Compactified Conformal Field Theories in Symplectic Manifolds

Gustavo Xavier Antunes Petronilo, A.E. Santana, S. Ulhoa

International Center of Physics
Institute of Physics - University of Brasília, DF, Brazil

December 15 2020
1 Motivation

2 Galilean Covariance

3 Hilbert Space and Symplectic Structure

4 Symplectic Quantum Mechanics and the Galilean Covariance

5 Compactified field theory

6 Conclusion
Motivation

- Wigner Function $\rightarrow$ Compactified fields
  - Galilean Covariance
  - Star Product
  - Symplectic Representations \(^1\)

Consider a free particle of mass \( m \), the dispersion relation is given by

\[
p^2 - 2mE = 0,
\]

we can then define a 5-vector, \( p^\mu = (p_x, p_y, p_z, m, E) = (p^i, m, E) \), with \( i = 1, 2, 3 \), such that the norm is defined as

\[
p_\mu p_\nu g^{\mu\nu} = p_i p_i - p_4 p_5 - p_5 p_4 = p^2 - 2mE = k^2,
\]

where \( g^{\mu\nu} \) is the metric of spacetime to be constructed, and \( p_\nu g^{\mu\nu} = p_\mu \).
Galilean Covariance

Let $q^\mu$ the set of canonical coordinates associated with $p^\mu$, we write $q^\mu = (q, q^4, q^5)$.

- $q$ is the canonical coordinate associated with $p$;
- $q_4$ is the canonical coordinate associated with $E$, and thus can be considered as the time coordinate;
- $q_5$ is the canonical coordinate associated with $m$ explicitly given in terms of $q$ e $q^4$

$$q^\mu q_\mu = q_\mu q_\nu g^{\mu \nu} = q - 2q_4 q_5 = s^2.$$ Since $p^\mu p_\mu = 0$, we have to take $s = 0$

$$q^5 = \frac{q^2}{2t};$$ or infinitesimally, we obtain $\delta q^5 = \nu \cdot \delta \frac{q}{2}$.

Therefore, the fifth component is defined by velocity.
This can be seen as a special case of a scalar product in $G$ denoted as

$$(x|y) = g^{\mu\nu} x_\mu y_\nu = \sum_{i=1}^{3} x_i y_i - x_4 y_5 - x_5 y_4,$$

where $x^4 = y^4 = t$, $x^5 = \frac{x^2}{2t}$ and $y^5 = \frac{y^2}{2t}$.

we can introduce the metric

$$(g_{\mu\nu}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}.$$
Consider $\mathcal{G}$ an analytical manifold where each point is specified by coordinates $q_\mu$, with $\mu = 1, 2, 3, 4, 5$ and metric specified by (6). The coordinates of each point in the cotangent-bundle $T^*\mathcal{G}$ will be denoted by $(q_\mu, p_\mu)$. The Space $T^*\mathcal{G}$ is equipped with a symplectic structure via a 2-form.

$$\omega = dq^\mu \wedge dp_\mu$$

called the symplectic form (sum over repeated indices is assumed). We consider the following bidifferential operator on $C^\infty(T^*\mathcal{G})$ functions,

$$\Lambda = \frac{\left< \partial \right>}{\partial q_\mu} \frac{\left< \partial \right>}{\partial p_\mu} - \frac{\left< \partial \right>}{\partial p_\mu} \frac{\left< \partial \right>}{\partial q_\mu}$$
such that for $C^\infty$ functions, $f(q, p)$ and $g(q, p)$, we have

$$\omega(f\Lambda, g\Lambda) = f\Lambda g = \{f, g\}$$

where

$$\{f, g\} = \frac{\partial f}{\partial q^\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p^\mu} \frac{\partial g}{\partial q_\mu}$$

is the Poison bracket and $f\Lambda$ and $g\Lambda$ are two vector fields given by $h\Lambda = X_h = -\{h, \}$.

