Topological Bounds on Certain Moduli of Curves

Ph.D Thesis Defence

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Organization

1. Definitions and Motivation
2. Background on Moduli of Curves
3. Results
4. Techniques

Affine Stratification Number

An **affine stratification** of a scheme X is a finite decomposition

$$X = \bigsqcup_{k=0}^{n} \bigsqcup_{i} Y_{k,i} \qquad \text{where} \quad \overline{Y}_{k,i} \setminus Y_{k,i} \subset \bigcup_{k'>k,j} Y_{k',j}$$

and $Y_{k,i}$ are locally closed and affine. The number n, is the length of the stratification and the **minimum** over all affine stratifications will be called the **affine stratification number** of X, denoted as $\operatorname{asn} X$.

Theorem (Roth, Vakil)

If $\operatorname{asn} X = m$ *then there exists an affine stratification* $X = \bigsqcup_{k=0}^{m} Z_k$, *such that*

- $\overline{Z}_k = \cup_{k' \geq k} Z_{k'}$, *and* Z_k *dense open affine in* $\overline{Z}_k$
- $\overline{Z}_k$ *is pure co-dimension 1 in* Z_{k-1}.

We call this the **affine cell decomposition** of X.

Equidimensional case

If X is equi-dimensional, then an affine stratification of X will be

$$X = \bigsqcup_{k=0}^{n} Y_k$$

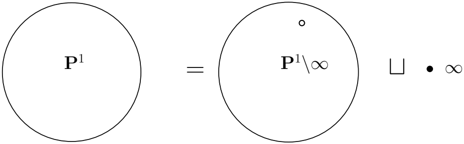
where Y_i is locally closed and co-dimension i in X, further

$$\overline{Y}_i \backslash Y_i \subset \bigsqcup_{k=i+1}^{n} Y_k$$

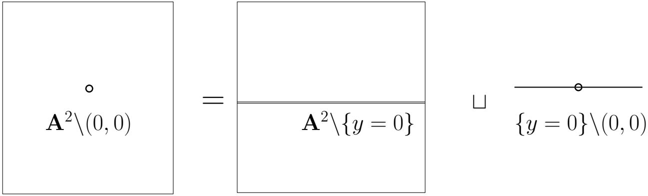
n is the length of the stratification and the length of the shortest affine stratification is the affine stratification number of X (denoted $\operatorname{asn} X$).

Examples

Projective Line: $\mathbb{P}^1 = \mathbb{A}^1 \sqcup \{\infty\}$



Punctured plane: $\mathbb{A}^2\backslash(0,0) = (\mathbb{A}^2\backslash\{y=0\}) \sqcup (\{y=0\}\backslash(0,0))$



Properties

Intuitively $\operatorname{asn} X$ measures how far the scheme X is from being affine and how close it is to being proper.

Let $a = \operatorname{asn} X$ and $d = \dim X$. We have the following bounds.

1. $a \leq d$, and equality holds if X has a proper d-dimensional component.
2. If $\mathcal{F}$ is a quasi-coherent sheaf on X, then $H^i(X, \mathcal{F}) = 0$ for $i > \operatorname{asn} X$.
3. If Z is a proper closed sub-scheme, then $\dim Z \leq a$.
4. If $\mathcal{F}$ is a constructible sheaf then $H^i(X, F) = 0$ for $i > \operatorname{asn} X + \dim X$.

As a consequence of (4) when the base field is $\mathbb{C}$, note that the Betti cohomology vanishes above degree $a + d$ (although the real dimension of X is $2d$).

Constructible Cohomological Dimension

Let X be a complex variety. Recall that an abelian sheaf $\mathcal{F}$ on X is said to be **constructible** if there exists a locally finite partition of X into subsets that are locally closed for the Zariski topology such that the restriction of $\mathcal{F}$ to each member of that partition is locally constant for the Euclidean topology (its stalks may be arbitrary).

Constructible cohomological dimension, $\mathrm{ccd}(X)$, of a variety X is the smallest integer d with the property that

$$H^n(X, \mathcal{F}) = 0 \text{ for } n > d$$

for every constructible sheaf $\mathcal{F}$ on X.

