Entropy production and thermal fluctuations in higher-order dissipative hydrodynamics

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Motivation and Outline: Part I

- Relativistic hydrodynamics: Description using thermodynamic parameters \( (T, P, \mu) \) and transport coefficients \( (\eta, \zeta, \sigma) \).

- Ideal hydrodynamics is unable to describe a system away from local equilibrium. Hence dissipation in hydro is required.

- Formulation of dissipative hydro is not settled. Developed kinetic theory approach for consistent formulation of dissipation.

- Derivation of higher-order entropy to quantify deviation from local equilibration.

- Comparison with exact solutions of Boltzmann equation in analytically solvable systems (Bjorken and Gubser flows).
Relativistic Dissipative hydrodynamics

- Local conservation equations: $d_\mu T^{\mu\nu} = 0, \ d_\mu N^\mu = 0$.

- The variables of hydro:

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Dissipative</th>
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<tbody>
<tr>
<td>$T^{\mu\nu} = \epsilon u^\mu u^\nu - P \Delta^{\mu\nu}$</td>
<td>$T^{\mu\nu} = \epsilon u^\mu u^\nu - P \Delta^{\mu\nu} + \pi^{\mu\nu}$</td>
</tr>
<tr>
<td>Unknows: $\epsilon, P, u^\mu = 5$</td>
<td>$\epsilon, P, u^\mu, \pi^{\mu\nu} = 10$</td>
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<tr>
<td>$1+1+3$</td>
<td>$1+1+3+5$</td>
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Equations: $d_\mu T^{\mu\nu} = 0, \ EOS = 5$

Closed set of equations $4 + 1$ required

5 more equations required

where, $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$. 
Relativistic Kinetic theory [S.R. de Groot, W.A. van Leeuwen, Ch.G. van Weert, Relativistic Kinetic Theory]

- Description based on single particle distribution function $f(x, p)$.

- $f(x, p)\Delta^3x\Delta^3p$ gives the mean number of particles $\Delta N$ that are in the phase space volume element.

- $T^{\mu\nu}(x)$ in terms of phase-space distribution function,

$$T^{\mu\nu}(x) = \int dP\, p^\mu p^\nu f(x, p) \equiv \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu},$$

where $dP \equiv d^3p/[(2\pi)^3 E_p]$.

- Entropy current using $f(x, p)$,

$$S^\mu = -\int dP\, p^\mu f (\ln f - 1)$$
Evolution of $f(x, p)$ governed by the Boltzmann equation,

$$p^\mu \partial_\mu f = C[f],$$

the collision kernel specifies the effect of interactions.

Usually the collision kernel takes into account binary collisions.

We consider a simplistic collision kernel: the relaxation-time approximation,

$$p^\mu \partial_\mu f = -(u \cdot p) \frac{f - f_{eq}}{\tau_R},$$

$\tau_R$ is the relaxation time.
Solving Boltzmann equation approximately

- Write BE as: \( f = f_{eq} - \tau_R p^\mu \partial_\mu f \)

- Assume that mean-free path is small compared to the space-time gradients (small Knudsen number): \( \tau_R \frac{\partial}{\partial x^\mu} \) is of \( O(\varepsilon) \):
  \[
  f = f_{eq} - \varepsilon \tau_R p^\mu \partial_\mu f
  \]

- Consider a perturbative series (Chapman-Enskog expansion):
  \[
  f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots
  \]

- Solve order-by-order in \( \varepsilon \),
  \[
  f_0 = f_{eq} = e^{-\beta (u \cdot p)}, \quad f_1 = -\frac{\tau_R}{u \cdot p} p^\mu \partial_\mu f_{eq},
  \]
  \[
  f_2 = \frac{\tau_R}{u \cdot p} p^\mu p^\nu \partial_\mu \left( \frac{\tau_R}{u \cdot p} \partial_\nu f_{eq} \right)
  \]

- Obtain macroscopic quantities:
  \[
  \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dP \: p^\alpha p^\beta (f_0 + f_1 + \cdots)
  \]
Out of equilibrium entropy four-current,

\[ S^\mu(x) = - \int dp \ p^\mu f (\ln f - 1) \]

