DEDICATION

Prof. Varadharajan Muruganandam
Large time behaviour of heat propagator on Damek–Ricci spaces

Muna Naik

Harish-Chandra Research Institute, Prayagraj
India

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Joint work with Dr. Rudra P. Sarkar and Dr. Swagato K. Ray
Notation:

- **Volume mean value operator** $B_r f$ is given by

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$$B_r f(x) := \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) \, dy = f * m_r(x)$$

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Theorem 1 (Repnikov and Éidel’man, [5], 1966).

Let $f \in L^\infty(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$ be fixed. Then

$$\lim_{r \to \infty} f \ast m_r(x_0) = L \text{ if and only if } \lim_{t \to \infty} f \ast h_t(x_0) = L.$$
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The above result was generalized by Li [3] to complete \( n \)-dimensional Riemannian manifolds \( M \) with nonnegative Ricci curvature satisfying

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\liminf_{r \to \infty} \frac{|B(x, r)|}{r^n} > 0. \tag{1}
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The above result was generalized by Li [3] to complete $n$-dimensional Riemannian manifolds $M$ with nonnegative Ricci curvature satisfying

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Using Bishop–Gromov comparison theorem one can show that the geodesic ball $B(x, r)$ has polynomial volume growth.

The proof of Li’s result relies on the above result of Repnikov et al.

Let $S$ be a Damek–Ricci space and $f \in L^\infty(S)$. Then for any $x_0 \in S$,

$$\lim_{r \to \infty} f \ast m_r(x_0) = L \implies \lim_{t \to \infty} f \ast h_t(x_0) = L,$$

where $L$ is a constant.

The converse of above theorem is not true in Damek–Ricci space.

Question: Can one replace the boundedness condition of $f$ in Repnikov and ´Eidel’mann’s Theorem 1 by any other suitable growth condition?

Let $S$ be a Damek–Ricci space and $f \in L^\infty(S)$. Then for any $x_o \in S$,

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Let $S$ be a Damek–Ricci space and $f \in L^\infty(S)$. Then for any $x_0 \in S$,

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The converse of above theorem is not true in Damek–Ricci space.

Question: Can one replace the boundedness condition of $f$ in Repnikov and Éidel’man’s Theorem 1 by any other suitable growth condition?

Let \( f, g \) be measurable functions on \( S \) such that \( f \in L^\infty(S) \) and

\[
\lim_{r \to \infty} f * m_r(x) = g(x), \quad \text{for almost every } x \in S.
\]

Then \( \Delta g = 0 \).

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Then $\Delta g = 0$.

Proof.

- Applying Theorem 2 we get

\[ \lim_{s \to \infty} f \ast h_s(x) = g(x), \quad (2) \]

for almost every $x \in S$. 

Let \( f, g \) be measurable functions on \( S \) such that \( f \in L^\infty(S) \) and

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Proposition 1.

Let $f \in L^\infty(\mathbb{R}^n)$ and $x_o \in \mathbb{R}^n$ be fixed.
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Let \( f \in L^\infty(\mathbb{R}^n) \) and \( x_o \in \mathbb{R}^n \) be fixed. If

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\lim_{r \to \infty} f \ast m_r(x_o) = L
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for a constant \( L \), then for any \( x \in \mathbb{R}^n \)

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f \ast m_r(0) - f \ast m_r(x) = 1 \left| B(o, r) \right| \left( \int_{B(o, r)} f(y) \, dy - \int_{B(x, r)} f(y) \, dy \right)
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\begin{align*}
f \ast m_r(0) - f \ast m_r(x) &= 1 \left| B(o, r) \right| 
\left( \int_{B(o, r)} f(y) \, dy - \int_{B(x, r)} f(y) \, dy \right) \\
&= \frac{1}{\left| B(o, r) \right|} \left[ \left( \int_{B(o, r) \setminus B(x, r)} f(y) \, dy + \int_{B(o, r) \cap B(x, r)} f(y) \, dy \right) \right]
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= \frac{1}{|B(o, r)|} \left( \int_{B(o, r) \setminus B(x, r)} f(y) \, dy - \int_{B(x, r) \setminus B(o, r)} f(y) \, dy \right)
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\[ |f \ast m_r(0) - f \ast m_r(x)| \]
\[ |f * m_r(0) - f * m_r(x)| \leq \frac{1}{|B(o, r)|} \int_{B(o, r) \triangle B(x, r)} |f(y)| \, dy \]
\[ |f \ast m_r(0) - f \ast m_r(x)| \leq \frac{1}{|B(o, r)|} \int_{B(o, r) \triangle B(x, r)} |f(y)| \, dy \leq \|f\|_\infty \frac{|B(o, r) \triangle B(x, r)|}{|B(o, r)|} \]
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\leq \|f\|_{\infty} \frac{|B(o, r) \triangle B(x, r)|}{|B(o, r)|}
\]