Cohomological Excess

The *cohomological excess* of a non-empty variety X, denoted $\mathrm{ce}(X)$, is

$$\mathrm{ce}(X) = \max\{\mathrm{ccd}(W) - \dim W \mid W \text{ closed subvariety of } X\}$$

It can be easily seen that

$$0 \leq \mathrm{ce}(X) \leq \dim X$$

If X is affine $\mathrm{ce}(X) = 0$, and when projective $\mathrm{ce}(X) = \dim X$.

Cohomological Excess is a weaker but better behaved invariant than the affine stratification number.

Moduli of Curves

Let $M_{g,n}$ be the moduli space of algebraic curves of genus g with n marked points (over $\mathbb{C}$). (I would like to consider the course moduli spaces but one can also think of them as Deligne-Mumford stacks).

Let $\overline{M}_{g,n}$ be it's Deligne-Mumford compactification. This space parametrizes isomorphism classes of stable curves of genus g with n marked points. We call such a curve a stable curve of type (g, n).

$M_{g,n}$ is a dense open subset of $\overline{M}_{g,n}$, and the complement is a divisor with normal crossings.

Conjecture (Original)

M_g can be covered by $g-1$ open affines.

Recent progress by Fontanari and Pascolutti showing that this is true for $g \leq 5$.

Weaker conjecture due to Looijenga,

Conjecture (Looijenga)

$$\operatorname{asn} M_g \leq g-2 \qquad \textit{when} \quad g \geq 2$$

He proves the conjecture for $g \leq 5$. From this conjecture it follows that

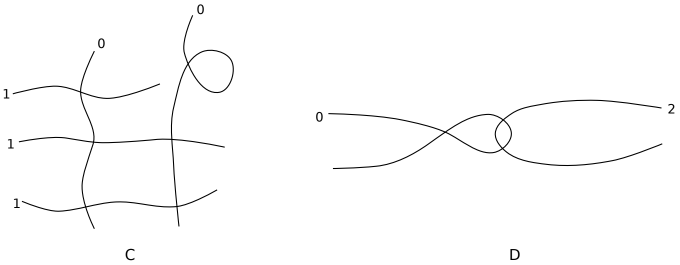
$$\operatorname{asn} M_{g,n} \leq g-1 \quad \text{for} \quad n \geq 1$$

Recently Looijenga showed that: $\operatorname{ce}(M_g) \leq g-2$ and $\operatorname{ce}(M_{g,n}) \leq g-1$ for all $g \geq 2$.
The bounds are sharp due to a result by Harer.

A Filtration

There is a filtration on $\overline{M}_{g,n}$ given by the number of rational components of a curve.

Let $\overline{M}_{g,n}^{\leqslant k}$ be the open set in $\overline{M}_{g,n}$ parametrizing stable curves with at most k rational components (components of geometric genus 0).



Curve C has 2 rational components where as D has just 1. We have

$$\overline{M}_{g,n}^{\leqslant 0} \subset \overline{M}_{g,n}^{\leqslant 1} \subset \ldots \subset \overline{M}_{g,n}^{\leqslant 2g-2+n} = \overline{M}_{g,n}$$

There is a generalization of Looijenga's conjecture by Roth and Vakil.

Conjecture (Roth and Vakil)

$$\operatorname{asn} \overline{M}_{g,n}^{\leqslant k} \leq g - 1 + k \quad \textit{for} \quad g > 0, k \geq 0$$

This conjecture is still open, but recently Looijenga showed the same bound for cohomological excess.

Theorem

$\operatorname{ce}(\overline{M}_{g,n}^{\leqslant k}) \leq g - 1 + k.$

Sharpness of these bounds are not known.

I study analogous questions for the hyperelliptic locus.

Hyperelliptic Locus

Let $H_g \subset M_g$ be the hyperelliptic locus, that is the subspace parametrizing isomorphism classes of hyperelliptic curves. A hyper-elliptic curve of genus g is a double cover of $\mathbb{P}^1$, ramified over $2g+2$ points. Let $\overline{H}_g$ be the closure of H_g in $\overline{M}_g$.

