Ladder operators: The angular momentum eigenvalue equations (5) can also be solved by introducing ladder operators very similar to the one applied to SHO,

\[ L_\pm = L_x \pm i L_y, \]  

(28)

The commutation relations involving \( L_\pm \) and components of angular momentum are derived using the relations (4),

\[
\begin{align*}
[L_z, L_\pm] &= [L_z, L_x \pm i L_y] = \pm \hbar L_\pm \\
[L^2, L_\pm] &= 0 \\
[L_\pm, L_{\mp}] &= \pm 2\hbar L_z.
\end{align*}
\]

(29)

The ladder operators \( L_\pm \) also satisfy,

\[
L_\pm L_\mp = (L_x \pm i L_y)(L_x \mp i L_y) = L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x) \\
= L^2 - L_z^2 \pm \hbar L_z
\]

(30)

For the present purpose, let the angular momentum eigenvalue equations be written in the following form,

\[
L^2 f = \lambda \hbar^2 f \quad \text{and} \quad L_z f = \mu \hbar f
\]

(31)

Just as in SHO, if \( f \) is an eigenfunction of \( L^2 \) and \( L_z \), so also is \( L_\pm f \),

\[
L^2 (L_\pm f) = L_\pm (L^2 f) = L_\pm (\lambda \hbar^2 f) = \lambda \hbar^2 (L_\pm f) \\
L_z (L_\pm f) = \pm \hbar (L_\pm f) + L_\pm (L_z f) = (\mu \pm 1)\hbar (L_\pm f)
\]

(32)

Interestingly, \( L_\pm f \) is indeed an eigenfunction of \( L_z \) but with new eigenvalues, raised or lowered by one unit of \( \hbar \). Hence, \( L_+ \) (\( L_- \)) operator is called raising (lowering) operator and together are called ladder operator. Therefore, each time we apply \( L_+ \) or \( L_- \), the eigenvalue \( \mu \) increases or decreases by \( \hbar \). But can this continue indefinitely? The restriction on \( \mu \) follows from the observation,

\[
\begin{align*}
\langle L^2 - L_z^2 \rangle &= (\lambda - \mu^2)\hbar^2, \\
\langle L^2 - L_z^2 \rangle &= \langle L_x^2 + L_y^2 \rangle \\
&= \langle L_x^2 \rangle + \langle L_y^2 \rangle \\
&= \int d\tau f^* L_x L_x f + \int d\tau f^* L_y L_y f \\
&= \int d\tau (L_x f)^2 + \int d\tau (L_y f)^2 \quad \text{using hermiticity of } L_x, L_y \\
&\geq 0
\end{align*}
\]

(33)

where \( d\tau \) is some appropriate volume element. Comparing (33) and (34), we get,

\[
\lambda \geq \mu^2.
\]

(35)

Therefore, raising or lowering of \( \mu \) must stop to obey (35). Say raising must stop at some \( f_{\text{max}} \) and maximum value for \( \mu \) be, say, \( l\hbar \),

\[
L_\pm f_{\text{max}} = 0 \quad \Rightarrow \quad L_z f_{\text{max}} = l \hbar f_{\text{max}} \quad \text{and} \quad L^2 f_{\text{max}} = \lambda \hbar^2 f_{\text{max}}
\]

(36)
Using (30), it follows that,

\[ L^2 f_{\text{max}} = \lambda \hbar^2 f_{\text{max}} \Rightarrow (L_- L_+ + L_z^2 + \hbar L_z) f_{\text{max}} = (0 + \ell^2 \hbar^2 + \hbar^2) f_{\text{max}} \]

and therefore, the eigenvalue \( \lambda \) of \( L^2 \) in terms of maximum eigenvalue of \( L_z \) is,

\[ \lambda = l(l + 1) \hbar^2 \]  \hspace{1cm} (37)

which is exactly what we got by solving angular part of Schrödinger equation in (21). Similarly, lowering also should end at some \( f_{\text{min}} \) and minimum value for \( \mu \) be \( \bar{l} \hbar \),

\[ L_- f_{\text{min}} = 0 \Rightarrow L_z f_{\text{min}} = \bar{l} \hbar f_{\text{min}} \text{ and } L^2 f_{\text{min}} = \lambda \hbar^2 f_{\text{min}} \]  \hspace{1cm} (38)

