Multiparton webs beyond 3 loops

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Outline of the talk

- Introduction
- Soft function and Webs
- Multiparton webs beyond 3-loops
- Summary
Introduction
Factorization of Multileg amplitudes

ref: Mueller (81), Sen (83), Botts Sterman (89), Kidonakis Oderda Sterman (98), Catani (98), Tejeda-Yeomans Sterman (02), Kosower (03), Aybat Dixon Sterman (06), Becher Neubert (09), Gardi Magnea (09)
Soft approximation and Wilson lines

- Definition of soft function:

\[ S = \langle 0 | \prod_{i=1}^{n} \phi_{\beta_i}(\infty, 0) | 0 \rangle \]

- Evolution equation:

\[
\mu \frac{d}{d\mu} S(\rho_{ij}, \alpha_s(\mu^2), \epsilon) = - \Gamma^S(\rho_{ij}, \alpha_s(\mu^2, \epsilon)) \times S(\rho_{ij}, \alpha_s(\mu^2), \epsilon)
\]

- Solution:

\[
S(\rho_{ij}, \alpha_s(\mu^2), \epsilon) = \mathcal{P} \exp \left\{ - \frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma^S(\rho_{ij}, \alpha(\lambda^2, \epsilon)) \right\}
\]
Soft Function and Webs
Webs in non abelian theory

- Soft function in terms of web $W$ is $S = \exp(W)$
- A web in the multiparton case is a set of diagrams which differ only by the order of the gluon attachment on each Wilson line.

![Diagram of webs]

- If a diagram is $D = F(D)C(D)$ a Web $W$ is expressed as a sum of diagrams in terms of mixing matrix $R$.

$$W = \sum_D F(D)\tilde{C}(D) = \sum_{D,D'} F(D)R_{DD'}C(D)$$

[ref: Gardi et. al. (10)]
Idempotence: \( R^2 = R \), eigenvalues 1 or 0.

Zero-sum rows. [ref: Gardi et. al. (10)]

Conjecture: \( \sum_D c(D)s(D) = 0 \). [ref: Gardi et. al. (11)]
Multiparton webs beyond three loops

Neelima Agarwal, Abhinava Danish, Lorenzo Magnea, SP, Anurag Tripathi

JHEP 05 (2020) 128
Challenges at 4-loop

- Calculation of number of Webs

- At 4-loop we have 21 webs connecting 4-legs and 9 webs connecting 5-legs

- The largest dimension of the mixing matrix for the web is $24 \times 24$

- Results available for 3-loops has largest dimension of mixing matrix as $16 \times 16$
Webs and Cwebs

- A **web** in the multiparton case is a set of diagrams which differ only by the order of the gluon attachment on each Wilson line.

- A **Cweb** is a set of skeleton diagrams, built out of connected gluon correlators attached to Wilson lines, closed under shuffles of gluon attachments to each Wilson line.
Webs and Cwebs

\[ \text{Permutation}(CDE) = \{CDE, CED, DCE, DEC, ECD, EDC\} \]

\[ C \sqcup \sqcup DE = \{CDE, DCE, DEC\} \]
Enumeration using Cwebs

- One loop Cweb
Enumeration using Cwebs

- 2 loop Cweb:
  - Add a propagator to 1-loop Cweb.

- Connect a \( m \) point correlator to Wilson line and turn them into a \((m + 1)\) point correlator

- Connect a \( m \) point correlator to a \( n \) point correlator, if you have more than one correlator.
- Discard double counted Cwebs.
4-line webs

$W_{4}^{(1,0,1)}(1, 1, 2, 2)$

<table>
<thead>
<tr>
<th>Diagrams</th>
<th>Sequences</th>
<th>S-factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>${{BA}, {CD}}$</td>
<td>1</td>
</tr>
<tr>
<td>$C_2$</td>
<td>${{BA}, {DC}}$</td>
<td>0</td>
</tr>
<tr>
<td>$C_3$</td>
<td>${{AB}, {CD}}$</td>
<td>0</td>
</tr>
<tr>
<td>$C_4$</td>
<td>${{AB}, {DC}}$</td>
<td>1</td>
</tr>
</tbody>
</table>