There is the filtration on $\overline{H}_g$, induced by the filtration on $\overline{M}_g$.

$$\overline{H}_g^{\leqslant k} = \overline{H}_g \cap \overline{M}_g^{\leqslant k}$$

We are interested in the affine stratification number and cohomological excess of the varieties $\overline{H}_g^{\leqslant k}$.

Results

We show Looijenga's bounds hold for these varieties.

Theorem (C)

$$\operatorname{asn} \overline{H}_g^{\leqslant k} \leq g - 1 + k \quad \textit{for all } g, k$$

This provides evidence towards the actual conjecture of Roth and Vakil. The effectiveness of the bound is not known, but when $k = 0$, we show

Theorem (C)

$$\operatorname{ce}(\overline{H}_g^{\leqslant 0}) \geq g - 1 \quad \textit{for} \quad g \geq 2$$

This proves

1. $\operatorname{asn} \overline{H}_g^{\leqslant 0} = \operatorname{ce}(\overline{H}_g^{\leqslant 0}) = g - 1.$
2. $\operatorname{ce}(\overline{M}_g^{\leqslant 0}) = g - 1.$

The second result follows since $\overline{H}_g^{\leqslant 0}$ is a closed subvariety of $\overline{M}_g^{\leqslant 0}$, and due to Looijenga's upper bound.

Reduction to question on $\overline{M}_{0,n}$

The Hurwitz space, $\mathcal{H}ur_{g,d}$, parametrizes genus g, d-sheeted simply branched covers of $\mathbb{P}^1$ (with a numbering of branch points). By Riemann-Hurwitz, the number of branch points is $2g + 2d - 2$. The Hurwitz space can be constructed as a finite étale cover

$$q : \mathcal{H}ur_{g,d} \to M_{0,2g+2d-2}$$

There is also a natural map

$$\pi : \mathcal{H}ur_{g,d} \to M_g$$

In particular $\mathcal{H}ur_{g,2} \cong M_{0,2g+2}$ and there's a finite map $\mathcal{H}ur_{g,2} \to H_g$. So we get a map $\pi : M_{0,2g+2} \to H_g \subset M_g$ that extends to the compactifications.

$$\pi : \overline{M}_{0,2g+2} \to \overline{H}_g$$

Admissible covers

Harris and Mumford introduced a compactification of the Hurwitz space using admissible covers. So that now there is a finite map

$$q : \overline{\mathcal{H}ur}_{g,d} \to \overline{M}_{0,2g+2d-2}$$

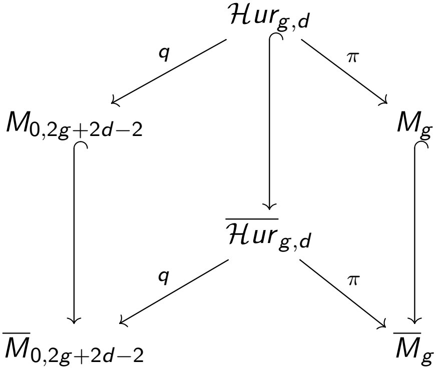
An **admissible cover** of genus g is a semi-stable curve C, that is a degree d cover of a stable genus 0 curve D with $2g + 2d - 2$ marked points. So that

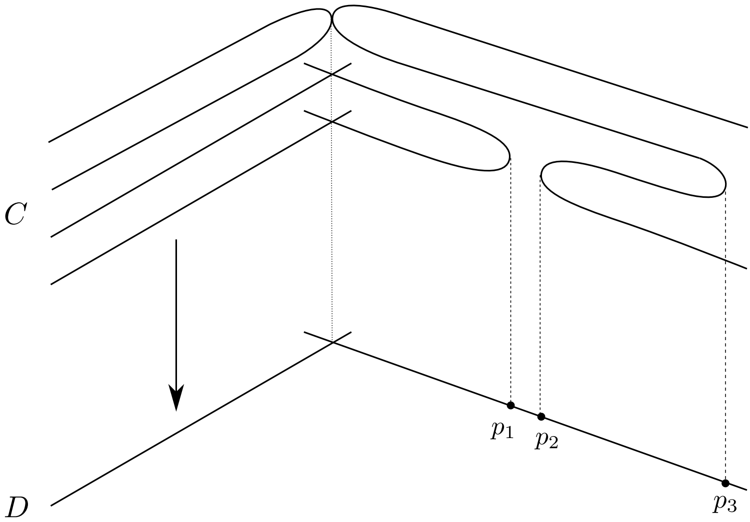
- The pre-image of the smooth locus of D is the smooth locus of C, pre-image of nodes of D are nodes in C.
- There is simple branching over the marked points.
- At nodes the branching can be more complicated but the ramification along the two components meeting at the node have to be the same.