In the same way as with \( L_+ \), using (30), it follows that,

\[ L^2 f_{\text{min}} = \lambda \hbar^2 f_{\text{min}} \Rightarrow (L_+ L_- + L_z^2 - \hbar L_z) f_{\text{min}} = (0 + \ell^2 \hbar^2 - \bar{l} \hbar^2) f_{\text{min}} \]

and therefore, the eigenvalue \( \lambda \) of \( L^2 \) in terms of minimum eigenvalue of \( L_z \) is,

\[ \lambda = \bar{l}(\bar{l} - 1) \hbar^2. \]  \hspace{1cm} (39)

Since eigenvalue \( \lambda \) of \( L^2 \) does not change with action of \( L_\pm \), comparing equations (37) and (39), we see that \( l(l + 1) = \bar{l}(\bar{l} - 1) \) and solving for \( \bar{l} \) we get,

\[ \bar{l} = -l \text{ and } \bar{l} = l + 1. \]

But \( \bar{l} = l + 1 \) is absurd since minimum eigenvalue cannot be larger than maximum eigenvalue, so the acceptable solution is

\[ \bar{l} = -l \Rightarrow -l \leq \mu \leq +l \]  \hspace{1cm} (40)

which the limit of \( \mu \) (the magnetic quantum number) we talked about before without actually showing it. So for a given value of \( l \), there are \( 2l + 1 \) different values of \( \mu \) and, since \( \mu \) changes by one unit of \( \hbar \), it goes from \( -l \) to \( +l \) (or the other way round) in \( N \) (say) integer steps,

\[ l = -l + N \Rightarrow l = N/2 \]

i.e. \( l \) must be an integer or a half-integer, \( l = 0, 1/2, 1, 3/2, 2, \ldots \) However, we have seen before when angular part of the Schrödinger equation was solved explicitly that \( l \) is an integer. Thus here we have determined the eigenvalues of generic angular momentum operator without even knowing its eigenfunctions.

The ladder operator when acted upon the eigenfunctions of \( L^2 \) and \( L_z \) changes the eigenvalues of \( L_z \) by one unit which can be represented as,

\[ L_+ f_\mu = c_\lambda f_{\mu + 1} \text{ and } L_- f_\mu = d_\lambda f_{\mu - 1}. \]  \hspace{1cm} (41)
The coefficient $c_{\lambda \mu}$ can be determined as,

$$L_+ f_\mu = c_{\lambda \mu} f_{\mu+1} \Rightarrow \int d\tau f_\mu^* L_- L_+ f_\mu = \int d\tau (L_+ f_\mu)^*(L_+ f_\mu) = |c_{\lambda \mu}|^2 \int d\tau |f_{\mu+1}|^2 = |c_{\lambda \mu}|^2$$

or,

$$\int d\tau f_\mu^* L_- L_+ f_\mu = \int d\tau f_\mu^* (L^2 - L_z^2 - \hbar L_z) f_\mu$$

$$= \left[ l(l+1)\hbar^2 - \mu^2 \hbar^2 - \mu \hbar^2 \right] \int d\tau |f_\mu|^2$$

$$= (l-\mu)(l+\mu+1)\hbar^2$$

$$\Rightarrow c_{\lambda \mu} = \sqrt{(l-\mu)(l+\mu+1)\hbar}$$  \hspace{1cm} (42)

In the same way we can get $d_{\lambda \mu} = [(l+\mu)(l-\mu+1)]^{1/2}\hbar$. In short,

$$c_{\lambda \mu} f_\mu = \sqrt{(l-\mu)(l+\mu+1)} \hbar f_{\mu+1} \hspace{1cm} (43)$$

$$d_{\lambda \mu} f_\mu = \sqrt{(l+\mu)(l-\mu+1)} \hbar f_{\mu-1}. \hspace{1cm} (44)$$