$R = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
-\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\
-\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\
-\frac{1}{2} & 0 & 0 & \frac{1}{2}
\end{pmatrix}$

$D = (1_3, 0)$

$$(YC)_{1} = if^{abg} f^{cdg} f^{edh} T^{a}_{1} T^{b}_{2} T^{c}_{3} T^{d}_{4} - if^{abg} f^{cdg} f^{cej} T^{a}_{1} T^{b}_{2} T^{j}_{3} T^{d}_{4} T^{e}_{4},$$

$$(YC)_{2} = -if^{abg} f^{cdg} f^{cej} T^{a}_{1} T^{b}_{2} T^{j}_{3} T^{d}_{4} T^{e}_{4},$$

$$(YC)_{3} = if^{abg} f^{cdg} f^{edh} T^{a}_{1} T^{b}_{2} T^{e}_{3} T^{c}_{4} T^{h}_{4} - f^{abg} f^{cdg} f^{cej} f^{edh} T^{a}_{1} T^{b}_{2} T^{j}_{3} T^{4}_{4}. $$
5-line webs

\[ W_5^{(0,2)}(1, 1, 1, 1, 2) \]

<table>
<thead>
<tr>
<th>Diagrams</th>
<th>Sequences</th>
<th>S-factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>{{DA}}</td>
<td>1</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>{{AD}}</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ R = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \]

\[ (YC)_1 = -if^{abc}f^{def}f^{dah}T_1^aT_2^bT_3^cT_4^dT_5^f \]
General color structures

[ref: SP et. al. (2020), Becher et. al. (2019)]
Summary and outlook
Summary

- Soft function diagrammatically exponentiate in terms of webs.
- We have introduced CWebs using gluon correlators.
- CWebs are useful for enumeration of webs at higher order.
- We have calculated all the exponentiated color factors that can connect 4 and 5 Wilson lines at 4-loops.
- We find that there are only two diagrams present at 4-loops.
Calculation of all the exponentiated color factors for two and three legs.

The next step is to apply color conservation to all the exponentiated color factors.

Applying color conservation we will have a color basis.

We will apply bootstrap technique to determine the kinematic factors.
THANK YOU!
Backup slides
All exponentiated color factors at 3-loops order are calculated. [ref: Gardi et. al. (15)]

Color conservation technique \( \sum_i T_i^a = 0 \) is used to the exponentiated color factors.

Bootstrap technique is used to determine the kinematics. [ref: Almelid et. al. (15)]
$W_{3,1}^{(1,0,1)}(1, 2, 3)$

The $R$, $Y$ and $D$ matrices are given by

$$R = \begin{pmatrix}
1 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\
0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\
0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1
\end{pmatrix}, \quad
Y = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}, \quad
D = (1_5, 0).
\[(YC)_1 = \text{if}^{afk} f^{bcg} f^{efg} T^b_1 T^a_1 T^c_2 T^e_3 T^k_3 + \text{if}^{aeh} f^{bcg} f^{efg} T^b_1 T^a_1 T^c_2 T^h_3 T^f_3 - \text{if}^{abm} f^{bcg} f^{efg} T^m_1 T^c_2 T^e_3 T^f_3 T^a_3\]

\[(YC)_2 = \text{if}^{afk} f^{bcg} f^{efg} T^b_1 T^a_1 T^c_2 T^e_3 T^k_3 - \text{if}^{abm} f^{bcg} f^{efg} T^m_1 T^c_2 T^e_3 T^f_3 T^a_3\]

\[(YC)_3 = -\text{if}^{abm} f^{bcg} f^{efg} T^m_1 T^c_2 T^e_3 T^f_3 T^a_3\]

\[(YC)_4 = \text{if}^{afk} f^{bcg} f^{efg} T^a_1 T^b_1 T^c_2 T^e_3 T^k_3 + \text{if}^{aeh} f^{bcg} f^{efg} T^a_1 T^b_1 T^c_2 T^h_3 T^f_3\]

\[(YC)_5 = \text{if}^{afk} f^{bcg} f^{efg} T^a_1 T^b_1 T^c_2 T^e_3 T^k_3\]
$W_2^{(0,2)}(3, 4)$
Mixing matrix