The map π extends to the compactification, but stabilization involved.

$$\pi : \overline{\mathcal{H}ur}_{g,d} \to \overline{M}_g$$

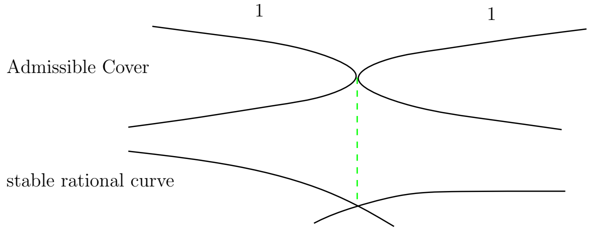
We have

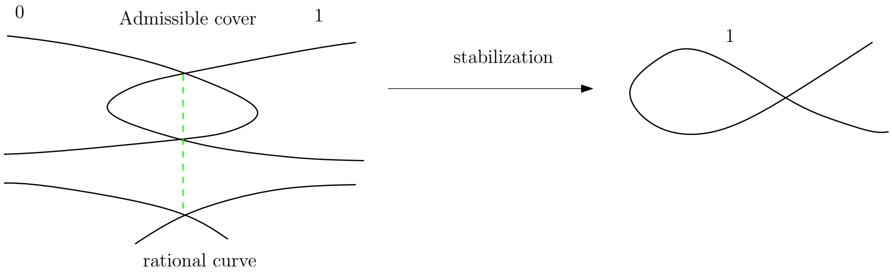

$\mathcal{H}ur_{g,d}$
q
π
$M_{0,2g+2d-2}$
M_g
$\overline{\mathcal{H}ur}_{g,d}$
q
π
$\overline{M}_{0,2g+2d-2}$
$\overline{M}_g$



C
D
p_1
p_2
p_3

Examples


1
1
Admissible Cover
stable rational curve


0
Admissible cover
1
stabilization
1
rational curve

Isomorphism

In our case we have,

$$\pi : \overline{\mathcal{H}ur}_{g,2} \cong \overline{M}_{0,2g+2} \to \overline{H}_g$$

Here π is the quotient map under the action of the symmetric group S_{2g+2} which acts by permuting the fixed points. Hence we have

$$\overline{H}_g \cong \overline{M}_{0,2g+2}/S_{2g+2}$$

Let

$$\overline{M}^{(k)}_{0,2g+2} = \pi^{-1}\overline{H}^{\leqslant k}_g$$

An affine stratification of $\overline{M}^{(k)}_{0,2g+2}$ gives an affine stratification of $\overline{H}^{\leqslant k}_g$ of the same length.

Dual Graphs of Stable curves

The dual graph of a marked stable graph is obtained by placing a vertex for each component, an edge for each node and a leg for each marked point. Further the vertices are labelled by the geometric genus of the component it represents.

We denote the set of isomorphism classes of dual graphs of genus g, with n marked points by $\Gamma(g,n)$.