For \( f = f_{eq}(1 + \phi_1 + \phi_2 + ..) \) to third-order in gradients,

\[ S^\mu = s_{eq} u^\mu - \int dp \ p^\mu f_{eq} \left( \frac{\phi_1^2}{2} + \phi_1 \phi_2 - \frac{\phi_1^3}{6} \right) \]

Substituting \( \phi_1 \) and \( \phi_2 \),

\[ S^\mu = s_{eq} u^\mu - \left[ \frac{\beta}{4 \beta_\pi} \pi^{\alpha \beta} \pi_{\alpha \beta} u^\mu + \frac{5 \beta}{42 \beta_\pi^2} \pi_{\alpha \gamma} \pi_{\beta}^{\gamma} \pi^{\alpha \beta} u^\mu \right] + \frac{\beta \tau_{\pi}}{7 \beta_\pi} \left[ \frac{18}{5} \dot{u}^\rho \pi_{\rho \gamma} \pi^{\mu \gamma} + \frac{2}{5} \pi^{\mu \gamma} \nabla_{\rho} \pi_{\rho \gamma} - \frac{1}{2} \pi^{\alpha \beta} \nabla^\mu \pi_{\alpha \beta} \right] + 3 \dot{u}^\mu \pi_{\alpha \beta} \pi^{\alpha \beta} - \pi^{\alpha \gamma} \Delta^\mu_{\rho} \nabla_{\alpha} \pi_{\rho \gamma} \]
Third-order entropy four-current

- Entropy density, \( s \equiv u_\mu S^\mu \),

\[
s = s_{eq} - \frac{\beta}{4\beta_\pi} \pi^{\alpha\beta} \pi^{\alpha\beta} - \frac{5\beta}{42\beta_\pi^2} \pi^{\alpha\gamma} \pi^{\gamma\beta} \pi^{\alpha\beta}
\]

- The entropy flux, \( S^{(\mu)} \equiv \Delta_\nu^{\mu} S^\nu \) is,

\[
S^{(\mu)} = \frac{\beta \tau_\pi}{7\beta_\pi} \left[ \frac{18}{5} \dot{u}^\rho \pi^{\rho\gamma} \pi^{\mu\gamma} + \frac{2}{5} \pi^{\mu\gamma} \nabla^\rho \pi^{\rho\gamma} - \frac{1}{2} \pi^{\alpha\beta} \nabla^\mu \pi^{\alpha\beta} 
\right.
\]

\[
\left. + 3 \dot{u}^\mu \pi^{\alpha\beta} \pi^{\alpha\beta} - \pi^{\alpha\gamma} \Delta^{\mu\rho} \nabla^\alpha \pi^{\rho\gamma} \right]
\]

- Entropy flow not necessarily along energy flow
Application I: Bjorken flow [J.D. Bjorken, PRD, 27, 140 (1983)]

For a system undergoing longitudinal boost-invariant expansion, \( v^z = \frac{z}{t}, \ v^x = v^y = 0. \)

All scalar functions depend only on the proper time \( \tau = \sqrt{t^2 - z^2}. \)

In Milne coordinate system \((\tau, x, y, \eta_s)\) where \( \eta_s = \tanh^{-1}(z/t), \) fluid four-velocity \( u^\mu = (1, 0, 0, 0). \)
Results: Bjorken flow [C.C., U. Heinz, S. Pal, G. Vujanovic, PRC, 97, 064909 (2018)]

Proper-time evolution of (a) normalised shear pressure, (b) pressure anisotropy (c) entropy content and (d) normalised entropy density.

- \( v^z = z/t, \quad u^r(x) \neq 0, \quad u^\phi(x) = 0; \) has transverse dynamics.
- Suitably described in de Sitter coordinates \((\rho, \theta, \phi, \eta)\), in which \( u^\mu = (1, 0, 0, 0) \),

\[
\rho = - \sinh^{-1} \left( \frac{1 - q^2 \tau^2 + q^2 r^2}{2q \tau} \right), \quad \theta = \tan^{-1} \left( \frac{2qr}{1 + q^2 \tau^2 - q^2 r^2} \right),
\]

\( 1/q \approx \) transverse size.

