Cosmology of Bianchi type-I metric using renormalization group approach for quantum gravity

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Plan of the talk

- Renormalization group approach to quantum gravity
- Exact Renormalization Group equation
- Bianchi type-I cosmological model with running $G$ and $\Lambda$
- Radiation
- Stiff Matter
- Summary
Renormalization group approach to quantum gravity

- Traditional approach to QG is as follows: Einstein-Hilbert action is regarded as a fundamental action which should be quantized along the same lines as the familiar renormalizable field theories.

- Difficult both technically and conceptually.

- Non-renormalizability hampers a meaningful perturbative analysis.

- However, one can argue that gravity should not be quantized at all, because Einstein gravity is an effective theory resulting from quantizing some yet unknown fundamental theory.

- The RG approach regards Einstein gravity to be valid near a certain non-zero momentum scale $k$.

- This means that it arises from some fundamental theory by a “partial quantization” in which only excitations with momenta larger than $k$ are integrated out, while those with momenta smaller than $k$ are not included.
An effective theory at scale $k$, when evaluated at tree level, should correctly describe all gravitational phenomena which involve a typical scale $k$ acting as a physical infra-red cutoff.

Generating functional of Euclidean correlation functions

$$Z[J] = \int D\phi \ e^{-S[\phi]+(J,\phi)} , \ (J,\phi) = \int d^d x \ J(x)\phi(x).$$

Schwinger functional $W[J] = \log_e Z[J] \rightarrow$ connected correlation functions.

Effective action is obtained by Legendre transform of $W[J]$

$$\Gamma[\varphi] = (J,\varphi) - W[J] , \ \varphi(x) = \frac{\delta W[J]}{\delta J(x)}.$$
Renormalization group approach to quantum gravity

- Scale-dependent generating functional
  \[ Z_k[J] = \int D\phi \ e^{-S[\phi] + (J,\phi) - \Delta S_k[\phi]} \]

- Scale-dependent Schwinger functional
  \[ W_k[J] = \log_e Z_k[J] \]

- Regulator: quadratic functional
  \[ \Delta S_k[\phi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ \phi^*(p) R_k(p) \phi(p) \]

- The scale dependent modified effective action is of the form
  \[ \Gamma_k[\phi] = (J,\phi) - W_k[J] - \Delta S_k[\phi] \]
Exact Renormalization Group Equation

- **Exact non-perturbative flow equation**

\[
\partial_t \Gamma_k = \frac{1}{2} Tr \left[ \partial_t R_k \left( \Gamma_k^{(2)}[\Phi] + R_k \right)^{-1} \right]
\]

where \( t \equiv \ln k \).

- For gravity the flow equation takes form as

\[
\partial_t \Gamma_k[g, \bar{g}] = \frac{1}{2} Tr \left[ \left( \kappa^{-2} \Gamma_k^{(2)}[g, \bar{g}] + R_k^{grav}[\bar{g}] \right)^{-1} \partial_t R_k^{grav}[\bar{g}] \right] - \frac{1}{2} Tr \left[ \left( M[g, \bar{g}] + R_k^{gh}[\bar{g}] \right)^{-1} \partial_t R_k^{gh}[\bar{g}] \right]
\]

where \( \Gamma_k^{(2)}[g, \bar{g}] \) is the Hessian of \( \Gamma_k[g, \bar{g}] \) with respect to \( g_{\mu\nu} \) and \( \kappa = (32\pi \bar{G})^{-\frac{1}{2}} \), where \( \bar{G} \) is the value of \( G(k) \) as \( k \to \infty \).
The form of the cut-off function is taken to be

\[ R_k(p^2) \propto k^2 R^{(0)} \left( \frac{p^2}{k^2} \right) \]

where the function

\[ R^{(0)}(z) = z \left[ \exp(z) - 1 \right]^{-1} \]

is smooth and satisfies the conditions \( R^{(0)}(0) = 1 \) and \( R^{(0)}(z) \to 0 \) for \( z \to \infty \). This implies \( R_k(p^2) \sim k^2 \) for \( p^2 << k^2 \), that is the regulator screens the IR modes in a mass like fashion. The regulator function is zero for \( p^2 >> k^2 \) implying that all the fluctuations with momenta \( p^2 >> k^2 \) are included in the effective action.

Truncation up to \( \sqrt{g} \) and \( \sqrt{g} R \) requires to consider an effective action of the form

\[ \Gamma_k[g, \bar{g}] = (16\pi G(k))^{-1} \int d^d x \sqrt{g} \left\{ -R(g) + 2\Lambda(k) \right\} + S_{gf}[g, \bar{g}]. \]
Flow equations for $G$ and $\Lambda$

- This now leads to the flow equations for $G(k)$ and $\Lambda(k)$.