The Space $T^*_G$ endowed with this symplectic structure is called the phase-space, and will be denoted by $\Gamma$. 
To associate the Hilbert Space with the phase-space $\Gamma$, we will consider the set of complex functions of integrable square, $\phi(q, p)$ in $\Gamma$, such that

$$
\int dpdq \phi^\dagger(q, p)\phi(q, p) < \infty
$$

is called the real bilinear form. In this case $\phi(q, p) = \langle q, p|\phi \rangle$ is written with the aid of

$$
\int dpdq |q, p\rangle\langle q, p| = 1
$$

where $\langle \phi| \ $ is the dual vector of $|\phi\rangle$.

This symplectic Hilbert Space is denoted by $H(\Gamma)$. 

Compactified Conformal Field Theories in Symplectic Manifolds
Consider the unitary transformations $U: \mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma)$ such that $\langle \psi_1 | \psi_2 \rangle$ is invariant. Using the $\Lambda$ operator, we define a mapping $e^{i\Lambda/2} = \ast: \Gamma \times \Gamma \to \Gamma$ called as Moyal (or star) product, defined by.

$$f \ast g = f(q, p) \exp \left[ \frac{i}{2} \left( \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial p_\mu} - \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q^\mu} \right) \right] g(q, p)$$

where $\hbar = 1$. 
The generators of $U$ can be introduced by the following (Moyal-Weyl) star-operators:

$$\hat{F} = f(q, p)^\star = f \left( q^\mu + \frac{i}{2} \frac{\partial}{\partial p_\mu}, p^\mu - \frac{i}{2} \frac{\partial}{\partial q_\mu} \right).$$
To construct a representation of Galilei algebra in $\mathcal{H}$, we define the following operators,

\[
\hat{P}^\mu = p^\mu_\star = p^\mu - \frac{i}{2} \frac{\partial}{\partial q^\mu},
\]

\[
\hat{Q}^\mu = q_\star = q^\mu + \frac{i}{2} \frac{\partial}{\partial p^\mu}.
\]

and

\[
\hat{M}_{\nu\sigma} = M_{\nu\sigma}_\star = \hat{Q}_\nu \hat{P}_\sigma - \hat{Q}_\sigma \hat{P}_\nu.
\]

The generators of the algebra of $\mathcal{G}$ are $\hat{M}_\mu$ and $\hat{P}_\mu$. 
Consider the massless dispersion relation

\[ p^\mu p_\mu = 0. \]

This equation is conformally invariant, using the light-cone coordinates defined by the metric \( g_{\mu\nu} \), the dispersion relation becomes

\[ p_i p_i - p_5 p_4 - p_4 p_5 = 0. \]

This is the non-relativistic dispersion relation. Since the original dispersion relation has conformal symmetry, this means that the symmetry group of the above equation is a subgroup of the conformal group.
The conformal algebra is

\[
\begin{align*}
[\tilde{M}_{\mu\nu}, \tilde{M}_{\rho\sigma}] &= -i(\eta_{\nu\rho}\tilde{M}_{\mu\sigma} - \eta_{\mu\rho}\tilde{M}_{\nu\sigma} + \eta_{\mu\sigma}\tilde{M}_{\nu\rho} - \eta_{\nu\sigma}\tilde{M}_{\mu\rho}), \\
[\tilde{P}_\mu, \tilde{M}_{\rho\sigma}] &= -i(\eta_{\mu\rho}\tilde{P}_\sigma - \eta_{\mu\sigma}\tilde{P}_\rho), \\
[\tilde{B}_\mu, \tilde{M}_{\rho\sigma}] &= i(\eta_{\mu\rho}\tilde{B}_\sigma - \eta_{\mu\sigma}\tilde{B}_\rho), \\
[\tilde{P}_\mu, \tilde{D}] &= i\tilde{P}_\mu, \\
[\tilde{D}, \tilde{B}_\mu] &= i\tilde{B}_\mu \\
[\tilde{P}_\mu, \tilde{B}_\nu] &= -2i(\eta_{\mu\nu}\tilde{D} - \tilde{M}_{\mu\nu}),
\end{align*}
\]

where, \(\tilde{D}\) generates scaling transformations (also known as dilatation) and \(\tilde{B}_\mu\) generates the special conformal transformations.
Doing the following identification