Weyl rescaled unitless quantities,

\[
\epsilon(\tau, r) = \frac{\hat{\epsilon}(\rho)}{\tau^4},
\]

\[
\pi_{\mu\nu}(\tau, r) = \frac{1}{\tau^2} \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \frac{\partial \hat{x}^\beta}{\partial x^\nu} \hat{\pi}_{\alpha\beta}(\rho).
\]

Du. et al [2019]

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Perturbative expansion in CE breaks down at large times when system is highly out of equilibrium.
Part II: Thermal Fluctuations

- Finite number of particles in each fluid cell
  \[\implies\text{Hydrodynamic Fluctuations}\]
- Thermal noise in higher-order dissipative theories
- Correlations induced in a static fluid
- Effects on an expanding medium
- Numerical simulation of noise
Consider a system whose macro-states are labelled by \( \{x_1, x_2, \ldots, x_N\} \).

The probability of the system being in state \( \{x_a\} \):
\[
P(\{x_a\}) \propto \exp[S(\{x_a\})].
\]

In thermal equilibrium, i.e., \( x_a = \bar{x}_a \), the entropy is a maximum :
\[
\text{Max}[S(\{x_a\})] = S(\bar{x}_a).
\]

Consider small fluctuations : \( x_a = \bar{x}_a + \delta x_a \).

The first correction to entropy is quadratic in \( \delta x_a \):
\[
S(x_a) = S(\bar{x}_a) - \frac{1}{2} \beta_{ik} \delta x_i \delta x_k
\]

The fluctuation then becomes
\[
\langle \delta x_i \delta x_j \rangle = \beta_{ik}^{-1}.
\]
Fluid in global equilibrium

- Using $\epsilon = \alpha T^4$ and $P = \epsilon/3$, expand $T^{00} = (\epsilon + P)(u^0)^2 - P \eta^{\mu\nu}$ and $s^0 = (\epsilon + P)u^0$ to $O(\delta^2)$

\[
\Delta T^{00} = \frac{\alpha \bar{T}^3}{3} \left( 12 \frac{\delta T}{\bar{T}} + 4\delta \bar{v}^2 + 18 \frac{(\delta T)^2}{\bar{T}^2} \right)
\]

\[
\Delta (s^0 u^0) = \frac{4\alpha \bar{T}^3}{3} \left( \frac{3}{2} \frac{\delta T}{\bar{T}} + \frac{\delta \bar{v}^2}{\bar{T}^2} + 3 \frac{(\delta T)^2}{\bar{T}^2} \right)
\]

- Using $\Delta S_{tot} = V \left( \Delta T^{00} - \bar{T} \Delta (s^0 u^0) \right) / \bar{T}$ we have

\[
\text{Prob}(\delta T, \delta v^i) \propto e^{-V \left( 2\alpha \bar{T}(\delta T)^2 + \frac{2}{3} \alpha \bar{T}^3 \delta \bar{v}^2 \right)}
\]

- Using this, the statistically independent fluctuations in energy density and flow velocity is,

\[
\langle (\delta \epsilon)^2 \rangle = \frac{(\bar{\epsilon} + \bar{P}) \bar{T}}{c_s^2 V}, \langle (\delta \bar{v})^2 \rangle = \frac{\bar{T}}{(\bar{\epsilon} + \bar{P}) V}
\]

- Langevin equation, \( \frac{dp}{dt} = -\lambda p + \xi \) (drag + diffusion).

- To get \( \langle p^2(t) \rangle = 2MT \), the noise should have
  \( \langle \xi(t)\xi(t') \rangle = 2\lambda MT\delta(t - t') \).