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\
0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
## Comparison

<table>
<thead>
<tr>
<th></th>
<th>Old code</th>
<th>New code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimension</td>
<td>$24 \times 24$</td>
<td>$36 \times 36$</td>
</tr>
<tr>
<td>Runtime</td>
<td>$\approx 8$ days</td>
<td>$\approx 14$ hours</td>
</tr>
</tbody>
</table>

Efficiency increased **30 times**
Quantum Chromodynamics

\[ \mathcal{L}_{QCD} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\Psi - \frac{1}{4}(F_{\mu\nu}^{a})^{2} + g\bar{\Psi}\gamma^{\mu}T^{a}\Psi A_{\mu}^{a} \]

- Dirac Lagrangian
- free gluon field
- quarks-gluon interaction

\[ [ a = 1, \ldots, 8 \ ; \ T^{a} = \text{SU}(3) \text{ generators} ] \]

\[ F_{\mu\nu}^{a} = \partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a} + g f^{abc}A_{\mu}^{b}A_{\nu}^{c} \]

\[ [ T^{a}, T^{b} ] = if^{abc} T^{c} \]
QCD Feynman Rules

\[ i \frac{(p+m)}{p^2 - m^2 + i\epsilon} \delta_{ij} \]

\[ -ig_{\mu\nu} + (1 - \eta) \frac{p_\mu p_\nu}{p^2} \delta_{ab} \]

\[ ig\gamma^\mu t^a_{ji} \]

\[ gf^{abc} [g^{\mu\nu} (k - p)^\rho + \text{Cyclic Permutations}] \]

\[ -ig^2 [f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + C.P.] \]

\[ i\delta_{ab} \frac{p^2}{p^2 + i\epsilon} \]

\[ -gf^{abc} p^\mu \]
Asymptotic freedom

\[ \alpha_s(Q) \]

- Deep Inelastic Scattering
- \( e^+e^- \) Annihilation
- Hadron Collisions
- Heavy Quarkonia

\[ QCD \quad \alpha_s(MZ) = 0.1189 \pm 0.0010 \]
The Nobel Prize in Physics 2004

"for the discovery of asymptotic freedom in the theory of the strong interaction"

David J. Gross
1/3 of the prize
USA
Kavli Institute for Theoretical Physics,
University of California
Santa Barbara, CA, USA
b. 1941

H. David Politzer
1/3 of the prize
USA
California Institute of Technology
Pasadena, CA, USA
b. 1949

Frank Wilczek
1/3 of the prize
USA
Massachusetts Institute of Technology
Cambridge, MA, USA
b. 1951
Singularities in QFT

Singularity

- UV singularity
  - $k \to \infty$
  - $d=4-\epsilon$
  - Renormalization

- IR singularity
  - $k \to 0$
  - $d=4+\epsilon$
IR singularity

\[
\frac{1}{(p + k)^2} = \frac{1}{2p.k} = \frac{1}{2 \parallel p \parallel \parallel k \parallel (1 - \cos \theta)}
\]

Two types of divergences

1. **Soft divergences** \([k \to 0]\)

2. **Collinear divergences** \([\theta \to 0]\)

\[d = 4 + \epsilon\]
Jet functions

- Partonic jet: \( J_i \left( \frac{(2p_i \cdot n_i)^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = \langle p | \bar{\psi}(0) \phi_{\beta_1}(0, -\infty) | 0 \rangle \)

