Workshop on
Geometry, Analysis and Mathematical Physics

July 24 – August 02, 2023

School of Mathematical Sciences

National Institute of Science Education and Research Bhubaneswar

(An Autonomous Institute under Department of Atomic Energy, Government of India)

Organizers: Ramesh Manna, Chitrabhanu Chaudhuri, Ritwik Mukherjee

Mini-courses

List of speakers

1. Mahuya Dutta, ISI Kolkata
2. Somnath Basu, IISER Kolkata
3. Subhojoy Gupta, IISc Bangalore
4. Rudra Sarkar, ISI Kolkata
5. Kingshook Biswas, ISI Kolkata
6. Pranav Pandit, ICTS Bangalore
7. Utsav Choudhury, ISI Kolkata
8. Parasar Mohanty, IIT Kanpur
9. Suparna Sen, University of Calcutta
10. Muna Naik , HRI, Prayagraj

Titles and Abstracts

1. **Convex Integration and h-principle**

   Speaker: Mahuya Dutta, ISI Kolkata

   Abstract: The convex integration technique has its origin in the work of Nash and Kuiper on C^1 isometric immersions. Taking a cue from Kuiper's approach, Gromov developed the theory of convex integration which has a wide range of applications to partial differential relations which appear in geometry. There has been a renewed interest in the theory due to the work of C. De Lellis and L. Székelyhidi Jr. on equations of Fluid dynamics.

   Prerequisites:

   (a) Smooth manifolds. Tangent spaces. Smooth maps between manifolds, derivatives of smooth maps.

   (b) Tangent and cotangent bundles. Vector fields and flows. Differential forms, exterior differentiation of forms. Lie derivative.

   (c) Basic language of vector bundles and fibre bundles. Jet bundles.

2. **Theory of manifolds ala Morse**

   Speaker: Somnath Basu, IISER Kolkata

   Lecture 1: Theory of manifolds ala Morse

   Abstract: We shall review the theory of smooth manifolds, followed by the viewpoint pioneered by Marston Morse. The basic idea is to probe manifolds by studying functions on it.

   References:

(a) J. Milnor : Topology from a differentiable viewpoint,

(b) J. Milnor : Morse theory,

(c) J. Hirsch : Differential Topology

Lecture 2: Morse lemma

Abstract: We shall discuss the basic lemma of Morse theory and its consequences. There will be plenty of visually appealing examples.

Lecture 3: Reeb's Theorem & exotic spheres

Abstract: We will prove Reeb's Theorem and use it to outline the construction of exotic 7-spheres, due to Milnor.

Lecture 4: Reeb's Theorem revisited

Abstract: We shall discuss the theory of Morse-Bott functions, a generalization of Morse functions. We shall then state and explain a generalization of Reeb's result for Morse-Bott functions.

3. **Quadratic differentials and their geometry**

Speaker: Subhojoy Gupta, IISc Bangalore

Abstract: Holomorphic quadratic differentials on a Riemann surface arise in various contexts, and have been an important part of Teichmüller theory since its inception. In this mini-course, I plan to discuss some of the ways they arise, and talk about the flat geometry induced by such a differential, which has played a major role in many of the developments. I plan to include a discussion of meromorphic quadratic differentials, which is of recent interest.

Some highlighted topics that I plan to cover in each lecture are as follows:

Lecture 1: Half-translation surfaces, extremal length problems, Strebel differentials

Lecture 2: Teichmüller maps, measured foliations and their trajectory structure

Lecture 3: The Hubbard-Masur theorem and Gardiner-Masur theorem

Lecture 4: Generalizations to the case of meromorphic quadratic differentials

4. **Peter-Weyl Theorem**

Speaker: Rudra Sarkar, ISI Kolkata

Abstract: For any $n \in \mathbb{Z}$, the function $e_n : \theta \to e^{in\theta}$ on $\mathbb{R}$ is periodic of period 2π, i.e. $e_n(\theta) = e_n(\theta + 2\pi)$ for all $\theta \in \mathbb{R}$. Thus e_n can be considered as a function on the unit circle, denoted by T.

A primary motivation of Fourier analysis on T is the desire to express an arbitrary (e.g. continuous) function f on T as a (possibly infinite) linear combination of e_n. This linear combination is called the Fourier series of the function and coefficient of e_n in

this linear combination is called the n-th Fourier coefficient of f. We ask, where else can we play this game and who will play the role of e_n there? A circle is a metric space, a topological space, a group, a compact topological group and is equipped with a canonical measure $\frac{d\theta}{2\pi}$.