The genus of the curve can be obtained by the formula

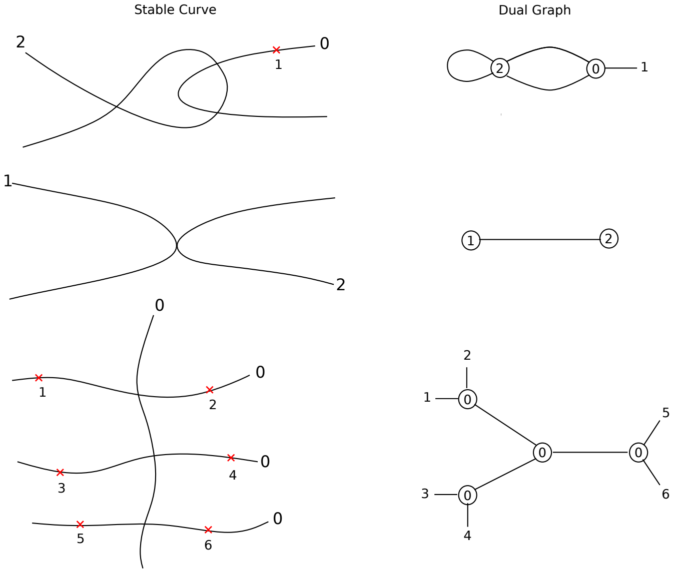
$$g = \sum_{v \in \text{ vertices of } G} g_v + b_1(G)$$

For a graph G, denote by M_G, the locus of curves whose dual graph is G, and by $\overline{M}_G$ its closure. Then

$$\overline{M}_{g,n} = \bigcup_{G \in \Gamma(g,n)} M_G$$

$\overline{M}_G$ is a subvariety of codimension equal to the number of edges of G.

Examples of Dual Graphs



Coloring of graphs

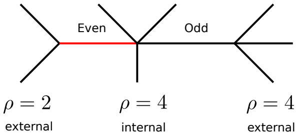
We define **parity** of graph T of type $(0, 2k)$.

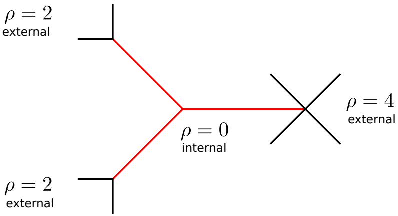
- Legs of T are odd.
- Edges are odd if deleting it produces two trees each with odd number of marked points otherwise even.

Let the ramification number of a vertex v of T be the number of legs and odd edges of the vertex. We denote it by $\rho(v)$.

We call a vertex of a graph *internal* if it has more than one edge.

If a graph is not-stable we can stabilize it by suitably deleting the unstable vertexes.





Going between strata

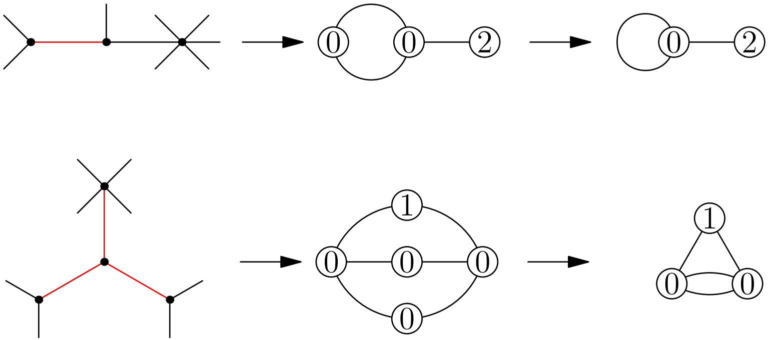
We have a stratification of the Moduli of curves by dual graphs. To go between the strata of $\overline{M}_{0,2g+2}$ and that of $\overline{H}_g$, we describe the following algorithm.

Algorithm

Given a tree T corresponding to a curve C in $\overline{M}_{0,2g+2}$, the dual graph of $\pi([C]) \in \overline{M}_g$ is the stabilization of the graph G defined as follows.

- *There are two edges in G for each even edge of T, and one edge in G for each odd edge of T. (The leaves of T do not contribute flags to G.)*
- *A vertex v of T contributes a single vertex to G, of genus $(\rho(v) - 2)/2$, unless $\rho(v) = 0$, in which case it contributes two vertices of genus 0.*

Illustration



Here the even edges are drawn in red, where as the legs and the odd edges are drawn in black.

Strata in $\overline{M}^{(k)}_{0,2g+2}$

To determine which strata are contained in $\overline{M}^{(k)}_{0,2g+2}$, we have the following lemma, which follows easily from the Algorithm we described.

