A Gibbs ILC Algorithm to Estimate CMB Posterior over Large Angular Scales of the Sky

Vipin Sudevan\textsuperscript{1}

in collaboration with

Dr. Rajib Saha\textsuperscript{2}

\textsuperscript{1}Leung Center for Cosmology and Particle Astrophysics, NTU, Taiwan
\textsuperscript{2}Indian Institute of Science Education & Research, Bhopal.

DAE-HEP, 2020

December 14, 2020

ILC Method - In Pixel Space

- Total number of available frequency bands be $n$
- A cleaned map ($\hat{y}$) is obtained by forming a linear superposition of these input maps ($x^i$),

$$\hat{y} = \sum_{i=1}^{n} w_i x^i , \quad (1)$$

with weights constrained by

$$\sum_{i=1}^{n} w_i = 1$$

- Weights are computed by minimizing the variance in the cleaned maps.
- Choice of Weights which minimizes the variance of cleaned map is given by,

$$W = \frac{e A^\dagger}{e A^\dagger e^T} \quad (2)$$

where, $A_{ij} = x_i^T x_j$

and $[e] = (1, 1, \ldots, 1)_{1 \times n}$.

Figure 1: Standard deviation map obtained from the difference of cleaned map and corresponding randomly generated input CMB map using 200 MC simulations of foreground minimization following usual pixel space ILC approach at large angular scales.
Why there is excess residuals in simple ILC method?

We consider two cases:

- **Case 1**
  \[
  \hat{\mathbf{C}} = \hat{\mathbf{C}}_{fc} + \hat{\mathbf{C}}_f + \hat{\mathbf{C}}_N
  \]
  
  The weights are obtained as follows,
  \[
  \mathbf{W} = \frac{\mathbf{e}\hat{\mathbf{A}}^\dagger}{\mathbf{e}\hat{\mathbf{A}}^\dagger \mathbf{e}^T}
  \]  
  
  where \( \hat{\mathbf{A}} = \hat{\mathbf{C}}_{fc} + \hat{\mathbf{C}}_f + \hat{\mathbf{C}}_N \).

- **Case 2**
  \[
  \hat{\mathbf{C}} = \hat{\mathbf{C}}_f + \hat{\mathbf{C}}_N
  \]
  
  The weights are obtained as follows,
  \[
  \mathbf{W} = \frac{\mathbf{e}\hat{\mathbf{A}}^\dagger}{\mathbf{e}\hat{\mathbf{A}}^\dagger \mathbf{e}^T}
  \]  
  
  where \( \hat{\mathbf{A}} = \hat{\mathbf{C}}_f + \hat{\mathbf{C}}_N \).
A Global ILC Method

- We incorporate the CMB Covariance structure information in the usual ILC method in pixel space.
- Instead of minimizing the cleaned map variance, we propose a new measure,

\[ \sigma^2 = y^T C^{-1} y, \]  

(5)

where \( C \) is the Theoretical CMB covariance matrix,

\[ C_{ij} = \sum_{l=2}^{l_{max}} \frac{2l + 1}{4\pi} C_l B_l^2 P_l(\cos(\gamma_{ij})) P_l^2 \]  

(6)

where

\[ \cos(\gamma_{ij}) = \cos(\theta_i)\cos(\theta_j) + \sin(\theta_i)\sin(\theta_j)\cos(\phi_i - \phi_j) \]

Figure 4: Standard deviation map obtained following Global ILC Method.

A Gibbs ILC Approach in harmonic Space

Gibbs Sampling

- Draw a sample, $S^{i+1}$, from from conditional density of CMB signal $S$ given both, the data $D$ and some chosen CMB theoretical angular power spectrum, $C^i_\ell$. Symbolically,

  $$S^{i+1} \leftarrow P_1(S|D, C^i_\ell).$$

- Now draw a sample of $C^{i+1}_\ell$ from the conditional density of $C_\ell$ given both, $D$ and $S^{i+1}$, which was obtained in the first step above. In symbols,

  $$C^{i+1}_\ell \leftarrow P_2(C_\ell|D, S^{i+1}).$$

At this stage one has a pair of samples $S^{i+1}, C^{i+1}_\ell$.

- Repeat above two basic steps for $i = 1$ to $\mathcal{N}$, where $\mathcal{N}$ is a large number, by replacing first $C^i_\ell$ by $C^{i+1}_\ell$ in step 1 above, and then replacing $S^{i+1}$ in Eqn. 8 by the one obtained in Eqn. 7.