- The flow equation for $G(k)$ and $\Lambda(k)$ are

$$k \partial_k G(k) = \eta_N G(k)$$

$$k \partial_k \Lambda(k) = \eta_N \Lambda(k) + \frac{1}{2\pi} k^4 G(k) \left[ 10 \Phi_2^1 (-2\Lambda(k)/k^2) - 8 \Phi_2^1(0) - 5 \eta_N \Phi_2^1 (-2\Lambda(k)/k^2) \right].$$


- The solutions $G(k)$ and $\Lambda(k)$ can be written as power series

$$G(k) = G_0 \left[ 1 - \omega G_0 k^2 + \omega_1 G_0^2 k^4 + \mathcal{O}(G_0^3 k^6) \right]$$

$$\Lambda(k) = \Lambda_0 + G_0 k^4 \left[ \nu + \nu_1 G_0 k^2 + \mathcal{O}(G_0^2 k^4) \right].$$
The anisotropic cosmology described by Bianchi Type-I metric is given by
\[ ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2. \]

The energy-momentum tensor of a perfect fluid reads
\[ T_{\mu\nu} = (p + \rho)v_{\mu}v_{\nu} + pg_{\mu\nu}. \]

Einstein’s field equations of general relativity with time-varying \( G \) and \( \Lambda \) reads
\[ R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} = -8\pi G(t)T_{\mu\nu} + \Lambda(t)g_{\mu\nu}. \]

The equations for scale factors read
\[ -\frac{\ddot{b}}{b} - \frac{\ddot{c}}{c} - \frac{\dot{b}\dot{c}}{bc} = 8\pi Gp - \Lambda. \]
The covariant conservation of the energy-momentum tensor yields

\[ \dot{\rho} + (p + \rho) \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right) = 0. \]

The consistency equation reads

\[ 8\pi G \dot{\rho} + \dot{\Lambda} = 0. \]
The solution of the scale factors are

\[ a(t) = m_1 \mathcal{R}(t) \exp \left[ \frac{l(2 + \beta)}{3} \int \frac{dt}{\mathcal{R}^3(t)} \right] \]
\[ b(t) = m_2 \mathcal{R}(t) \exp \left[ \frac{l(\beta - 1)}{3} \int \frac{dt}{\mathcal{R}^3(t)} \right] \]
\[ c(t) = m_3 \mathcal{R}(t) \exp \left[ -\frac{l(1 + 2\beta)}{3} \int \frac{dt}{\mathcal{R}^3(t)} \right] \]

where \( m_1, m_2, m_3 \) are arbitrary constants of integration satisfying \( m_1 m_2 m_3 = 1 \) and \( \mathcal{R}^3(t) = abc \).
The simplest behaviour of $k$ at late times is

$$k = \sum_n \frac{\xi_n}{t^n}.$$

We will keep terms up to $n = 3$ as we are interested in the behaviour of $G$ and $\Lambda$ up to $\mathcal{O}\left(\frac{t_{Pl}^4}{t^4}\right)$

$$k = \frac{\xi}{t} + \frac{\sigma}{t^2} + \frac{\delta}{t^3}.$$

The time dependent Newton’s gravitational constant and cosmological constant in the perturbative regime reads

$$G(t) = G_0 \left[ 1 - \frac{\tilde{\omega} G_0}{t^2} \left( 1 + \frac{2\tilde{\sigma}}{t} + \frac{2\tilde{\delta}}{t^2} + \frac{\tilde{\sigma}^2}{t^2} \right) + \frac{\omega_1 G_0^2}{t^4} + \mathcal{O}\left(\frac{t_{Pl}^6}{t^6}\right) \right]$$

$$\Lambda(t) = \Lambda_0 + \frac{G_0}{t^4} \left[ \tilde{\nu} \left( 1 + \frac{4\tilde{\sigma}}{t} + \frac{4\tilde{\delta}}{t^2} + \frac{6\tilde{\sigma}^2}{t^2} \right) + \tilde{\nu}_1 G_0 + \mathcal{O}\left(\frac{t_{Pl}^4}{t^4}\right) \right].$$
The equation of state relating the pressure $p$ and the energy density $\rho$ reads
\[ p(t) = \Omega \rho(t). \]

The average scale factor reads
\[ R(t) = \left[ -\frac{\mathcal{M}\dot{G}}{\dot{\Lambda}} \right]^{\frac{1}{3+3\Omega}}. \]

The energy density up to $\mathcal{O}(G_0)$ takes the following form in terms of $t$
\[ \rho(t) = \frac{1}{4\pi} \left( \tilde{\nu} \right) \frac{1}{G_0 t^2} \left\{ 1 + \frac{2\tilde{\sigma}}{t} + \frac{2\tilde{\delta}}{t^2} + \frac{\tilde{\sigma}^2}{t^2} + \left( \frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}} \right) \frac{G_0}{t^2} \right\}. \]
By comparing inverse powers of $t$ in the consistency condition, we get

\[
\frac{\tilde{\omega}}{\tilde{\nu}} = \frac{8}{3}
\]

\[
4 \left( \frac{\tilde{\nu}}{\tilde{\omega}} \right) \tilde{\sigma} = \frac{3}{2} \tilde{\sigma} - \frac{l^2 \alpha^2 (\beta^2 + \beta + 1)}{3}.
\]

It immediately follows that

\[
l^2 \alpha^2 (\beta^2 + \beta + 1) = 0.
\]

If $l \neq 0$, we must have $\beta^2 + \beta + 1 = 0$. This implies that the two roots of $\beta$ are complex.