\[ J_i = \frac{1}{2} \epsilon_{ijk} \tilde{M}_{jk}, \]
\[ P_\mu = \tilde{P}_\mu, \]
\[ K_i = \tilde{M}_{5i}, \]
\[ C = \frac{\tilde{B}_5}{2}, \]
\[ D = \tilde{D} + \tilde{M}_{54}. \]
This forms a closed algebra, given by

\[
\begin{align*}
[J_i, J_j] &= i\epsilon_{ijk} J_k, \\
[D, K_i] &= iK_i, \\
[P_4, D] &= 2iP_4, \\
[P_i, K_j] &= i\delta_{ij} P_5, \\
[P_4, K_i] &= iP_i, \\
[J_i, K_j] &= i\epsilon_{ijk} K_k, \\
[J_i, P_j] &= i\epsilon_{ijk} P_k, \\
[P_4, C] &= iD, \\
[C, D] &= -2iC, \\
[P_i, D] &= iP_i, \\
[P_i, C] &= -iK_i.
\end{align*}
\]

All others commutations relation being zero. This algebra is the Schrödinger-Lie algebra. And the Casimir invariants are \( I_1 = p^\mu p_\mu \), \( I_2 = p_5 \), \( I_3 = W_{5\mu} W_5^\mu \).
Using the Casimir invariants $I_1$ and $I_2$ and applying in $\Psi$, 

$$\hat{P}_\mu \hat{P}^\mu \Psi = 0$$

$$\hat{P}_5 \Psi = -m \Psi$$

From this we obtain

$$(p^2 - ip \cdot \nabla - \frac{1}{4} \nabla^2) \Psi = 2\left(p_4 - \frac{i}{2} \partial_t\right)\left(p_5 - \frac{i}{2} \partial_5\right) \Psi,$$

with $\hat{P}_\mu = p^{\mu*} = p^{\mu} - \frac{i}{2} \frac{\partial}{\partial q^{\mu}}$. A solution for this equation is

$$\Psi = e^{-2i[(p_5+m)q_5+(p_4+E)t]} \Phi(q, p).$$

Thus, $\frac{1}{2m} (p^2 - ip \cdot \nabla - \frac{1}{4} \nabla^2) \Phi = E \Phi$, 

with $\Phi$ the wave function.
This equation, and its complex conjugate, can also be obtained by the Lagrangian density in phase-Space (we use $d^\mu = d/dq_\mu$)

$$\mathcal{L}_0 = \partial^\mu \psi(q, p) \partial \psi^*(q, p) + \frac{i}{2} p^\mu [\psi(q, p) \partial^\mu \psi^*(q, p)$$

$$- \psi^*(q, p) \partial^\mu \psi(q, p)] + \frac{p^\mu p_\mu}{4} \psi = 0.$$  

The association of this representation with the Wigner formalism is given by

$$f_w(q, p) = \psi(q, p) \star \psi^\dagger(q, p)$$

where $f_w(q, p)$ is the Wigner function.
We start with the (5+1) manifold with the following metric

$$g_{MN} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

The flat metric is consistent with a six-dimensional de Sitter space having the product $\mathcal{G} \otimes S^1$, where $\mathcal{G}$ is the Galilean flat space and $S^1$ is a circle of radius $R$. 

**Compactified field theory**
The equation of motion of a scalar field, $\varphi$, in configuration space has the form

$$\partial^M \partial_M \varphi = 0.$$  

Making $x_M = (x_\mu, y)$, with $\mu = 1, \ldots, 5$, so that $y \in [0, 2\pi R]$. Then, the equation of motion becomes

$$\partial^\mu \partial_\mu \varphi + \partial_y^2 \varphi = 0.$$
Since $\varphi(x, y) = \varphi(x, y + 2\pi R)$, so we can write a normalized solution as

$$\varphi(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_n \varphi_n(x) e^{iny/R}.$$  

Substituting solution, we get

$$\partial^\mu \partial_\mu \varphi_n(x) + \frac{n^2}{R^2} \varphi_n(x) = 0.$$  

or

$$\left( i \partial_t + \frac{1}{2m} \nabla^2 \right) \psi_n(x) + \frac{n^2}{2mR^2} \psi_n(x) = 0.$$
Therefore $\varphi_n$ is a five-dimensional scalar field in the light cone of de Sitter space, and $n^2/(2mR^2)$ is a shift in the Energy $E$. The ground state $\varphi_0$ has no energy shift, after it the shift will increase with $n$. 