\[
\leq \|f\|_{\infty} \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|}
\]

where \(A(r - |x|, r + |x|)\) is the **annulus** centered at \(o\) with inner radius \(r - |x|\) and outer radius \(r + |x|\).
\begin{align*}
|f \ast m_r(0) - f \ast m_r(x)| & \leq \frac{1}{|B(o, r)|} \int_{B(o, r) \triangle B(x, r)} |f(y)| \, dy \\
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where \( A(r - |x|, r + |x|) \) is the annulus centered at \( o \) with inner radius \( r - |x| \) and outer radius \( r + |x| \).

\( op = xp - ox = r - |x| \).

\( oq = ox + xq = r + |x| \).

\( B(o, r) \triangle B(x, r) \subseteq A(r - |x|, r + |x|) \).
Thus we get

\[ |f \ast m_r(0) - f \ast m_r(x)| \leq \|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|}. \]
Thus we get

\[ |f * m_r(0) - f * m_r(x)| \leq \|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|} \cdot (r + |x|)^n - (r - |x|)^n \]

Since

\[ \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|} = \frac{(r + |x|)^n - (r - |x|)^n}{r^n} \]

goes to zero as \( r \to \infty \), by taking \( \lim \sup \) in both sides of (3) we get our desired result. \[\square\]
Thus we get

\[ |f \ast m_r(0) - f \ast m_r(x)| \leq \|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|}. \tag{3} \]

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goes to zero as \( r \to \infty \), by taking lim sup in both sides of (3) we get our desired result. \( \square \)

- In a Damek–Ricci space,

\[ \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|} \lesssim \frac{e^{\alpha(r + |x|)} - e^{\alpha(r - |x|)}}{e^{\alpha r}} = e^{\alpha |x|} - e^{-\alpha |x|} \]

doesn’t go to zero as \( r \to \infty \).
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\[ |f * m_r(0) - f * m_r(x)| \leq \|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|}. \] (3)

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- Let $\rho$ denotes the half of the limit of the mean curvature of geodesic spheres as radius of the sphere tends to infinity.
- For $\lambda \in \mathbb{C}$, Elementary spherical function $\varphi_\lambda$ is the unique smooth radial eigenfunction of $\Delta$ with
  \[
  \Delta \varphi_\lambda = -(\lambda^2 + \rho^2) \varphi_\lambda, \quad \varphi_\lambda(o) = 1.
  \]
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Let $\rho$ denotes the half of the limit of the mean curvature of geodesic spheres as radius of the sphere tends to infinity.

For $\lambda \in \mathbb{C}$, **Elementary spherical function** $\varphi_{\lambda}$ is the unique smooth radial eigenfunction of $\Delta$ with

$$\Delta \varphi_{\lambda} = -(\lambda^2 + \rho^2) \varphi_{\lambda}, \quad \varphi_{\lambda}(o) = 1.$$ 

$\varphi_{\lambda} = \varphi_{-\lambda}$ and $\varphi_{i\rho} = \varphi_{-i\rho} = 1$. 
Notation:

- We fix the identity element $e$ of the group $S$ as the origin $o$.
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- $|\varphi_\lambda(x)| \leq 1$ for $|\Re \lambda| \leq \rho$.  

