]^{-1},$$

- Note that in absence of information about dissipative fluxes, $f_{\mathrm{ME}} \to f_{eq}$.

A pleasant surprise

- Expand f_{ME} around equilibrium:

$$f_{ME} \approx f_{eq}\left[1-\left(c_\lambda\lambda_\Pi+c_{\mu\nu}\gamma^{\langle\mu\nu\rangle}\right)(u\cdot p)+\lambda_\Pi\frac{p_{\langle\mu\rangle}p^{\langle\mu\rangle}}{u\cdot p}-\gamma^{\langle\mu\nu\rangle}\frac{p_{\langle\mu}p_{\nu\rangle}}{u\cdot p}\right].$$

- Plug δf_{ME} in definitions for shear and bulk,

$$\pi^{\mu\nu}=\Delta^{\mu\nu}_{\alpha\beta}\int dP\, p^\alpha\, p^\beta\, \delta f_{ME}, \quad \Pi=-\frac{1}{3}\Delta_{\mu\nu}\int dP\, p^\mu\, p^\nu\, \delta f_{ME},$$

and invert.

- δf_{ME} to linear order in dissipative quantities,

$$\delta f_{ME}=f_{eq}\left[\frac{\beta}{3\beta_\Pi}\left((1-3c_s^2)(u\cdot p)^2-m^2\right)\frac{\Pi}{u\cdot p}+\frac{\beta}{2\beta_\pi}\frac{p_\mu p_\nu\pi^{\mu\nu}}{u\cdot p}\right].$$

- Solve RTA BE: $p^\mu\partial_\mu f=-(u\cdot p)\delta f/\tau_R$ in the small Knudsen number approximation (Chapman-Enskog like iteration); the δf_{CE} obtained matches exactly with δf_{ME}! Mere coincidence?

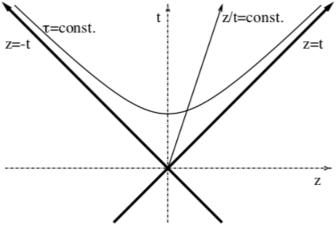
Features of maximum-entropy distribution

- Positive-definite for all momenta.
- Non-linear dependence on $(\Pi, \pi^{\mu\nu})$; exact matching to $T^{\mu\nu}$ for entire range of viscous stresses allowed by kinetic theory.
- Reduces to linearized Chapman-Enskog δf of Boltzmann eq. in the relaxation-time approximation for weak dissipative stresses.
- Can be systematically improved by adding information of higher-order moments via other Lagrange multipliers.

Application I: Bjorken flow [J.D. Bjorken, PRD, 27, 140 (1983)]

- Bjorken flow is valid during the early stages of ultra-relativistic heavy-ion collisions.
- The fluid is assumed to be homogeneous in $(x - y)$ direction.
- The medium expands boost-invariantly along beam $(z-)$ direction: $v^x = 0,\ v^y = 0,\ v^z = z/t$.

- Switch to Milne coordinates $(\tau, x_\perp, \phi, \eta_s)$ where $\tau \equiv \sqrt{t^2 - z^2}$, and $\eta_s \equiv \tanh^{-1}(z/t)$.
- Fluid appears static, $u^\mu = (1, 0, 0, 0)$. However, has finite expansion rate, $\theta = 1/\tau$.



Consequences of Bjorken symmetries

- $T^{\mu\nu} = \mathrm{diag}(e, P_T, P_T, P_L)$, has 3 independent variables where $P_T = P + \Pi + \pi/2$, $P_L = P + \Pi - \pi$. All functions depend only on proper time τ.
- Bjorken symmetries constrain phase-space dependence of distribution: $f(x, p) = f(\tau; p_T, p_\eta)$ [Baym '84, Florkowski, Strickland et al. '13, '14].
- The maximum-entropy distribution has 3 Lagrange parameters:

$$f_{\mathrm{ME}} = \exp\left(-\Lambda p^\tau - \frac{\lambda_\Pi}{p^\tau}\left(p_T^2 + p_\eta^2\right) - \frac{\gamma\left(p_T^2/2 - p_\eta^2\right)}{p^\tau}\right),$$

where $p^\tau = \sqrt{p_T^2 + p_\eta^2 + m^2}$.