Looking at the results and proofs of the classical Fourier analysis on circle we try to comprehend what are the structures needed to carry out the corresponding tasks e.g. expressing an abstract function in terms of concrete objects and what are those concrete objects which replaces e_n.

Peter-Weyl theorem (1927) is an attempt towards this, as it endeavours to push through, as much as possible, the basics of Fourier analysis on circle to any compact topological group G. But unlike circle the group G may not be commutative. Thus this theorem marks also the beginning of non-commutative Fourier analysis.

5. **Moebius maps and marked length spectrum rigidity**

Speaker: Kingshook Biswas, ISI Kolkata

Abstract: Moebius maps of Euclidean space are the maps given by composing reflections in hyperplanes and inversions in spheres. They can be characterized as the homeomorphisms of (the one-point compactification of) Euclidean space which preserve Euclidean cross-ratios of quadruples of distinct points. It is a classical fact that any Moebius map of Euclidean space extends to an isometry of the hyperbolic space of one dimension higher (viewing the n-dimensional Euclidean space as the boundary of the $(n+1)$-dimensional hyperbolic space), and in fact all isometries of hyperbolic space arise in this way. This fact is useful in the proofs of various rigidity theorems for hyperbolic manifolds, for example in Mostow Rigidity, and in the Ending Lamination Conjecture.

For complete, simply connected manifolds of variable negative curvature, one can define an ideal boundary of the space, known as the visual boundary, such that every geodesic ray converges to a unique point in the boundary. Topologically, the visual boundary is an $(n-1)$-sphere. There is a canonical cross-ratio on this boundary, which is a positive function of quadruples of distinct points in the boundary. More generally, a visual boundary and cross-ratio can also be defined for more general metric spaces known as CAT(-1) spaces, which are metric spaces with a synthetic notion of curvature bounded above by -1. A map between boundaries is defined to be Moebius if it preserves cross-ratios. It is straightforward to see that any isometry between CAT(-1) spaces extends to a Moebius map between their boundaries.

Unlike the classical case of hyperbolic space however, in this setting the converse is an open problem: does every Moebius map between boundaries of CAT(-1) spaces extend to an isometry between the spaces? The motivation for this problem, which we call the Moebius rigidity problem, comes from another well-known rigidity problem for negatively curved manifolds known as the Marked length spectrum rigidity conjecture.

We describe the connection between these two problems, as well some partial results on the Moebius rigidity problem.

6. **Divergent Series**

   Speaker: Pranav Pandit, ICTS, Bangalore

   Abstract: This mini-course will be an introduction to divergent formal power series. We will learn how to sum up the terms of certain divergent series using the Borel-Laplace method. This will be used to study the behaviour of solutions to ordinary differential equations in the complex domain. Applications of these methods to geometry and physics will be discussed.

7. **Gromov-Witten Invariants and Enumerative Geometry**

   Speaker: Utsav Choudhury, ISI Kolkata

   Abstract: The goal of the lectures is to give an exposition about Gromov-Witten Classes and its applicatoion to enumerative geometry following the article [1].

   Lecture 1: We will very briefly review the basics of (coarse)-Moduli spaces, Moduli Stacks necessary for Enumerative Geometry and recollect some basic defor- mation theory.

   Lecture 2: After briefly recollecting intersection theory on Moduli spaces, we describe the axioms of Gromov-Witten classes and give the geometric intuition behind them following [1][Section 2].

   Lecture 3: In this lecture we will prove that tree level Gromov-Witten classes can be reconstructed, under suitable hypothesis on the cohomology of the ambient algebraic manifold, starting with basic codimension 0 classes. We will give another formalism of cohomological field theory, based upon a version of operads ([1][Section 6]).

   Lecture 4: Based on the formalism developed in the previous lecture, we will compute the cohomology of moduli space of genus 0 curves and then formally prove the existence of Gromov-Witten classes for projective spaces.

   References: [1] M. Kontsevich, Yu. Manin, Gromov-Witten classes, quantum cohomology and Enumerative Geometry, Commun. Math. Phys. 164, 525-562(1994).

8. **Calderón-Zygmund Theory for Non-Doubling Measure**

   Speaker: Parasar Mohanty, IIT Kanpur

   Abstract: It was widely assumed that the Calderón-Zygmund theory was only true for homogeneous regions i.e. the underlying measure is doubling. The perception shifted 25 years ago. The scope of this theory was expanded to include non-doubling measures. In this series of lectures, we will trace some of the earliest developments, with a particular emphasis on the following topics:

(a) Centred and Un-centred Hardy-Littlewood Maximal function

(b) Calderón Zygmund Decomposition

(c) Boundedness of Calderón Zygmund operator

(d) Cotlar's Inequality and Boundedness of Maximal Calderón Zygmund operator

Some helpful resources are

(a) Nazarov, F.; Treil, S.; Volberg, A. Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces. Internat. Math. Res. Notices 1997, no. 15, 703-726.