- For multi-variable problem,
  \[
  \dot{x}_a = -\sum_b \gamma_{ab}x_b + \xi_a
  \]
  \( X_b \equiv -\partial S/\partial x^b \) are “driving” forces, \( \xi_a \) are random fluctuations

- Rate of change of entropy \( S(x) \),
  \[
  \dot{S} = -\sum_a \dot{x}_a x_a
  \]

- As the probability of fluctuating variables are \( e^S \)
  \[
  \langle \xi_a(t_1)\xi_b(t_2) \rangle = (\gamma_{ab} + \gamma_{ba})\delta(t_1 - t_2)
  \]
Fluctuating higher-order hydrodynamics

- Second-order Chapman-Enskog equation for $\pi^{\mu\nu}$,

\[
\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi^{\langle\mu}_{\gamma}\omega^{\nu\rangle}_{\gamma} - \frac{10}{7}\pi^{\langle\mu}_{\gamma}\sigma^{\nu\rangle}_{\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta,
\]

[A. Jaiswal, R.S. Bhalerao and S. Pal, Phys. Rev. C 87, 021901(R) 2013]

- Second-order entropy,

\[
S^\mu = s_{eq}u^\mu - \frac{\beta_2}{2T}u^\mu\pi^{\alpha\beta}\pi_{\alpha\beta}
\]

- Using $\partial_\mu S^\mu$,

\[
\frac{dS}{dt} = \int d^3x \frac{\pi^{\mu\nu}}{T} \left[ \nabla_\mu u_\nu - \beta_2 \dot{\pi}_{\mu\nu} - \frac{4}{3}\beta_2 \theta \pi_{\mu\nu} \right].
\]
In analogy to
\[ \dot{S} = - \sum_a \dot{x}_a X_a, \quad \dot{x}_a = -\gamma_{ab} X_b + \xi_a \]

Identify
\[ \dot{x}_a \rightarrow \pi^{\mu\nu} \]
\[ X_a \rightarrow - \frac{1}{T} \left[ \nabla_\mu u_\nu - \beta_2 \pi^{\mu\nu} - \frac{4}{3} \beta_2 \theta \pi^{\mu\nu} \right] \Delta V \equiv X_{\mu\nu} \]

Add fluctuations to \( \dot{x}_a \),
\[ \pi^{\mu\nu} = -\gamma^{\mu\nu\alpha\beta} X_{\alpha\beta} + \xi^{\mu\nu} \]
\[ \pi^{\mu\nu} = 2\eta_\nu \left[ -\beta_2 \pi^{(\mu\nu)} + \sigma^{\mu\nu} + 2\beta_2 \pi^{(\mu} \omega^{\nu)} - \beta_2 \frac{10}{7} \pi^{(\mu} \sigma^{\nu)}\gamma \right. \\
\left. - \frac{4}{3} \beta_2 \pi^{\mu\nu} \theta \right] \]
Constraints on $\gamma_{\mu \nu \alpha \beta}$

[C.C., R.S. Bhalerao, S. Pal, PRC 97, 054902 (2018)]

- Symmetries of $\pi_{\mu \nu}$: $\gamma_{\mu \nu \alpha \beta} = \gamma_{\nu \mu \alpha \beta}$, $\gamma_{\mu \alpha \beta} = 0$, and $\gamma_{\mu \nu \alpha \beta} u_\mu = 0$.

- Identification of $X_{\mu \nu}$ is not unique: $X_{\mu \nu} \rightarrow X_{\mu \nu} + H_{\mu \nu}$, keeps $dS/dt$ invariant, if $H_{\mu \nu}$ is orthogonal to $\pi_{\mu \nu}$.

- To obtain an autocorrelation which is insensitive to such transformations, $\gamma_{\mu \nu \alpha \beta} = \gamma_{\mu \nu \beta \alpha}$, $\gamma_{\mu \nu \alpha} = 0$, and $\gamma_{\mu \nu \alpha \beta} u_\alpha = 0$.