Bjorken flow: evolution equations

- As before, consider a simplified RTA Boltzmann collisional kernel,

$$\frac{\partial f}{\partial \tau} = -\frac{1}{\tau_R}\left(f - f_{eq}\right),$$

and use it to obtain the evolution of energy and effective pressures:

$$\frac{de}{d\tau} = -\frac{e + P_L}{\tau}, \quad \text{where } e = \int dP\, (p^\tau)^2\, f,$$

$$\frac{dP_L}{d\tau} = -\frac{P_L - P}{\tau_R} + \frac{\bar{\zeta}_z^L}{\tau}, \quad \frac{dP_T}{d\tau} = -\frac{P_T - P}{\tau_R} + \frac{\bar{\zeta}_z^\perp}{\tau}$$

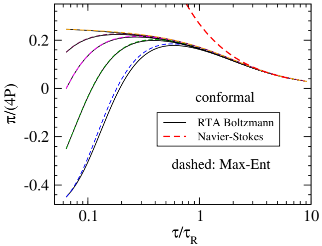
- The couplings $\bar{\zeta}_z^L$ and $\bar{\zeta}_z^\perp$ involve non-hydro moments:

$$\bar{\zeta}_z^L = -3P_L + \int dP\, (p^\tau)^{-2}\, p_\eta^4\, f, \quad \bar{\zeta}_z^\perp = -P_T + \frac{1}{2}\int dP\, (p^\tau)^{-2}\, p_\eta^2\, p_T^2\, f,$$

- To truncate, we replace $f \to f_{\mathrm{ME}}$. This makes $\bar{\zeta}_z^L$ and $\bar{\zeta}_z^\perp$ functions of (e, P_L, P_T). Now, solve 3 equations; same complexity as hydro.

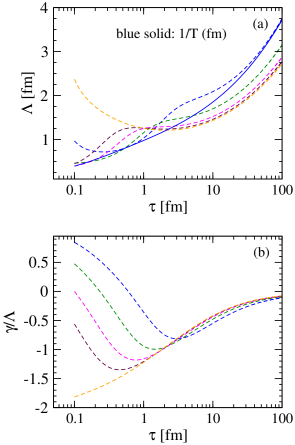
Case I: conformal dynamics (manuscript in preparation)

- In conformal case, $e = 3P$, $\Pi = 0$, and $\tau_R = 5(\eta/s)/T$.

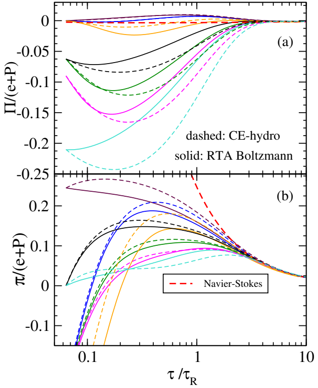


- Good agreement between Max-Ent truncated BE and exact solution of BE even far-off-equilibrium.
- At late times, $\mathrm{slope}(\Lambda) \approx \mathrm{slope}(1/T)$, and anisotropy $\gamma \to 0$.

- Evolution of Lagrange parameters:

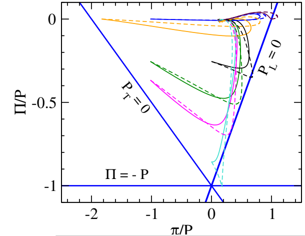


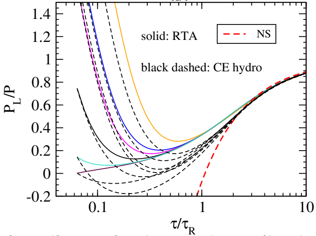
Case IIa: non-conformal CE hydro [S. Jaiswal, C.C., et al. '22]



- CE hydrodynamics not in good agreement for shear and bulk inverse Reynolds numbers far-off-equilibrium.

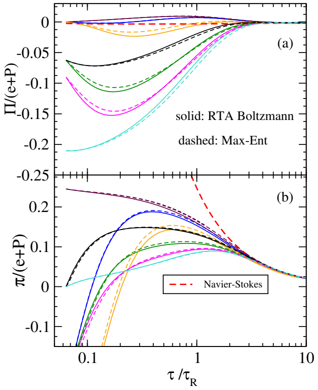
- Can generate dissipative stresses outside the domain allowed by KT:



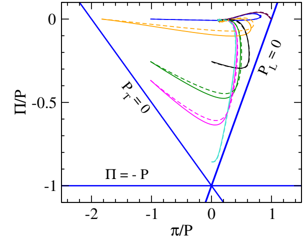


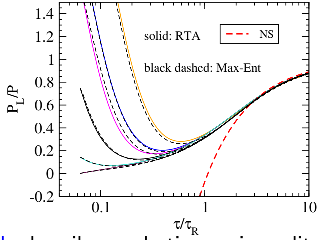
does not describe early-time universality in P_L/P.