(b) Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory. Progress in Mathematics, 307. Birkhuser/Springer, Cham, 2014.

9. **Harmonic Analysis on $\mathbb{H}^2$**

Speaker: Suparna Sen (University of Calcutta) and Muna Naik (HRI, Prayagraj)

Abstract: The aim of our lectures is to develop harmonic analysis on the real hyperbolic space ($\mathbb{H}^2$). It is well-known that the Riemannian manifold $\mathbb{H}^2$ is a Riemannian Symmetric space i.e. at each point, $\mathbb{H}^2$ is equipped with special kind of isometry called geodesic inversion. Our lectures will be intended as an introduction to harmonic analysis on symmetric space $\mathbb{H}^2$. Precisely, we plan to cover the following topics:

(a) Geometry of Poincare disk and upper half plane,

(b) Helgason Fourier transform,

(c) Radon transform,

(d) Abel transform,

(e) Inversion and Plancherel formula.

Reference: Topics in harmonic analysis on homogeneous spaces, 1981, S. Helgason.

Research Seminars

List of speakers

1. Senthil Raani, IISER Berhampur
2. Sreedhar Bhamidi, HRI Prayagraj
3. Rukmini Dey, ICTS, Bangalore
4. Soumen Sarkar, IIT Madras
5. Ramesh Sreekantan, ISI Bangalore

Title and Abstract

1. **Distance sets of sparse sets**

   Speaker: Senthil Raani, IISER Berhampur

   Abstract: The distance set of a set E consists of all non-negative numbers that represent distances between pairs of points in E. In this talk, we will describe the properties of distances in Lebesgue null sets that are of large Hausdorff dimension. This talk is based on joint work with Malabika Pramanik.

   Prerequisite: Basic Fourier analysis, definitions and properties of Hausdorff measure

2. **Hochschild homology and matrix factorization categories**

   Speaker: Sreedhar Bhamidi, HRI Prayagraj

   Abstract: We will discuss the definition of matrix factorization categories and motivate why they are important. We will discuss a Hirzebruch-Riemann-Roch (HRR) type theorem for matrix factorization categories of Deligne-Mumford stacks. which is a consequence of a Hochschild-Kostant-Rosenberg type isomorphism. This talk is based on a joint work with Dongwook Choa and Bumsig Kim.

3. **Berezin-type quantization on compact even dimensional manifolds and pull-back coherent states**

   Speaker: Rukmini Dey

   Abstract: We first give a local description of Berezin quantization of $\mathbb{C}P^d$. We show that a Berezin-type quantization can be achieved on a compact even dimensional manifold M^{2d} by removing a skeleton M_0 of lower dimension such that what remains is diffeomorphic to $\mathbb{R}^{2d}$ which we identify with $\mathbb{C}^d$ and embed in $\mathbb{C}P^d$. A local Poisson structure and Berezin-type quantization are induced from $\mathbb{C}P^d$. This construction depends on the diffeomorphism. We study the possibility of this construction to be extended to the whole of M. We have a similar construction where we consider an arbitrary complex manifold and use local coordinates to induce the quantization from $\mathbb{C}P^d$. We study the possibility of defining a global Berezin quantization on compact complex manifolds. We give a similar construction of Berezin-Toeplitz quantization. Finally we give a simple construction of pullback coherent states on compact smooth manifolds.

4. **Some properties of polynomial vector fields on $S^1 \times S^2$**

   Speaker: Soumen Sarkar

   Abstract: Let X be a vector field on $S^p \times S^q$. In this talk, I'll characterize linear, quadratic, cubic Kolmogorov and homogeneous vector fields on $S^1 \times S^2$ and $S^2 \times S^1$. Then, I'll compute the number of certain invariant algebraic subsets of $S^p \times S^q$ for the vector field X if either $p > 1$ or $q > 1$.

5. **Algebraic cycles and values of Green's functions**

   Speaker: Ramesh Sreekantan

   Abstract: In this talk we will discuss the construction of certain motivic cycles in families of $K3$ surfaces. The construction is inspired by classical theorems of Kummer and Humbert and some ideas from enumerative geometry. We then show that these cycles can be used to prove that the values of certain Green's functions are logarithms of algebraic numbers. Gross, Kohnen and Zagier had made conjectures along similar lines and we will relate our work with theirs.