Generalized uncertainty principle in resonant detectors of gravitational waves

Sukanta Bhattacharyya

Department of Physics,
West Bengal State University, Barasat, Kolkata 700126, India

[Classical and Quantum Gravity 37 (2020) 19, 195006]

December 13, 2020
Plan of the talk

1. Introduction
2. Brief Methodology
3. Transition probabilities for different types of GW
   - periodic linearly polarized GW
   - Periodic circularly polarized GW
   - Aperiodic linearly polarized GW: Burst
4. Summary
Introduction

- Existence of Planck-length leads to a modification of the Heisenberg uncertainty principle (HUP) to the generalized uncertainty principle (GUP).
- This modified Heisenberg algebra is given by
  \[
  [\hat{q}_i, \hat{p}_j] = i\hbar(\delta_{ij} + \beta \delta_{ij} \hat{p}^2 + 2\beta \hat{p}_i \hat{p}_j). \tag{1}
  \]
- GUP parameter $\beta$ defined as $\beta = \frac{\beta_0}{(M_{pl}c)^2}$.
- A lot of effort has been put to find an upper bound of the dimensionless GUP parameter $\beta_0$.
- Resonant frequencies of a mechanical oscillator and an optomechanical scheme indicate a good possibility of detecting the signature of GUP.
• Gravitational waves are ‘ripples’ in the fabric of space-time.
• The study of resonant-bar detectors is fundamental because it focuses on how GW interacts with elastic matter.
• The present day gravitational wave (GW) detectors strive to detect the length variation $\mathcal{O} \sim 10^{-18} - 10^{-21}$ meter.
• The normal modes of the ton-scale GW detector AURIGA, has been analyzed to probe any possible planck-scale modification on the ground state energy of an oscillator.
• Such effects may only appear near the string/Planckian scale, it is plausible that some low energy relics may exist and their phenomenological consequences may be important at the level of quantum mechanics.
GW-matter interaction in the GUP framework that can anticipate the GUP effects in GW detection events.

Demonstrate how the presence of GUP modifies the responding frequency of the resonant mass detectors of GW and the corresponding probabilities of GW induced transitions that the phonon modes of the resonant mass detectors undergo.

The vibrations are nothing but quantum mechanical forced Harmonic Oscillators (HO).

Thus the response of a resonant detector to GW can be quantum mechanically described as GW-HO interaction.

With this motivation, we have studied the interaction of GW(s) with simple matter systems in a GUP platform.
Brief Methodology

- The Lagrangian of the system
  \[ \mathcal{L} = \frac{1}{2} m (\dot{q}_j)^2 - m \Gamma_{0k}^j \dot{q}_j q^k - \frac{1}{2} m \bar{\omega}^2 (q_j)^2 \]  
  (2)

- The Hamiltonian therefore reads
  \[ H = \frac{1}{2m} \left( p_j + m \Gamma_{0k}^j q^k \right)^2 + \frac{1}{2} m \bar{\omega}^2 q_j^2. \]  
  (3)

- The gravitational wave is taken care of by \( \Gamma_{0k}^j = \frac{h_{jk}}{2} \)

- The linearly polarized GW can be expressed as
  \[ h_{jk} (t) = 2f \left( \varepsilon_\times \sigma_{jk}^1 + \varepsilon_+ \sigma_{jk}^3 \right) \]  
  (4)

- where \( 2f \) is the amplitude of the GW and \((\varepsilon_\times, \varepsilon_+)\) are the two possible polarization states of the GW satisfying the condition \( \varepsilon_\times^2 + \varepsilon_+^2 = 1 \) for all \( t \).
The position and momentum operators upto first order in $\beta$ obeying eq.(1) reads

$$\hat{q}_i = q_{0i} \quad , \quad \hat{p}_i = p_{0i}(1 + \beta p_0^2)$$  \hspace{1cm} (5)

We treat the resonant bar detectors as a one-dimensional system (length $L \equiv 3$ m and radius $R \equiv 30$ cm).

The Hamiltonian become

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 , \quad H_1 = \frac{\beta}{m} p^4$$

$$H_2 = \frac{1}{2} \Gamma_{01}^1 (pq + qp) + \frac{\beta}{2} \Gamma_{01}^1 (p^3 q + qp^3) .$$

$H_0$ stands for the Hamiltonian of ordinary HO. $H_1$ and $H_2$ are the time independent and time dependent part of the Hamiltonian.
• We now define raising and lowering operators in terms of the oscillator frequency $\nu$

$$q_j = \sqrt{\frac{\hbar}{2m\nu}} (a_j + a_j^\dagger); \quad p_j = \sqrt{\frac{\hbar m\nu}{2i}} (a_j - a_j^\dagger). \quad (6)$$