Case IIb: Max-Ent truncated kinetic theory (in preparation)



- Generates dissipative stresses within domain allowed by KT.



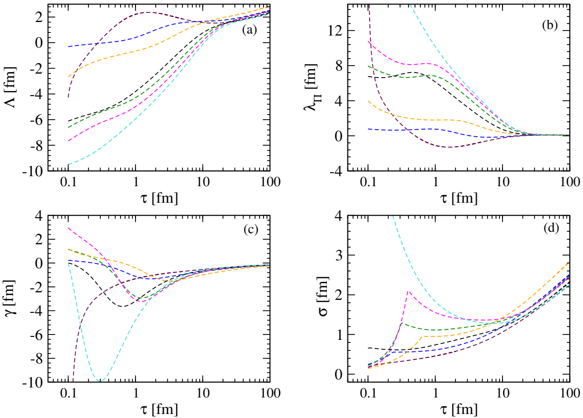


- Max-Ent truncated kinetic theory provides good agreement for shear and bulk inverse Reynolds numbers throughout evolution.

Accurately describes early-time universality in P_L/P.

Evolution of Lagrange parameters (in preparation)

- In far-off-equilibrium regimes, $\Lambda < 0$!
- Should not be identified with $\sim$ inverse temperature at early times.
- At late times, $\Lambda > 0$ and $(\lambda_\Pi, \gamma) \to 0$.



- The quantity $\sigma \equiv \Lambda + \lambda_\Pi - |\min(\gamma/2, -\gamma)|$ is positive definite.
- Ensures $f_{ME}(\vec{p}) \to 0$ at large momenta.

Large negative bulk pressure and $\Lambda < 0$

- The total isotropic pressure P_r is,

$$P + \Pi = \frac{1}{3}\int dP\, \vec{p}^2\, f$$

- $\Pi \sim -P$ can be attained by populating low momentum states with large number of particles, $f \sim A\,\delta(|\vec{p}|)/\vec{p}^2$.
- At low momenta $f_{\mathrm{ME}} \approx \exp(-\Lambda\, m)$. Enhancement of occupation in low momentum modes is facilitated by $\Lambda < 0$.
- The aHydro ansatz, $f_a = \exp(-\sqrt{p_T^2/\alpha_T^2 + p_\eta^2/\alpha_L^2 + m^2}/\lambda)$, cannot generate $\Pi \sim -P$ for $m/T \lesssim 1$; requires the introduction of a fugacity factor [C.C., S. Jaiswal et al, PLB 2020].

Application II: Gubser Flow [S.S. Gubser, PRD, 82, 085027 (2010)]

- Gubser flow is longitudinally boost-invariant: $v^z = z/t$, and has $u^\phi = 0$. But it has transverse dynamics: $u^r(x) \neq 0$.
- Re-scale metric, $ds^2 \to d\hat{s}^2 = ds^2/\tau^2$, followed by coordinate transform: $(\tau, r, \phi, \eta) \to (\rho, \theta, \phi, \eta)$,

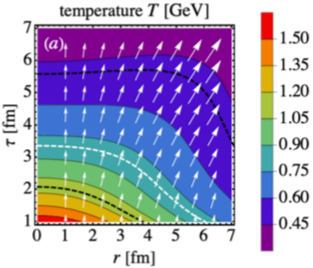
$$\rho = -\sinh^{-1}\left(\frac{1 - q^2\tau^2 + q^2 r^2}{2q\tau}\right), \; \theta = \tan^{-1}\left(\frac{2qr}{1 + q^2\tau^2 - q^2 r^2}\right),$$

such that $\hat{u}^\mu = (1, 0, 0, 0)$.