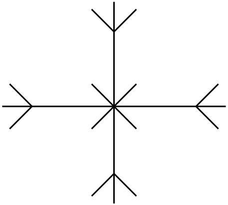
Lemma

Let C be a curve in $\overline{M}_{0,2g+2}$ with corresponding dual graph G. Then the image $\pi([C]) \in \overline{H}_g$ has a rational component if and only if G has an internal vertex v with $\rho(v) \leq 2$.
Furthermore the number of rational components of $\pi([C])$ is given by

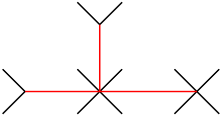
$$2 \times \#\{v \in V(G) \mid \rho(v) = 0\} + \#\{v \in V(G) \mid v \text{ internal and } \rho(v) = 2\}$$

Hence a curve in $\overline{M}_{0,2g+2}$ belongs to $\overline{M}^{(0)}_{0,2g+2}$ if its dual graph has only internal vertexes of ramification 4 or more. We call such graphs **good** graphs.

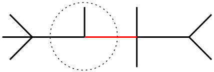
Examples of good trees



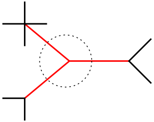
S (good)



T (good)



U (not good)



V (not good)

An affine stratification

The dual graph stratification of $\overline{M}^{(k)}_{0,2g+2}$ gives an affine stratification, since the strata corresponding to dual graphs, in genus 0 are affine.

This stratification is of length $g - 1 + k$, which follows from

Lemma

For a curve C in $\overline{M}_{0,2g+2}$, if $[C] \in \overline{M}^{(k)}_{0,2g+2}$ then C has at most $g + k - 1$ nodes or equivalently the dual graph has at most $g - 1 + k$ edges.

This affine stratification gives an affine stratification on $\overline{H}_g^{\leqslant k}$, through the map π, showing that:

$$\operatorname{asn} \overline{M}^{(k)}_{0,2g+2} = \operatorname{asn} \overline{H}_g^{\leqslant k} \leq g - 1 + k$$

Sharpness for $k = 0$

Consider the constant sheaf $\underline{\mathbb{C}}$ on $\overline{M}^{(0)}_{0,2g+2}$, and let $\mathcal{L} = \pi_* \underline{\mathbb{C}}$. Then $\mathcal{L}$ is a constructible sheaf on $\overline{H}^{\leqslant 0}_g$. Note that

$$H^i(\overline{H}^{\leqslant 0}_g, \mathcal{L}) \cong H^i(\overline{M}^{(0)}_{0,2g+2}, \underline{\mathbb{C}})$$

Since $\dim \overline{H}^{\leqslant 0}_g = 2g - 1$ if we can show that

$$H^{3g-2}(\overline{H}^{\leqslant 0}_g, \mathcal{L}) \cong H^{3g-2}(\overline{M}^{(0)}_{0,2g+2}, \underline{\mathbb{C}}) \neq 0$$

that would imply $\mathrm{ce}\left(\overline{H}^{\leqslant 0}_g\right)$ is at least $g - 1$.

Lemma (C)

The cohomology group $H^{3g-2}(\overline{M}^{(0)}_{0,2g+2}, \underline{\mathbb{C}})$ *is non-zero and has a pure Hodge structure of weight* $2(2g-1)$ *and* $H^k(\overline{M}^{(0)}_{0,2g+2}) = 0$ *for* $k > 3g - 2$.

A Spectral Sequence

Let $\Gamma_p(0, n)$ be the set of dual graphs of stable curves of genus 0, with n marked points and p nodes. Then we have the spectral sequence.

$$_nE_1^{p,q} = \bigoplus_{[T]\in\Gamma_{-p}(0,n)} H_c^{p+q}(M_T)$$

This spectral sequence converges to $H^{p+q}(\overline{M}_{0,n})$. In fact the spectral sequence is in the category of mixed Hodge structures, and by a purity argument for the cohomology of $\overline{M}_{0,n}$, it can be seen that

$$H^{2i}(\overline{M}_{0,n}) \cong {}_nE_2^{-(n-3-i),n-3+i}$$

and ${}_nE_2^{p,q} = 0$ if $p + q \neq 2(n-3)$.