$\text{Muna Naik (HRI, Prayagraj)}$
For $\lambda \in \mathbb{C}$, let $\hat{f}(\lambda)$ denotes the spherical Fourier transform of $f$ at $\lambda$.
Definition

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$$\hat{f}(\lambda) := \int_S f(x) \varphi_{\lambda}(x) \, dx.$$
**Definition**

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Let $\psi_{\lambda}(r) := \frac{1}{|B(o, r)|} \int_{B(o, r)} \varphi_{\lambda}(x) \, dx$.
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$$\left( \text{Recall: } m_r(y) = \frac{1}{|B(o,r)|} \chi_{B(o,r)}(y) \right).$$

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For $\Im \lambda < 0$ and $t > 0$, we have the following asymptotic estimate of $\psi_\lambda$,

$$\lim_{t \to \infty} e^{-(i\lambda - \rho)t} \psi_\lambda(t) = c(\lambda)$$

where $c(\lambda)$ is an analogue of Harish-Chandra $c$-function. It is also known that $c$-function has neither zero nor pole in the region $\Im \lambda < 0$. Let $0 \neq \alpha \in \mathbb{R}$ be fixed. Claim: $\psi_{\alpha - i\rho}(r)$ does not converge to any value as $r \to \infty$ and is oscillatory. Reason: $e^{i\alpha r} = 1 e^{-(i(\alpha - i\rho) - \rho)r} \psi_{\alpha - i\rho}(r)$.
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**Claim:** \( \psi_{\alpha - i\rho}(r) \) does not converge to any value as \( r \to \infty \) and is oscillatory.

**Reason:**

\[
e^{i\alpha r} = \frac{1}{e^{-[i(\alpha - i\rho) - \rho]r} \psi_{\alpha - i\rho}(r)} \psi_{\alpha - i\rho}(r).
\]
For a radial measure \( \mu \), one can show that

\[
\varphi_\lambda \ast \mu(x) = \hat{\mu}(\lambda) \varphi_\lambda(x).
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For a radial measure $\mu$, one can show that

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Thus we get

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From above, it is clear that for any $x \in S$, $\varphi_{\alpha - i\rho} \ast h_t(x) \to 0$ as $t \to \infty$. 

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From above, it is clear that for any $x \in S$, $\varphi_{\alpha - i\rho} * h_t(x) \to 0$ as $t \to \infty$. 
Similarly we have, \( \varphi_\lambda \ast m_r(x) = \hat{m}_r(\lambda) \varphi_\lambda(x) \).
Similarly we have, $\varphi_\lambda \ast m_r(x) = \hat{m}_r(\lambda) \varphi_\lambda(x)$

$= \psi_\lambda(r) \varphi_\lambda(x)$. 

Since $\psi_\lambda(r)$ does not converge to any value as $r \to \infty$, it follows that $\varphi_\lambda \ast h_t(x)$ does not converge to any value as $t \to \infty$. But since $\varphi_\lambda \ast h_t(x) \to 0$ as $t \to \infty$, it follows that the function $\varphi_\lambda$ forms a counterexample for Repnikov et al’s theorem in Damek–Ricci space.
Similarly we have, \( \varphi_\lambda \ast m_r(x) = \widehat{m_r}(\lambda)\varphi_\lambda(x) \)
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Thus we get, \( \varphi_{\alpha-i\rho} \ast m_r(x) = \psi_{\alpha-i\rho}(r)\varphi_{\alpha-i\rho}(x). \)
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But since \( \varphi_{\alpha-i\rho} \ast h_t(x) \to 0 \) as \( t \to \infty \), it follows that the function \( \varphi_{\alpha-i\rho} \) forms a counterexample for Repnikov et al’s theorem in Damek–Ricci space.
Let \( \varphi_{\alpha - i \rho}(x) = u(x) + i v(x) \).
Let $\varphi_{\alpha-i\rho}(x) = u(x) + i v(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$. 