Weyl rescaled unitless quantities,

$$e(\tau, r) = \frac{\hat{e}(\rho)}{\tau^4},$$

$$\pi_{\mu\nu}(\tau, r) = \frac{1}{\tau^2}\frac{\partial \hat{x}^\alpha}{\partial x^\mu}\frac{\partial \hat{x}^\beta}{\partial x^\nu}\hat{\pi}_{\alpha\beta}(\rho).$$



Du et al. [2019]

Evolution equations: Gubser flow

- The evolution of two independent components $(\hat{e}, \hat{P}_T)$ are given by:

$$\frac{d\hat{e}}{d\rho} = -2\tanh\rho\left(\hat{e} + \hat{P}_T\right),$$
$$\frac{d\hat{P}_T}{d\rho} = -\frac{1}{\hat{\tau}_R}\left(\hat{P}_T - \hat{P}\right) - 2\tanh\rho\,\hat{\zeta}^{\perp}, \text{ where } \hat{\tau}_R = \frac{5(\eta/s)}{\hat{T}}.$$

- Similar to the Bjorken case, the equations are not closed:

$$\hat{\zeta}^{\perp} = 2\,\hat{P}_T - \frac{1}{4}\int d\hat{P}\,(\hat{p}^{\rho})^{-2}\left(\frac{\hat{p}_{\Omega}}{\cosh\rho}\right)^4 f;$$

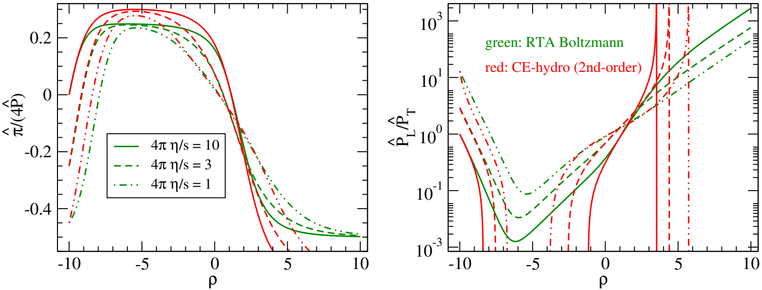
here $\hat{p}_{\Omega} = \sqrt{\hat{p}_{\theta}^2 + \hat{p}_{\phi}^2/\sin^2\theta}$ and $\hat{p}^{\rho} = \sqrt{\hat{p}_{\Omega}^2/\cosh^2\rho + \hat{p}_{\eta}^2}$.

- As before, we truncate by replacing $f \to f_{\mathrm{ME}}$:

$$f_{\mathrm{ME}} = \exp\left(-\hat{\Lambda}\,\hat{p}^{\rho} - \frac{\hat{\gamma}}{\hat{p}^{\rho}}\left(\frac{\hat{p}_{\Omega}^2}{\cosh^2\rho} - \hat{p}_{\eta}^2\right)\right)$$

Results: Breakdown of CE hydrodynamics I

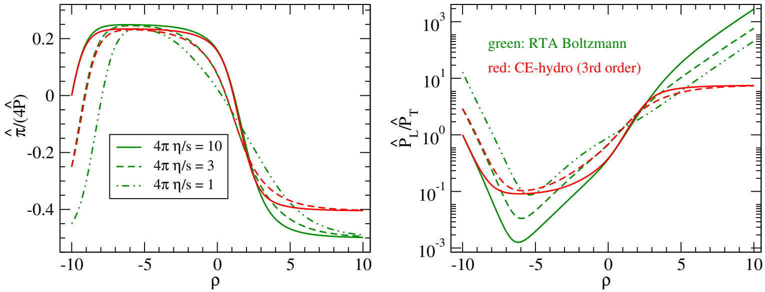
- Evolution of normalised shear and pressure anisotropy using second-order CE hydro (identical to Denicol et al. or DNMR):



- Rapid transverse expansion in Gubser flow at late times prevents system from thermalizing; fluid approaches transverse free-streaming : $\hat{P}_T \to 0$; not described by CE hydro.
- Second-order CE and DNMR yield negative $\hat{P}_L$ and $\hat{P}_T$.

Results: Breakdown of CE hydrodynamics II

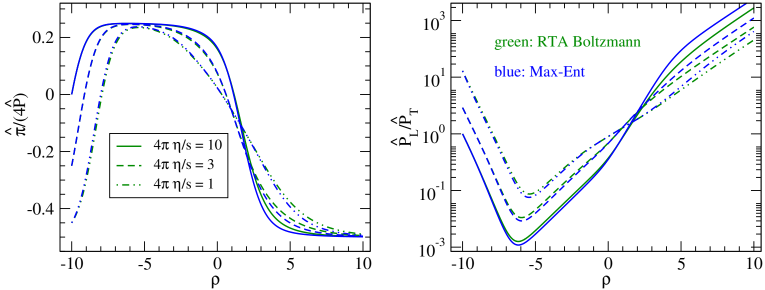
- Evolution of normalised shear and pressure anisotropy using third-order CE hydro [C.C., Heinz, et al. '18]:



- Third-order CE yields incorrect asymptotic value of $\hat{\pi}/(4\hat{P}) \approx -0.4$.
- For initialisations $\hat{\pi}/(4\hat{P}) \lesssim -0.4$, third-order CE equations become numerically unstable.

