Gauge symmetry in quantum mechanics

To understand gauge symmetry, both local and global, let us go through an extremely concise review of *gauge transformation* in classical electrodynamics since it is in that context it appears first. The entire classical electrodynamics in vacuum is described by the following four Maxwell's equations,

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\epsilon_0} && \text{Gauss's law} && (61)\\
\vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} && \text{Ampere's law} && (62)\\
\vec{\nabla}\cdot\vec{B} &= 0 && && (63)\\
\vec{\nabla}\times\vec{B} &= \mu_0\vec{j}+\frac{\partial\vec{E}}{\partial t} && \text{Faraday's law,} && (64)
\end{aligned}
$$

together with Lorentz force and continuity equation,

$$\vec{F} = q\left(\vec{E}+\vec{v}\times\vec{B}\right) \quad \text{and} \quad \frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{j} = 0. \tag{65}$$

The electric field $\vec{E}$ and magnetic field $\vec{B}$ can be written in terms of scalar and vector potentials (ϕ, $\vec{A}$),

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}. \tag{66}$$

However, ϕ and $\vec{A}$ do not uniquely specify $\vec{E}$ and $\vec{B}$. There is some amount of arbitrariness in the potentials and if they are transformed in the following manner,

$$\vec{A}' = \vec{A}-\vec{\nabla}\Lambda \quad \text{and} \quad \phi' = \phi+\frac{\partial\Lambda}{\partial t}. \tag{67}$$

Maxwell equations describing electrodynamics remain invariant. The transformation of the potentials is called *gauge transformation* and invariance of Maxwell equations under gauge transformation is known as *gauge invariance* or the theory of the fields $\vec{E}$ and $\vec{B}$ are said to have *gauge symmetry*. Λ is called the *gauge parameter*. It has been realized that gauge symmetry is one of the most desirable symmteries for all sorts of field theory.

However in context of quantum theory, gauge transformation is defined to be change in the phase of the fields, which may or may not depend on the local coordinates. If the phase is constant it is called *global gauge transformation* and called *local* if the phase depends on local space-time coordinates. Presently, the very first step to study is properties of Schrödinger equation under global gauge transformation. Let $\psi(x,t)$ be the solution of the Schrödinger equation describing motion of a quantum particle in a potential $V_0(x)$. Apply the gauge transformation,

$$\psi(x,t) \;\rightarrow\; \psi'(x,t) = e^{i\alpha}\,\psi(x,t), \tag{68}$$

where α is the gauge parameter which is constant. If we substitute ψ' for ψ in Schrödinger equation then we get,

$$i\hbar\frac{\partial\psi'(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi'(x,t)+V_0(x)\,\psi'(x,t) \tag{69}$$

which implies ψ' is also a solution of Schrödinger equation and the form of the Schrödinger equation *i.e.* the Hamiltonian remains invariant and so it probability denstiy $|\psi|^2 = |\psi'|^2$.

Hence we conclude that the Schrödinger equation has global gauge symmetry. Next thing to address is the invariance of Schrödinger equation or the corresponding Hamiltonian under local gauge transformation, which is,

$$\psi(x,t) \;\rightarrow\; \psi'(x,t) \;=\; e^{i\alpha(x,t)}\,\psi(x,t). \tag{70}$$

Now the time and space derivative on ψ' will be,

$$\begin{aligned}
\frac{\partial \psi'}{\partial t} &= i\,\frac{\partial \alpha}{\partial t}\,\psi' + e^{i\alpha}\,\frac{\partial \psi}{\partial t} \\
\nabla\,\psi' &= i\nabla\alpha\,\psi' + e^{i\alpha}\,\nabla\psi \\
\nabla^2\psi' &= i\nabla^2\alpha\,\psi' + 2i\nabla\alpha\,\nabla\psi\,e^{i\alpha} - (\nabla\alpha)^2\psi' + e^{i\alpha}\nabla^2\psi
\end{aligned}$$

Now assuming ψ' is a solution of Schrödinger equation, under gauge transformation ψ is no longer one,

$$i\hbar\,\frac{\partial\psi(x,t)}{\partial t} \;=\; -\frac{\hbar^2}{2m}\,\left[\nabla + i\,\nabla\alpha(x,t)\right]^2\psi(x,t) + \left[V_0(x) + \hbar\,\frac{\partial\alpha(x,t)}{\partial t}\right]\psi(x,t) \tag{71}$$

and more importantly, the equation (71) is no longer a Schrödinger equation. In other words, Schrödinger equation has no local gauge symmetry, although $|\psi|^2 = |\psi'|^2$ still holds. This gives us a clue that *Schrödinger equation in the present form or the Hamiltonian from which it is derived is not gauge invariant.* But if gauge invariance is demanded then Schrödinger equation requires re-writing by introducing in it space-time dependent functions $\phi(x,t)$ and $\vec{A}(x,t)$ hoping to cancel the unwanted terms as,

$$i\hbar\,\frac{\partial\psi(x,t)}{\partial t} \;=\; -\frac{\hbar^2}{2m}\,\left[\nabla - iq\,\vec{A}(x,t)\right]^2\psi(x,t) + \left[V_0(x) + \hbar\,\phi(x,t)\right]\psi(x,t) \tag{72}$$

Now, if local gauge transformation (70) is performed then the modified Schrödinger equation with $\vec{A}$ and ϕ is invarinat under,

$$\begin{aligned}
\psi(x,t) &\rightarrow e^{i\alpha(x,t)}\,\psi(x,t) \\
\phi(x,t) &\rightarrow \phi(x,t) - \frac{\partial\alpha(x,t)}{\partial t} \\
\vec{A}(x,t) &\rightarrow \vec{A}(x,t) + \frac{1}{q}\,\nabla\,\alpha(x,t).
\end{aligned} \tag{73}$$

Comparing this with the gauge transformation in electrodynamics (67), the $\vec{A}$ and ϕ are completely equivalent to standard gauge transformation of electrodynamics with, $\vec{A}$ and ϕ being the vector and scalar potentials satisfying Maxwell's equations. This is a spectacular conclusion – *local gauge symmetry of Schrödinger equation demands presence of em-fields and charge q!*