- Eikonal jet: \( J_i \left( \frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right) = \langle 0 | \phi_{\beta}(\infty, 0) \phi_n(0, -\infty) | 0 \rangle \)
Relation between $\Gamma^S$ and $\Gamma^\overline{S}$

$$\Gamma^\overline{S}_{lJ} = \Gamma^S_{lJ} - \delta_{lJ} \sum_{k=1}^{n} \gamma J_k$$
In $d$ dimensions

\[ \mu \frac{d\alpha}{d\mu} = \beta \]

\[ \frac{\mu}{m_R} \frac{d m_R}{d\mu} = \gamma \]
Reduced soft function

\[
\overline{S}_{LK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) = \frac{S_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon)}{\prod_i J_i(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon)}.
\]

\[
\rho_{ij} = \frac{n_i^2 n_j^2 (\beta_i \cdot \beta_j)^2}{4(\beta_i \cdot n_i)^2(\beta_j \cdot n_j)^2}.
\]
The RG equation of different quantities is given by:

$$
\mu \frac{d}{d\mu} \ln J_i \left( \frac{(2p_i \cdot n_i)^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = -\gamma J_i
$$

$$
\mu \frac{d}{d\mu} \ln \mathcal{J}_i \left( \frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right) = -\gamma \mathcal{J}_i
$$

$$
\mu \frac{d}{d\mu} S_{IK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) = -\Gamma_{IJ}^{S}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon)
$$

$$
\times S_{JK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon)
$$

$$
\mu \frac{d}{d\mu} \bar{S}_{IK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) = -\Gamma_{IJ}^{\bar{S}}(\rho_{ij}, \alpha_s(\mu^2), \epsilon)
$$

$$
\times \bar{S}_{JK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon)
$$
\[ \mathcal{M}_L \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \sum_K S_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) \]
\[ \times H_K \left( \frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2) \right) \]
\[ \times \prod_i J_i \left( \frac{(2p_i \cdot n_i)^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \]
\[ \prod_i J_i \left( \frac{2(\beta_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) \]
\[ = \mathcal{S}_{LK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) \]
\[ \times H_K \left( \frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2) \right) \]
\[ \times J_i \left( \frac{(2p_i \cdot n_i)^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \]
The soft singularities can be factorized from the hard part of the scattering amplitudes as-

\[
\mathcal{M}_L \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \sum_K S_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) \\
\times H_K \left( \frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2) \right) \\
\times \Pi_i \frac{J_i \left( \frac{(2p_i \cdot n_i)^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right)}{J_i \left( \frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right)}.
\]

Hard function $H$ is finite after UV renormalization.
Replica trick Algorithm

- Identify distinct connected pieces in the diagrams. Associate a replica variable to each connected piece which can take value from 1 to N.
- Determine the hierarchy between the replica variables, for example if there are two connected pieces then a hierarchy determines whether $i \leq j$, $i = j$, or $i > j$.
- Define an operator $R$ which will arrange the gluons on a particular leg according to the hierarchy. Given two attachments $i$ and $j$ to a leg $l$, the $R$ operation
  - does nothing if $i = j$
  - will order them as $T_i T_j$ if $i < j$ and as $T_j T_i$ if $i > j$. 

For each hierarchy compute its multiplicity, namely a combinational factor counting how many times this hierarchy arise when all i’s go over the range 1 to N. It is equal to the number of ways of choosing \( n(h) \) different replica numbers from N. \( M_N(h) = NC_n(h) \).
\[
\tilde{C}(D) = \sum_h M_n(h)R[C(D)\mid h] \text{ at order } N^1
\]
Calculation of $R$ using replica trick

| $h$  | $\mathcal{R}[(a)|h]$ | $M_N(h)$ | $\mathcal{O}(N^1)$ part of $M_N(h)$ |
|------|----------------------|-----------|-------------------------------------|
| $i = j$ | $C(a)$               | $N$       | $1$                                 |
| $i > j$ | $C(a)$               | $N(N-1)/2$ | $-\frac{1}{2}$                     |
| $i < j$ | $C(b)$               | $N(N-1)/2$ | $-\frac{1}{2}$                     |

$$\tilde{C}(a) = \frac{1}{2} [C(a) - C(b)]$$

$$\tilde{C}(b) = \frac{1}{2} [C(b) - C(a)]$$
At 3 loop when we can connect 4 lines through gulon structures then in the soft anomalous dimension formula we observe the presence of CICRs.

\[
\rho_{ijkl} = \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}
\]