After some initial pair of samples of signal and theoretical power spectrum are discarded (i.e., after initial burn-in period has completed) they represent the desired samples drawn from the posterior density, $P(S, C_\ell|D)$, under consideration.
Methodology

- A cleaned map is defined as
  \[ y = \sum_{i=1}^{n} w_i X_i \]  
  (9)

- The weights are obtained by minimizing
  \[ \sigma^2 = y^T C^\dagger y = WAW^T \]  
  (10)

  where
  \[ A_{ij} = X_i^T C^\dagger X_j = \sum_l (2l + 1) \frac{\sigma_{ij}^l}{C_l} \]  
  (11)

- The weights which minimizes \( \sigma^2 \) are obtained as
  \[ W = \frac{eA^\dagger}{eA^\dagger e^T} \]  
  (12)

- The relation between pixel space CMB Covariance matrix and that of harmonic space is
  \[ C^{pp'} = \sum_p \sum_{p'} Y_{l'm'}^p C_{l'm',lm} Y_{lm}^{p'} \]  
  (13)
Sample a new theoretical $C_l$ from the Planck 2015 CMB theoretical angular power spectrum using Gibbs sampling.

$$P(C_l|s, d) = \frac{e^{-\frac{(2l+1)\sigma_l^2}{2}}}{\sqrt{C_l^{2l+1}}}(14)$$

Obtain a full-sky cleaned CMB map using the Eq. 9.

Estimate a cleaned angular power spectrum $\sigma_l$ from the full sky cleaned CMB map.

Using $\sigma_l$ as the prior, sample a new Theoretical angular powers spectrum $C_l$.

Repeat the above steps till convergence is achieved.

In our analysis, we the number of steps was limited to 5000.

We have also checked the convergence using Gelmann-Rubin Statistics.

Results - Histogram Plots

- Cleaned map.

![Histogram plots for some selected pixels](image)

**Figure 5:** The normalized probability density of CMB pixel temperatures for some selected pixels are shown in red. The normalization for each density is such that the peak corresponds to a value of unity. The horizontal axes represent pixel temperatures in the unit of \(\mu K\) (thermodynamic). The positions of mean temperatures are shown by the blue vertical lines.

- Angular power spectrum.

![Angular power spectrum plots](image)

**Figure 6:** Normalized densities of the CMB theoretical angular power spectrum obtained by Gibbs sampling for different multipoles. The horizontal axis for each sub plot represents \(\ell(\ell + 1)C_\ell/(2\pi)\) in the unit 1000 \(\mu K^2\). The region within the two vertical lines represent 1 \(\sigma\) confidence interval for the theoretical angular power spectrum.
Results - Cleaned CMB Map

**Figure 7:** Top left panel shows the best-fit cleaned CMB map obtained from the Gibbs samples. In top right panel we show the difference between the best-fit and the mean cleaned CMB map. We see that both agrees quite well. In the bottom left and right panels, we show the difference between the best-fit cleaned map and the Commander map and standard deviation map obtained using the observed CMB maps respectively.
Figure 8: Top panel shows the best-fit CMB theoretical angular power spectrum along with the asymmetric error bars indicating 68.27% confidence intervals obtained from the Gibbs samples in brown line. The sky blue and deep blue points are the angular power spectrum estimated from NILC and Commander CMB maps respectively. The black line is the Planck 2015 theoretical power spectrum. The bottom panel shows the differences of Commander and NILC angular power spectra respectively from our best-fit angular power spectrum.
Gelman-Rubin Statistics

- Let there are \( M \) independent chains with length \( N \) each (for simplicity).
- Compares variance of each chain to the pooled variance

\[
B = \frac{N}{M-1} \sum_{m=1}^{M} (\hat{\theta}_m - \hat{\theta})^2
\]

\[
W = \frac{1}{M} \sum_{m=1}^{M} \hat{\sigma}_m^2
\]

\[
\hat{V} = \frac{N-1}{N} W + \frac{M-1}{MN} B
\]

\[
\hat{R}_c = \sqrt{\frac{\hat{V}}{W}}
\]  

(15)

- Gelman - Rubin diagnostic \( R_c \) is < 1.001
- For our analysis \( R_c \) is 1.00005
Simulation Results

**Figure:** Top right figure shows the best-fit cleaned CMB map obtained from Monte Carlo simulations. The bottom right figure shows the difference between the best-fit and the input CMB map used in the simulation. The top left figure shows the difference between the best-fit and mean CMB map. The bottom left figure shows the standard deviation map.
**Blackwell-Rao Estimates**

**Figure**: The Blackwell-Rao estimates of the likelihood functions for the multipoles $2 \leq \ell \leq 32$. 
Current Projects

- Generalize the Gibbs ILC method to handle partial sky maps.
- Implementing this method to higher resolution maps.