Results: Max-Ent truncated BE (in preparation)

- Evolution of shear inverse Reynolds number and pressure anisotropy using Max-Ent truncated kinetic theory:



- Max-Ent truncated BE correctly describes both longitudinal ($\hat{\bar{\pi}} \approx 0.25$) and transverse free-streaming ($\hat{\bar{\pi}} \approx -0.5$) domains.

Conclusions

- In order to derive macroscopic evolution equations from a Boltzmann equation, we propose a 'least-biased' distribution function to truncate the infinite tower of moment equations.
- This scheme does not introduce ad-hoc assumptions about the microscopic physics or the flow profile being modeled; uses information contained only within the hydrodynamic moments of the distribution function.
- Relaxation-type dynamics obtained with this procedure was shown to accurately predict the kinetic theory evolution of $T^{\mu\nu}$ in both free-streaming and hydrodynamic regimes for certain flow profiles.
- The description of $T^{\mu\nu}$ within this approach for flow profiles with less restricted symmetries remains to be seen.

Backup Slide 1: Applicability of classical kinetic theory [Jeon and Heinz, arXiv:1503.03931 (2015)]

- Hydro formulated as a series in velocity gradients: $\pi^{ij} \sim \eta \partial^i v^j$, $\Pi \sim -\zeta \partial \cdot v$.
- Three scales: Two microscopic: $l_{mfp} \sim 1/(\sigma v n)$, thermal wavelength $l_{th} \sim 1/T$, one macroscopic $1/L \sim \partial \cdot u$.
- $l_{mfp}/l_{th} \sim \eta/s,\ \zeta/s,\ T\kappa/s$
- Hydro applicable whenever microscopic and macroscopic scales are well-separated: $l_{mfp}\, \partial \cdot u \equiv Kn < 1$
  - Dilute gas regime: $l_{mfp}/l_{th} \sim \eta/s \gg 1$; Weakly coupled regime, Boltzmann equation applicable (on-shell particles).
  - Dense gas regime: $\eta/s \sim 1$; quasi-particle description in terms of Wigner functions.
  - Liquid regime: $\eta/s \ll 1$; strong-coupling regime, no valid kinetic description.

Backup slide 2: Lagrange multipliers of ME distribution

- 7 Lagrange parameters $(\Lambda, \lambda_\Pi, \gamma_{\langle\mu\nu\rangle}) \implies$ 7-d inversion problem; numerically expensive.
- However, matching condition implies that shear matrix $\boldsymbol{\pi} \equiv \pi^{ij}_{\rm LRF}$ is a power-series in $\boldsymbol{\gamma} \equiv \gamma^{ij}_{\rm LRF} \implies [\boldsymbol{\pi}, \boldsymbol{\gamma}] = 0$; simultaneously diagonalizable by spatial rotation.
- 3 of 5 d.o.f's of γ^{ij} fixed using common eigenvectors of π^{ij}; 4-d inversion problem.

Backup 3: Simplifying the non-linear problem

- The full (non-linear) problem requires an inversion for 7 parameters $(\Lambda, \lambda, \gamma_{\alpha\beta})$: numerically intractable.
- To match shear stress tensor,

$$\pi^{ij} = \Delta^{ij}_{kl} \int dP p^k p^l \exp\left(-\Lambda E_p - \frac{\lambda_0}{E_p} p^2\right) \exp\left(-\frac{\gamma_{rs} p^r p^s}{E_p}\right)$$

- We show,

$$\pi = \Gamma - \frac{1}{3} I \, \mathrm{tr}(\Gamma),$$

$$\tilde{c}_1 \gamma - \tilde{c}_2 \gamma^2 + \tilde{c}_3 \gamma^3 - \tilde{c}_4 \gamma^4 + \cdots \equiv \Gamma,$$

- The shear tensor and γ^{ij} commute, $[\pi, \gamma] = 0$; Simultaneously diagonalizable.