• The Hamiltonian in terms of raising and lowering operator can be recast as

$$H_0 = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right)$$

$$H_1 = \frac{\beta}{m} \left( \frac{\hbar m\omega}{2} \right)^2 \left[ aaaaa - aaaaa^\dagger - aaaa^\dagger a + aaaa^\dagger a^\dagger - aa^\dagger aa \\
+aa^\dagger aa^\dagger + aa^\dagger a^\dagger a - aa^\dagger a^\dagger a^\dagger - a^\dagger aaa + a^\dagger aaa^\dagger + a^\dagger aa^\dagger a \\
- a^\dagger aa^\dagger a^\dagger + a^\dagger a^\dagger aa - a^\dagger a^\dagger aa^\dagger - a^\dagger a^\dagger a^\dagger a + a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a \right]$$
and
\[ H_2 = i\hbar \dot{h}_{11} \left[ -\frac{1}{2} (aa - a^\dagger a^\dagger) + \frac{\beta \hbar m \omega}{4} \left( aaaa - a a^\dagger a - a a^\dagger a a^\dagger + a^\dagger a a^\dagger a^\dagger + a^\dagger a a^\dagger a^\dagger - a^\dagger a^\dagger a^\dagger a^\dagger \right) \right] . \] (7)

- It is to be noted that \( H_1 \) and \( H_2 \) are small compared to \( H_0 \).
- Our aim is to calculate the perturbed states, their corresponding energy levels and transition probabilities among them.
- First we calculate the time independent perturbation theory using the hamiltonian \( H_1 \).
The perturbed eigenstates read

\[ |0\rangle^β = |0\rangle + \frac{\Delta}{8} \left( 6\sqrt{2} |2\rangle - \sqrt{6} |4\rangle \right) \]

\[ |2\rangle^β = |2\rangle + \frac{\Delta}{8} \left( -6\sqrt{2} |0\rangle + 28\sqrt{3} |4\rangle - 3\sqrt{10} |6\rangle \right) \]

\[ |4\rangle^β = |4\rangle + \frac{\Delta}{8} \left( \sqrt{6} |0\rangle - 28\sqrt{3} |2\rangle + 22\sqrt{30} |6\rangle - 2\sqrt{105} |8\rangle \right) \]

with the corresponding energies

\[ E_0^β = \left( \frac{1}{2} + \frac{3}{4} \Delta \right) \hbar \omega \]

\[ E_2^β = \left( \frac{5}{2} + \frac{39}{4} \Delta \right) \hbar \omega \]

\[ E_4^β = \left( \frac{9}{2} + \frac{123}{4} \Delta \right) \hbar \omega \]

Here \( \Delta = \beta \hbar m \omega \) is the dimensionless parameter showing the GUP effect.
Now we calculate the transition probabilities between the perturbed states using the hamiltonian (7).

The transition amplitudes reads

\[ C_{0\beta \rightarrow 2\beta} = A \int_{-\infty}^{t\rightarrow+\infty} dt' \dot{h}_{11} e^{i(2+9\Delta)\omega t'} \]

\[ C_{0\beta \rightarrow 4\beta} = B \int_{-\infty}^{t\rightarrow+\infty} dt' \dot{h}_{11} e^{i(4+30\Delta)\omega t'} \]

\[ A = \left( \frac{1}{\sqrt{2}} + \frac{9}{4\sqrt{2}} \Delta \right), \quad B = -3\sqrt{6}\Delta \] are dimensionless constants.

The transition probabilities

\[ P_{i\rightarrow f} = |C_{i\rightarrow f}|^2. \]

For an ordinary HO only \(|0\rangle \rightarrow |2\rangle\) transition will occur. But due to the presence of the GUP, we get another transition \(|0\rangle^\beta \rightarrow |4\rangle^\beta\) with different amplitudes.
Now we consider the simple scenario of periodic GW with linear polarization. This can be written as

\[ h_{jk}(t) = 2f_0 \cos \Omega t \left( \varepsilon \times \sigma^1_{jk} + \varepsilon_+ \sigma^3_{jk} \right). \] (10)

The amplitude varies sinusoidally with a single frequency \( \Omega \).

Therefore, the transition rates become

\[
\lim_{T \to \infty} \frac{1}{T} P_{0^\beta \to 2^\beta} = (2\pi f_0 \Omega A \varepsilon_+)^2 \times [\delta (\omega (2 + 9\Delta) - \Omega)]
\]

\[
\lim_{T \to \infty} \frac{1}{T} P_{0^\beta \to 4^\beta} = (2\pi f_0 \Omega B \varepsilon_+)^2 \times [\delta (\omega (4 + 30\Delta) - \Omega)].
\]
From the above results we can make following observations:

- The transition rates show that the detector resonates with the GW at frequencies $\Omega_1 = \omega (2 + 9\Delta)$ and $\Omega_2 = \omega (4 + 30\Delta)$.
- We get two transitions instead of one. The transitions from the ground state to the higher excited states ($|0\rangle^\beta \rightarrow |4\rangle^\beta$) are only due to the presence of the GUP.
- The expressions of $A$ and $B$ show that terms both linear and quadratic in the dimensionless GUP parameter $\Delta$ will appear in the transition $P_{0\beta \rightarrow 2\beta}$.
- The resonant frequency due to the GUP must be less than the resonant frequency itself. The inequality $9\Delta \omega < 2\omega$ gives

$$\beta_0 < \frac{2}{9} \times \frac{M_p}{m} \times \frac{M_p c^2}{\hbar \omega} \quad (11)$$
The mass $m$ of the resonant bar detector is approx. one ton ($1.1 \times 10^3 \text{ kg}$), the frequency $\omega$ of the detector is of the order of 1kHz ($\omega/(2\pi) = 900 \text{ Hz}$), and the Planck mass $M_p c^2 = 1.2 \times 10^{19} \text{ GeV}$.