- Using these constraints,

$$
\gamma_{\mu \nu \alpha \beta} = 2\eta_v T \left( \Delta_{\mu \nu \alpha \beta} - \frac{10}{7} \beta_2 \Delta_{\mu \nu} \pi_{\gamma \Delta \kappa \gamma \alpha \beta} \right. \\
\left. + 2\tau_\pi \Delta_{\mu \nu \zeta \kappa \omega \gamma} \Delta_{\kappa \gamma \alpha \beta} \right) \times \frac{1}{\Delta V}
$$
Application I: Fluctuations in a static fluid

- Add noise to dissipative tensor: $T^{\mu\nu} = T_{id}^{\mu} + \pi^{\mu\nu} - \xi h \Delta^{\mu\nu}$

- Auto-correlations of noise:

$$\langle \xi(t_1, z_1) \xi(t_2, z_2) \rangle = \frac{2\nu}{h} \delta(\Delta t)\delta(\Delta z), \quad \nu = \frac{4}{3} \frac{\eta}{s}$$

- Linear perturbations around equilibrium: $\epsilon = \epsilon_0 + \delta\epsilon, \quad u^z = \delta u^z$

- Equal time two-point functions are short-ranged:

$$\langle \delta\epsilon(t, z_1)\delta\epsilon(t, z_2) \rangle = \frac{h_0 T_0}{c_s^2} \delta(z_1 - z_2)$$

- Unequal time correlations ($\eta/s \approx 0$) induced by sound modes:

$$\langle \delta\epsilon(t_1, z_1)\delta\epsilon(t_2, z_2) \rangle \rightarrow \frac{T_0 h_0}{2c_s^2} \left[ \delta(\Delta z - c_s \Delta t) + \delta(\Delta z + c_s \Delta t) \right]$$

- Viscosity dampens the correlations over time.
Fluctuations around expanding background:

\[ \epsilon = \epsilon_0(\tau) + \delta \epsilon(\tau, \eta), \quad u^\mu = (1, 0) + (0, \delta u^\eta) \]

Background evolution induces ‘dissipative-like’ dispersion. For \( \delta s/s_0 \),

\[ \omega(k) = \frac{i(1 - c_s^2)}{2} \pm \sqrt{c_s^2 k^2 - \left( \frac{1 - c_s^2}{2} \right)^2} \]

[R. S. Bhalerao, S. Gupta, PRC 79, 064901 (2009)]

Note, in static fluid, \( \omega = \pm c_s k \).

Perturbation propagates at speed of sound:

\[ \tau \frac{d \eta}{d \tau} = \pm c_s \implies \eta(\tau) = \eta_0 \pm c_s \log(\tau/\tau_0) \]
Singular and regular part of correlators in Bjorken [C.C., R.S. Bhalerao, S. Pal, PRC 97, 054902 (2018)]

Using $\tau_0 = 0.15$ fm/c, $T_0 = 600$ MeV, $\tau_f = 10$ fm
Conversion of fields to particles [F. Cooper, G. Frye, PRD 10,186(1974)]

- Number of particles crossing a hyper-surface $d\Sigma_\mu$:
  
  $$dN = d\Sigma_\mu N^\mu,$$

  $$E_p \frac{dN}{d^3p} = \int d\Sigma_\mu p^\mu f(x, p)$$

- We are interested in fluctuations of $dN/dy$,
  
  $$y = \tanh^{-1}(p^z/E_p)$$

- In Bjorken, the rapidity correlator is

  \[
  \langle \delta \frac{dN}{dy_1} \delta \frac{dN}{dy_2} \rangle = \sum_{X,Y=\rho,\omega} \int \int d\eta \, dz \, F_X(y) F_Y(\Delta y + \eta + z) C_{XY}(z)
  \]

  where $C_{XY}$ are equal-time correlators of hydro variables

  $$\langle X(\tau_f, \eta_1) Y(\tau_f, \eta_2) \rangle.$$
• The short-range peak decreases with inclusion of viscosity in background.
• Correlations become short-ranged with Lattice EoS due to lower speed of sound $c_s$. 
Two-particle rapidity correlations in (1+1)D expansion [C.C., S. Pal, PRC 98, 034901 (2018)]

- Correlations depend on rapidity $y_1$ and become short-ranged with Lattice EoS

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Summary

- Derived higher-order evolution of entropy four-current in conformal hydrodynamics using kinetic theory.
- Obtained higher-order evolution equations for Gubser flow.
- Derived fluctuation-dissipation relations in second-order Chapman-Enskog hydro.
- Analyzed the effects of thermal fluctuations in expanding medium (Bjorken and boost non-invariant) on particle correlations.
- In future we would like to extend the study of fluctuations to regimes where they may not be treated perturbatively.
Publications relevant to the thesis work:


I sincerely thank my Ph.D. advisor Prof. S. Pal for his guidance. I also thank Prof. U. Heinz, Prof. R. S. Bhalerao, Prof. A. Jaiswal, and members of the TIFR QCD Journal Club, especially, Prof. S. Gupta, Prof. R. Sharma and Prof. S. Datta for many insightful discussions.
Backup slide 1: Singular part of the correlators

- Work in Fourier coordinates,

\[ \tilde{X}(\tau, k) = - \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau'} \tilde{G}_X(\tau', \tau, k) \xi(\tau', k) \]

- The Green’s function is

\[ \langle X(\eta, \tau) Y(\eta', \tau) \rangle = 2 \frac{1}{A_\perp} \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau'^3} \frac{4\eta_v}{3sw_0(\tau')} G_{XY}(\eta - \eta'; \tau, \tau') \]

The Green’s function is

\[ G_{XY}(\eta - \eta'; \tau, \tau') = \int_{-\infty}^{\infty} \frac{d k}{2\pi} e^{ik(\eta - \eta')} \tilde{G}_{XY}(k; \tau, \tau') \]

\[ \tilde{G}_{XY}(k; \tau, \tau') \equiv \tilde{G}_X(k; \tau, \tau') \tilde{G}_Y(-k; \tau, \tau') \]
\[ \tilde{\mathcal{G}}_{\rho\rho}^{\text{sing}}(k; \tau, \tau') = (a_1 k^2 + b_1) + (a_2 k^2 + b_2) \cos(2c_s \gamma k) + \frac{a_3 k^2 + b_3}{k} \sin(2c_s \gamma k). \]

\[ \tilde{\mathcal{G}}_{\rho\omega}^{\text{sing}}(k; \tau, \tau') = d_1 k + d_2 k \cos(2c_s \gamma k) + (d_3 k^2 + d_4) \sin(2c_s \gamma k), \]

\[ \tilde{\mathcal{G}}_{\omega\omega}^{\text{sing}}(k; \tau, \tau') = (w_1 k^2 + w_2) + (w_3 k^2 + w_4) \cos(2c_s \gamma k) + \left( \frac{w_5 k^2 + w_6}{k} \right) \sin(2c_s \gamma k). \]
Under a conformal transformation $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu}$

Weyl covariant derivative $D_\mu T^{\mu\nu} \rightarrow e^{-w\phi} \tilde{D}_\mu \tilde{T}^{\mu\nu}$ if $T^{\mu\nu} \rightarrow e^{-w\phi} \tilde{T}^{\mu\nu}$

Using definition of $D$ we find

$$D_\mu T^{\mu\nu} = d_\mu T^{\mu\nu} + A^\nu T^\mu_\mu$$

where $A^\mu = \dot{u}^\mu - (\theta/3) u^\mu$

Hydrodynamic equations are conformal if $T^{\mu\nu}$ is traceless
Instead of translational invariance (whose generators are $\xi_i = \frac{\partial}{\partial x^i}$), Gubser uses invariance under the group $SO(3)_q$ whose generators are

$$\xi_i = \frac{\partial}{\partial x^i} + q^2 \left[ 2x^i x^\mu \frac{\partial}{\partial x^\mu} - x^\mu x_\mu \frac{\partial}{\partial x^i} \right]$$

These generators are easy to understand in $dS_3 \times R$

$$d\hat{s}^2 \equiv \frac{ds^2}{\tau^2} = d\rho^2 - \cosh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2) - d\eta^2,$$

where they correspond to rotations in $(\theta, \phi)$

The only time-like four vector invariant under these transformations $[\xi, \hat{u}] = 0$ is $\hat{u}^\mu = (1, 0, 0, 0)$. 
Backup slide 5: The Langevin equation

▶ Stochastic equation for Brownian motion of particle of mass $M$ in medium having temperature $T$:

$$\frac{dp}{dt} = -\lambda p + \xi,$$

where $\langle \xi(t)\xi(t') \rangle = 2\lambda M T \delta(t - t')$.