Since the scale factors must be real, it follows that $l = 0$.

Hence we see that all the directional Hubble parameters must be equal, that is, the universe must be FLRW in the presence of radiation.
For $\Omega = 1$, the scale factors are

\begin{align*}
a(t) &= m_1 R(t) t^{l(2+\beta)\alpha/3} \exp \left[-\frac{l(2+\beta)\alpha}{3} Q(t) \right] \\
b(t) &= m_2 R(t) t^{l(\beta-1)\alpha/3} \exp \left[-\frac{l(\beta-1)\alpha}{3} Q(t) \right] \\
c(t) &= m_3 R(t) t^{-l(1+2\beta)\alpha/3} \exp \left[\frac{l(1+2\beta)\alpha}{3} Q(t) \right]
\end{align*}

where

$$Q(t) = \tilde{\sigma} t^{-1} + \frac{1}{4} \left( 2\tilde{\delta} + \left( \frac{2\tilde{\omega}}{\tilde{\omega}} + \frac{3\tilde{\nu}}{2\tilde{v}} \right) G_0 \right) t^{-2}$$

and $\alpha = \left[ \frac{\mathcal{M} G_0}{2} (\tilde{\omega}/\tilde{\nu}) \right]^{-1/2}$ for $\Omega = 1$. 

\[\Omega = 1 \text{ (Stiff Matter)}\]
The consistency conditions are found to be

\[ \Lambda_0 = 0 \]
\[ 2 \left( \frac{\ddot{\nu}}{\ddot{\omega}} \right) = \frac{1}{3} \left( 1 - \alpha^2 l^2 (\beta^2 + \beta + 1) \right). \]

We can write \( l \) in terms of \( \beta \) as

\[ l = \frac{1}{\alpha} \frac{\sqrt{1 - 6 \left( \frac{\ddot{\nu}}{\ddot{\omega}} \right)}}{\sqrt{(\beta^2 + \beta + 1)}}. \]

As \( \frac{\dot{a}}{a}, \frac{\dot{b}}{b} \), and \( \frac{\dot{c}}{c} \) are real, we get the following condition from the above equation

\[ 1 - 6 \left( \frac{\ddot{\nu}}{\ddot{\omega}} \right) \geq 0. \]
* When $\tilde{\nu}/\tilde{\omega} = \frac{1}{6}$, $l^2(\beta^2 + \beta + 1) = 0$, which implies that $l = 0$ since $\beta$ must be real.

* We observe that for large $\beta$, we get a Kasner type solution, that is, there are expanding and contracting directions. This can be seen from the following equations

\[
\begin{align*}
\frac{\dot{a}}{a} &= \frac{\ddot{R}}{R} + \frac{l(2 + \beta)\alpha}{3} \bar{H}(t) \\
\frac{\dot{b}}{b} &= \frac{\ddot{R}}{R} + \frac{l(\beta - 1)\alpha}{3} \bar{H}(t) \\
\frac{\dot{c}}{c} &= \frac{\ddot{R}}{R} - \frac{l(1 + 2\beta)\alpha}{3} \bar{H}(t)
\end{align*}
\]

where

\[
\bar{H}(t) = \frac{1}{t} + \frac{\tilde{\sigma}}{t^2} + \frac{1}{2} \left( 2\tilde{\delta} + \left( \frac{2\tilde{\omega}_1}{\tilde{\omega}} + \frac{3\tilde{\nu}_1}{2\tilde{\nu}} \right) G_0 \right) \frac{1}{t^3}.
\]

* For large positive $\beta$, the expanding directions would involve the scale factors $a$, $b$ and contracting direction would involve $c$. 
We have studied the anisotropic Bianchi-I for late times cosmological model taking quantum gravitational effects into account.

The consistency conditions following from Einstein equations indicate that the Bianchi-I anisotropic cosmological universe eventually evolves into a FLRW universe at late times if filled with a perfect fluid with the equation of state $p = \Omega \rho$ for $0 < \Omega < 1$.

The scale factors $a(t), b(t)$ and $c(t)$ take the same form and expand in the same rate in all directions.

For $\Omega = 1$ which corresponds to stiff matter, we observe from the consistency conditions that the solution does not flow to the isotropic FLRW universe at late times.