Without loss of generality assume that $u * m_r(x)$ does not converge. Since $|\varphi_{\alpha-i\rho}(x)| \leq 1$, $|u(x)| \leq 1$. Let $f(x) = 2 - u(x)$. Clearly $f$ is a strictly positive function and $f * m_r(x)$ does not converge to any value as $r \to \infty$. On the other hand, since $u * h_t(x) + iv * h_t(x) = \varphi_{\alpha-i\rho} * h_t(x) \to 0$ as $t \to \infty$, it is clear that $u * h_t(x) \to 0$ as $t \to \infty$. Consequently we get

$$\lim_{t \to \infty} f * h_t(x) = 2 - \lim_{t \to \infty} u * h_t(x) = 2$$

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Let \( \varphi_{\alpha-i\rho}(x) = u(x) + iv(x) \). As \( \varphi_{\alpha-i\rho} \ast m_r(x) \) does not converge to any value, it follows that either \( u \ast m_r(x) \) or \( v \ast m_r(x) \) does not converge to any value as \( r \rightarrow \infty \). Without loss of generality assume that \( u \ast m_r(x) \) does not converge.
Let $\varphi_{\alpha-i\rho}(x) = u(x) + iv(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$. Without loss of generality assume that $u * m_r(x)$ does not converge.

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Consequently we get $\lim_{t \to \infty} f * h(x) = 2 - \lim_{t \to \infty} u * h(x) = 2$ for any fixed $x \in S$. 

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Let $f$ be a measurable function on $S$

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$$|f(x)| \leq Ae^{B|x|}, \text{ for almost every } x \in S, \quad (5)$$

and for constants $A > 0$ and $B \in \mathbb{R}$. 

Converse of the above theorem is not true.

The above theorem is not true for all complex number $\lambda$ with nonzero real and imaginary parts, i.e. $\lambda \notin (i\mathbb{R} \cup \mathbb{R})$. 

Let $f$ be a measurable function on $S$ satisfying

$$|f(x)| \leq Ae^{B|x|}, \quad \text{for almost every } x \in S,$$

and for constants $A > 0$ and $B \in \mathbb{R}$. Then for any $\lambda \in i\mathbb{R}$ and a point $x_0 \in S$,

$$\lim_{r \to \infty} \frac{1}{\psi_\lambda(r)} f \ast m_r(x_0) = L \implies \lim_{t \to \infty} e^{t(\lambda^2 + \rho^2)} f \ast h_t(x_0) = L,$$

where $L$ is a constant.

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- Converse of the above theorem is not true.

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and for constants $A > 0$ and $B \in \mathbb{R}$. Then for any $\lambda \in i\mathbb{R}$ and a point $x_0 \in S$,

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**Muna Naik (HRI, Prayagraj)**
Large time behaviour of heat propagator
5th January, 2022
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(a) $|f(x)| \leq B \phi(\lambda)(x)$ for almost every $x \in S$, for a constant $B > 0$,

(b) $f \in L^p(S)$ for some $p \in (2, \infty)$ satisfying $|\lambda| = (1 - 2/p) \rho$,

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Estimate of heat kernel

$$h_t(r) \approx t^{-\frac{3}{2}} (1 + r)^{-\frac{1}{4}} (1 + 1 + r t)^{n-\frac{3}{2}} e^{-r^2/4t - \rho^2 r}.$$  

$$(n = \dim(S), r \geq 0)$$

Exponential factors:

- $R_n: p_t(r) = e^{-r^2/4t},$ (peak is always at 0).
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\[ h_t(r) \propto t^{-\frac{3}{2}}(1 + r) \left(1 + \frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{-\rho r}, \quad (n = \dim(S), r \geq 0) \]
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Let $s(t)$ be a positive function such that $\frac{s(t)}{\sqrt{t}} \to \infty$ as $t \to \infty$. 

This means that heat produced initially at the origin $o \in S$ does not diffuse homogeneously but concentrates asymptotically in an annulus of width $s(t)$ moving to infinity with speed $2\rho$. Such behavior sharply contrasts to that of the Euclidean space $\mathbb{R}^n$. 

The above result was first proved by Davies et al for hyperbolic spaces [2] and by Anker and Setti for all symmetric spaces of noncompact type [1].
Let $s(t)$ be a positive function such that $\frac{s(t)}{\sqrt{t}} \to \infty$ as $t \to \infty$. Then

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