Backup 4: Simplifying the non-linear problem

- π^{ij} is symmetric; has real eigenvalues and admits orthogonal eigenvectors (can be diagonalised by spatial rotation, $\pi_D = R^T \pi R$).

- Diagonalise π; This diagonalises γ as well.

- Essentially, 3 of 5 independent degrees of freedom in the matrix γ can be fixed using the (common) eigenvectors of π^{ij}.

- Only two-dimensional root finding required to obtain $\gamma_D = \mathrm{diag}(\gamma_1, \gamma_2, -(\gamma_1 + \gamma_2))$ in terms of eigenvalues of π^{ij}.

- This property greatly simplifies the problem numerically.

Backup 4: Non-equilibrium entropy

- The canonical entropy $S = -\sum_i p_i \ln(p_i)$ for a continuous distribution:

$$S = -\int \frac{d^{3N}x\, d^{3N}p}{N!}\, \rho \ln(\rho),$$

where,

$$\rho(x_1, \cdots, x_N, p_1, \cdots, p_N) = \frac{\exp(-\beta H_N(x_1, \cdots, x_N, p_1, \cdots, p_N))}{Z(T, V, N)}$$

- Due to weak interaction,

$$H_N = \sum_i H_i, \quad Z(T, V, N) = Z(T, V, 1)^N = V^N n^N / N!,$$

where n is number density. Thus,

$$S = -\frac{\beta\, V^N}{Z(T, V, 1)^N} \int d^{3N}p\, H(p) \exp(-\beta H(p)) - \ln(Z(T, V, N))$$

Backup 5: Non-equilibrium entropy

- For large N, $\ln(Z(T, V, N)) \approx N$. Thus,

$$S = V \int d^3p \left(\beta\, H(p)\, f_{eq} + f_{eq}\right),$$

and the entropy density:

$$s = -\int d^3p\, f_{eq} \left(\ln(f_{eq}) - 1\right).$$

- Out of equilibrium, replace $f_{eq} \to f$. Relativistic version,

$$s = -\int dP\ (u \cdot p)\ f\ \left(\ln(f) - 1\right).$$

Backup 6: Conformal Hydrodynamics [R. Loganayagam, arXiv:0801.3701 (2008)]

- Equations of hydro are Lorentz covariant: admits rotationally and boost-invariant solutions.
- Hydro equations also have conformal invariance: should admit conformally invariant solutions.
- Under a conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu}$
- Weyl covariant derivative $\mathcal{D}_\mu T^{\mu\nu} \to e^{-w\phi} \tilde{\mathcal{D}}_\mu \tilde{T}^{\mu\nu}$ if $T^{\mu\nu} \to e^{-w\phi} \tilde{T}^{\mu\nu}$
- Using definition of $\mathcal{D}$ one can show

$$\mathcal{D}_\mu T^{\mu\nu} = d_\mu T^{\mu\nu} + A^\nu T^\mu_\mu$$

  where $A^\mu = \dot{u}^\mu - (\theta/3) u^\mu$
- Hydro equations are conformal if $T^\mu_\mu = m^2 \int dP\, f = 0$.

Backup 7: Gubser symmetries [S.S. Gubser, PRD, 82, 085027 (2010)]

- Instead of translational invariance (whose generators are $\xi_i = \frac{\partial}{\partial x^i}$), Gubser uses invariance under the group $SO(3)_q$ whose generators are $\partial/\partial\phi$, $\partial/\partial\eta$, and

$$\xi_i = \frac{\partial}{\partial x^i} + q^2 \left[2x^i x^\mu \frac{\partial}{\partial x^\mu} - x^\mu x_\mu \frac{\partial}{\partial x^i} \right], \quad (i = 1, 2)$$

  $1/q \approx$ transverse size

- These generators are easy to understand in $dS_3 \times R$

$$d\hat{s}^2 \equiv \frac{ds^2}{\tau^2} = d\rho^2 - \cosh^2\rho \left(d\theta^2 + \sin^2\theta d\phi^2 \right) - d\eta^2,$$

  where they correspond to rotations in (θ, ϕ):

$$\rho = -\sinh^{-1}\left(\frac{1 - q^2\tau^2 + q^2 r^2}{2q\tau} \right), \quad \theta = \tan^{-1}\left(\frac{2qr}{1 + q^2\tau^2 - q^2 r^2} \right),$$

- The only time-like four vector invariant under these transformations $[\xi, \hat{u}] = 0$ is $\hat{u}^\mu = (1, 0, 0, 0)$.