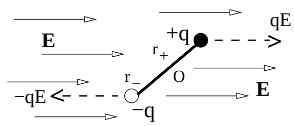
Dipole in uniform electric field: Consider a dipole in a uniform electric field $\vec{E}$ as shown in the figure below. Let the lengh of the dipole be s, such that $|\vec{r}_+| = |\vec{r}_-| = |\vec{s}/2|$, $\vec{s} = \vec{r}_+ - \vec{r}_-$ and the forces on $+q$ and $-q$ be $\vec{F}_+ = q\vec{E}$ and $\vec{F}_- = -q\vec{E}$.



The total force on the dipole in the field is zero, $\vec{F}_+ + \vec{F}_- = 0$ but there is a torque about the midpoint O of the dipole, which is

$$\begin{aligned} \vec{N} &= \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_- \\ &= \frac{\vec{s}}{2} \times q\vec{E} + \left(-\frac{\vec{s}}{2}\right) \times (-q\vec{E}) \\ &= q\,\vec{s} \times \vec{E} \\ \Rightarrow \vec{N} &= \vec{p} \times \vec{E} \end{aligned} \tag{5}$$

The direction of $\vec{N}$ is such as to line up $\vec{p}$ parallel to $\vec{E}$.

Dipole in non-uniform electric field: This time around the total force $\vec{F}_+ + \vec{F}_- \neq 0$, there is a net force on dipole in addition to torque. Let the electric field close to $+q$ is $\vec{E} + d\vec{E}$ while $\vec{E}$ at $-q$. The total force then is,

$$\vec{F} = \vec{F}_+ + \vec{F}_- = q(\vec{E} + d\vec{E}) - q\vec{E} = qd\vec{E}$$

We make use of the relation $d\phi = \nabla\phi \cdot d\vec{l}$ to re-write $d\vec{E}$ as $d\vec{E} = (d\vec{s} \cdot \nabla)\,\vec{E}$. Hence, we have $qd\vec{E} = q(d\vec{s} \cdot \nabla)\vec{E}$ and $\vec{p} = qd\vec{s}$ and therefore,

$$\vec{F} = (\vec{p} \cdot \nabla)\vec{E} \tag{6}$$

The torque in non-uniform field, using the notation $\vec{r}_+ = \vec{r}_- = \vec{r}$ is,

$$\begin{aligned} \vec{N} &= \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_- \\ &= q\left[\vec{r}_+ \times (\vec{E} + d\vec{E}) - \vec{r}_- \times \vec{E}\right] \\ &= q\left[(\vec{r}_+ - \vec{r}_-) \times \vec{E} + \vec{r}_+ \times d\vec{E}\right] \\ &= q\left[\vec{s} \times \vec{E} + \vec{r} \times d\vec{E}\right] \\ \Rightarrow \vec{N} &= \vec{p} \times \vec{E} + \vec{r} \times (\vec{p} \cdot \nabla)\vec{E} \end{aligned} \tag{7}$$

Energy of a dipole in electric field: We calculate the energy of a diple in a electric field $\vec{E}$ by bringing the dipole from infinity and placing it such that $-q$ charge is at $\vec{r}$. Work done is $(\vec{E} = -\nabla V(r))$,

$$U = -q\,V(r) + q\,V(r+s).$$

Now, since $r \gg s$, we can Taylor expand $V(r+s)$ as,

$$V(r+s) = V(r) + \vec{s} \cdot \nabla V(r) + \cdots$$

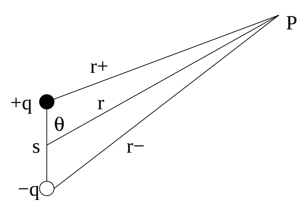
Putting back the above expression in U, we get

$$\begin{aligned} U &= -q\,V(r) + q\,V(r) + q\,\vec{s}\cdot\nabla V(r) \\ &= \vec{p}\cdot\nabla V(r) \\ \Rightarrow U &= -\vec{p}\cdot\vec{E} \end{aligned} \tag{8}$$

Work done to rotate a dipole: Rotating a dipole kept in a uniform electric field $\vec{E}$ by an angle θ_0 requires work,

$$W \;=\; \int_0^{\theta_0} N d\theta \;=\; \int_0^{\theta_0} p\,E\,\sin\theta\,d\theta \;=\; p\,E\,(1-\cos\theta_0) \tag{9}$$

Field due to di-polar object: Consider a dipole consists of two equal and opposite charges ($\pm q$) separated by a distance s. The object itself may be a dielectric or a molecule, but it is placed in vacuum, that is why we still be using ϵ_0. Let $r_\pm$ and r be the distances of the field point from the charges $\pm q$ and dipole center, where $r \gg s$.



The potential at point P is,

$$V_d(r) \;=\; \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

and from law of cosines,

$$r_\pm^2 \;=\; r^2 + (s/2)^2 \mp rs\cos\theta \;=\; r^2\left(1 \mp \frac{s}{r}\cos\theta + \frac{s^2}{4r^2}\right)$$

Since we are interested in the region $r \gg s$, the $\mathcal{O}(s^2)$ term is negligibly small and binomial expansion yields,

$$\frac{1}{r_\pm} \;\approx\; \frac{1}{r}\left(1 \mp \frac{s}{r}\cos\theta\right)^{-1/2} \;\approx\; \frac{1}{r}\left(1 \pm \frac{s}{2r}\cos\theta\right)$$

Using the above the expression, we get

$$\frac{1}{r_+} - \frac{1}{r_-} \;=\; \frac{s}{r^2}\cos\theta$$

and the potential for dipole at point P is, therefore, given by

$$V_d(r) \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{qs\cos\theta}{r^2} \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{\vec{p}\cdot\hat{r}}{r^2} \tag{10}$$