▶ Using $e^S \propto e^{-\beta p^2/(2M)}$,

▶ Thus,

$$\dot{S} = -\frac{\beta}{M} \dot{p} p \equiv -\dot{x} X.$$

Identify $x \rightarrow p$ and $X \rightarrow \frac{\beta}{M} p \equiv P$

▶

$$\frac{dp}{dt} = -\frac{\lambda M}{\beta} P + \xi,$$

where $\langle \xi(t)\xi(t') \rangle = 2\lambda M T \delta(t - t')$.
Backup slide 6: White noise vs Colored noise  [C.C., R.S. Bhalerao, S. Pal, PRC 97, 054902 (2018)]

- In the first-order Navier-Stokes theory,

\[ T^{\mu\nu} = T_{id}^{\mu\nu} + \pi^{\mu\nu} + \Xi^{\mu\nu}, \quad \pi^{\mu\nu} = 2\eta_v\sigma^{\mu\nu} \]

- Local auto-correlation function:

\[ \langle \Xi^{\mu\nu}(x_1)\Xi^{\mu\nu}(x_2) \rangle \propto \delta^4(x_1 - x_2) \]

- In higher order theories,

\[ T^{\mu\nu} = T_{id}^{\mu\nu} + \pi^{\mu\nu} + \Xi^{\mu\nu}, \quad \pi^{\mu\nu} = \tau_\pi \langle \mu\nu \rangle + 2\eta_v\sigma^{\mu\nu} + ... \]

- In the CE equation

\[ \dot{\Xi} \langle \mu\nu \rangle = -\frac{1}{\tau_\pi} (\Xi^{\mu\nu} - \xi^{\mu\nu}) - \frac{10}{7} \Xi^{\gamma\sigma\nu} \gamma - \lambda_\pi \Xi^{\mu\nu} \theta \]

For MIS,

\[ \dot{\Xi} \langle \mu\nu \rangle = -\frac{1}{\tau_\pi} (\Xi^{\mu\nu} - \xi^{\mu\nu}) - \lambda_\pi \Xi^{\mu\nu} \theta \]
Hydrodynamic ‘cell’ size $\sim b$ (‘coarse-graining’ size)

Microscopic scale (mean free path) $\sim l_{mfp}$: $b \gg l_{mfp}$

Hydro describes variation of coarse-grained densities at scale $\sim L$.

We have the hierarchy: $L \gg b \gg l_{mfp}$

There exists another microscopic scale, the correlation length $\xi \sim l_{mfp}$

In equilibrium, $\langle \phi(t, x_1) \phi(t, x_2) \rangle \sim \exp[-|\Delta x|/\xi]$, for $\Delta x \gg \xi$.

In hydro we approximate $b \gg \xi \sim 0$. Thus $\langle \phi(t, x_1) \phi(t, x_2) \rangle \propto \delta^3(\Delta x)$.

The scale of fluctuations is thus $\mathcal{O}(b)$. 
Three scales: Two microscopic: \( l_{mfp} \sim 1/(\sigma vn) \), thermal wavelength \( l_{th} \sim 1/T \), one macroscopic \( L \sim \partial \cdot u \).

\[ l_{mfp}/l_{th} \sim \eta/s, \zeta/s, T\kappa/s \]

Dilute gas regime: \( l_{mfp}/l_{th} \sim \eta/s \gg 1 \); Weakly coupled regime, Boltzmann equation applicable (on-shell particles)

Dense gas regime: \( l_{mfp}/l_{th} \sim \eta/s \sim 1 \); quasi-particle description in terms of Wigner functions

Liquid regime: \( l_{mfp}/l_{th} \sim \eta/s \sim 1 \ll 1 \); strong-coupling regime, no valid kinetic description

Hydro applicable whenever microscopic and macroscopic scales are well-separated: \( l_{mfp} \partial \cdot u \equiv Kn < 1 \)