In the case of phase space we have

\[ \hat{P}^M \hat{P}_M \varphi(q, p) = 0, \]

and letting \( q_M = (q_\mu, q_6) \) with \( q_6 \in [0, 2\pi] \), Eq. (23) becomes

\[ \hat{P}^\mu \hat{P}_\mu \varphi + \hat{P}_6^2 \varphi = 0. \]

As in the case of configuration space
\[ \varphi(q_\mu, q_6, p_M) = \varphi(q_\mu, q_6 + 2\pi R, p_M), \]
so

\[ \hat{P}_6 \varphi = \left( p_6 - \frac{i}{2} \partial_6 \right) \varphi = \frac{n}{R} \varphi. \]
Therefore, the general solution can be written as

\[ \varphi(q, p) = \sum_n \varphi_n(q_\mu, p_\mu) e^{-2i(p_6 - n/R)q_6}, \]

this gives a constraint in \( p_6 \)

\[ 4\pi Rp_6 = 2n\pi \]

\[ p_6 = \frac{n}{2R}. \]

Thus, the sixth momentum has a half-integer quantization and is inversely proportional to \( R \). The Schrödinger equation in phase space, in its explicit form, is

\[ \frac{1}{2m} \left( p^2 - ip \cdot \nabla - \frac{1}{4} \nabla^2 \right) \varphi_n = \left( E_n + \frac{n^2}{2mR^2} \right) \varphi_n. \]
Compactified field theory

A consistent solution of the equation for a particle confined in $q \in [0, L]$ is

$$\varphi_n = A \sqrt{\delta(p)} \frac{\sin \left( 2q \left( \sqrt{2mE_n} + p \right) \right)}{4 \left( \sqrt{2mE_n} + p \right)},$$

where

$$E_{n_1, n_2} = \frac{1}{2m} \left( \frac{\pi^2 n_1^2}{L^2} + \frac{n_2^2 (1 - R^2)}{R^2} \right),$$

and $0 \leq p \leq \sqrt{\frac{\pi n_1}{L^2}} - n_2$. Note that when $n_2 = 0$ we get the usual energy of the particle in a box.

The Wigner function is then

$$f_{W_{n_1, n_2}} = \varphi_{n_1, n_2} \star \varphi_{n_1, n_2}^\dagger = A^2(p) \sin^2 \left( -2q \left( \sqrt{2mE_n} \right) \right) \delta(p).$$
As the momentum is discreet, the sum of all momenta gives us

\[ A^2 \int dp \sin^2 \left( -2q \left( \sqrt{2mE_n} \right) \right) = A^2 \sin^2 \left( -2q \left( \sqrt{2mE_n} \right) \right) = |\psi(q)|^2, \]

as it should be.
the Wigner function for some values of $n_1$ and $n_2$ are plotted below, for convenience we set $m = L = 1$ and $R = 2$,

*Figure:* Wigner Function for $n_1 = 1$ and $n_2 = 0$

*Figure:* Wigner Function for $n_1 = 1$ and $n_2 = 1
Compactified field theory

**Figure:** Wigner Function for $n_1 = 2$ and $n_2 = 2$

**Figure:** Wigner Function for $n_1 = 3$ and $n_2 = 3$
We study the scalar particle equation, Schrödinger equation, in the context of Galilean covariance and then construct a phase-space formalism using such covariance.

We construct the formalism of the quantum mechanics of the Galilean covariant phase-space and we arrive at the representations of the spin 0 and study in a compactified field theory.
This work was supported by CAPES. The Brazilian universities and research centers are under attack, with cutting of budget, by the present days Brazilian Federal government. Due to this situation works like this may be impossible.
Inönü and Wigner, IL NUOVO CIMENTO 9, (1952) 706.


References

- de Melo, GR; de Montigny, M; Pompeia, PJ; Santos, Esdras S Int. J. Theor. Phys. **52**, (2013) 441.
- Amorim, RGG; Khanna, FC; Malbouisson, APC; Malbouisson, JMC; Santana, AE Int. J. M. Phys. A (2019)1950037.
Thank you!