- These values gives $\beta_0 < 1.4 \times 10^{28}$.
- The correction to the resonant frequency ($9\Delta \omega$) takes the value $\approx 1.3 \text{ kHz}$.
Periodic circularly polarized GW

- The simplest form of a periodic GW signal with circular polarization is

\[
h_{jk}(t) = 2f_0 \left[ \varepsilon_{\times}(t) \sigma_{jk}^1 + \varepsilon_+(t) \sigma_{jk}^3 \right]
\]

with \( \varepsilon_+(t) = \cos \Omega t \) and \( \varepsilon_{\times}(t) = \sin \Omega t \) and \( \Omega \) is the frequency of GW.

- The transition probabilities in this case are given by

\[
\lim_{T \to \infty} \frac{1}{T} P_{0\beta \to 2\beta} = (2\pi f_0 \Omega A)^2 \times \delta (\omega (2 + 9\Delta) - \Omega)
\]

\[
\lim_{T \to \infty} \frac{1}{T} P_{0\beta \to 4\beta} = (2\pi f_0 \Omega B)^2 \times \delta (\omega (4 + 30\Delta) - \Omega)
\]
Aperiodic linearly polarized GW: Burst

- we take a simple choice as the following

\[ h_{jk}(t) = 2f_0 g(t) \left( \varepsilon \sigma^1_{jk} + \varepsilon_+ \sigma^3_{jk} \right) \]  \hspace{1cm} (14)

- Let us take a Gaussian form for the function \( g(t) \)

\[ g(t) = e^{-t^2/\tau_g^2} \]  \hspace{1cm} (15)

with \( \tau_g \sim \frac{1}{f_{max}} \), where \( f_{max} \) is the maximum value of a broad range continuum spectrum of frequency.

- The transition probabilities take the form

\[ P_{0\beta \rightarrow 2\beta} = \left( 2\sqrt{\pi f_0 \varepsilon_+} A \tau_g (2\omega + 9\omega \Delta) \right)^2 e^{-2\left( \frac{2\omega + 9\omega \Delta}{2} \tau_g \right)^2} \]

\[ P_{0\beta \rightarrow 4\beta} = \left( 2\sqrt{\pi f_0 \varepsilon_+} B \tau_g (4\omega + 30\omega \Delta) \right)^2 e^{-2\left( \frac{4\omega + 30\omega \Delta}{2} \tau_g \right)^2} \]  \hspace{1cm} (16)
Lastly, consider a modulated Gaussian function $g(t)$ of the form

$$g(t) = e^{-t^2/\tau_g^2} \sin \Omega_0 t.$$  \hspace{1cm} (17)

The transition amplitudes take the forms as

$$P_{0\beta \rightarrow 2\beta} \approx e^{-(2\omega + 9\omega \Delta - \Omega_0)^2 \tau_g^2/2} \times \left\{ f_0 \varepsilon_+ A \sqrt{\pi \tau_g} (2\omega + 9\omega \Delta) \right\}^2$$

$$P_{0\beta \rightarrow 4\beta} \approx e^{-(4\omega + 30\omega \Delta - \Omega_0)^2 \tau_g^2/2} \times \left\{ f_0 \varepsilon_+ B \sqrt{\pi \tau_g} (4\omega + 30\omega \Delta) \right\}^2.$$  \hspace{1cm} (18)
Summary

- The resonant frequencies $\Omega_1 = \omega(2 + 9\Delta)$ and $\Omega_2 = \omega(4 + 30\Delta)$ of the resonant detector get modified by GUP parameter $\beta$.
- In presence of GUP, there are more than one transitions ($|0\rangle \rightarrow |2\rangle$ and $|0\rangle^\beta \rightarrow |4\rangle^\beta$) with different intensities.
- There are both the linear and quadratic terms in the GUP parameter $\Delta$.
- Both linear and circularly polarized GW are the good candidates to probe the presence of the generalized uncertainty principle in the resonant detectors. This is valid for both the periodic and aperiodic signals.
- We found an upper bound $\beta_0 < 10^{28}$. This is a much stronger bound than $\beta_0 < 10^{33}$. 
Reference

- AS, SG, SS, PRD 83, 025004, 2011
<table>
<thead>
<tr>
<th>Introduction</th>
<th>Brief Methodology</th>
<th>Transition probabilities for different types of GW</th>
<th>Summary</th>
</tr>
</thead>
</table>

THANK YOU