$\overline{M}_{0,8}$ example (${}_8E_1^{\bullet,\bullet}$)

χ	$p=-5$	$p=-4$	$p=-3$	$p=-2$	$p=-1$	$p=0$	
1	0	0	0	0	0	$\mathbb{C}(5)^1$	$q=10$
99	0	0	0	0	$\mathbb{C}(4)^{119}$	$\mathbb{C}(4)^{20}$	$q=9$
715	0	0	0	$\mathbb{C}(3)^{1918}$	$\mathbb{C}(3)^{1358}$	$\mathbb{C}(3)^{155}$	$q=8$
715	0	0	$\mathbb{C}(2)^{9450}$	$\mathbb{C}(2)^{13902}$	$\mathbb{C}(2)^{5747}$	$\mathbb{C}(2)^{580}$	$q=7$
99	0	$\mathbb{C}(1)^{17325}$	$\mathbb{C}(1)^{40950}$	$\mathbb{C}(1)^{33348}$	$\mathbb{C}(1)^{10668}$	$\mathbb{C}(1)^{1044}$	$q=6$
1	$\mathbb{C}(0)^{10395}$	$\mathbb{C}(0)^{34650}$	$\mathbb{C}(0)^{44100}$	$\mathbb{C}(0)^{26432}$	$\mathbb{C}(0)^{7308}$	$\mathbb{C}(0)^{720}$	$q=5$

Truncated Spectral Sequence

For a genus 0 graph G with $2g+2$ marked points, either $M_G \subset \overline{M}^{(0)}_{0,2g+2}$, if G is **good** or intersection is empty.

We have a spectral sequence in compactly supported cohomology.

$$ {}_gF_1^{p,q} = \bigoplus_{\substack{G \in \Gamma_{-p}(0,2g+2) \\ G \text{ good}}} H_c^{p+q}(M_G) $$

This spectral sequence converges to $H_c^{p+q}(\overline{M}^{(0)}_{0,2g+2})$.The spectral sequence has

- a natural mixed Hodge structure
- an action of S_{2g+2}

We use this spectral sequence to show $H_c^g(\overline{M}^{(0)}_{0,2g+2})$ is non-trivial and use Poincaré duality.

$$ H_c^g(\overline{M}^{(0)}_{0,2g+2}) \cong H^{3g-2}(\overline{M}^{(0)}_{0,2g+2})^\vee $$

The spectral sequence

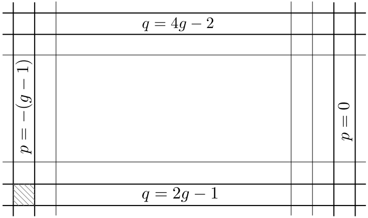


Figure: Support of ${}_gF_1^{\bullet,\bullet}$

	$p=-1$	$p=0$
$q=6$	0	s_6
$q=5$	$2s_6+2s_{4,2}+s_{5,1}$	$s_{4,2}$
$q=4$	$s_{3,3}+3s_{4,2}+2s_{5,1}+s_{2,2,2}+3s_{3,2,1}+2s_{4,1,1}$	$s_{4,1,1}+s_{3,2,1}$
$q=3$	$s_{5,1}+2s_{3,3}+s_{4,2}+3s_{3,2,1}+s_{4,1,1}+2s_{2,2,1,1}+s_{3,1,1,1}$	$s_{3,3}+s_{4,1,1}+s_{2,2,1,1}$

Table: Genus 2, ${}_2F_1^{\bullet,\bullet}$

First Proof

From the Spectral Sequence it is clear that

$$H_c^g(\overline{M}_{0,2g+2}^{(0)}) \cong {}_gF_\infty^{-g+1,2g-1} \cong {}_gF_2^{-g+1,2g-1}$$

In the first proof which is computational we analyze the S_{2g+2} action ${}_gF_1^{p,q}$.

Ezra Getzler calculated the S_n equivariant cohomology of $M_{0,n}$ an $\overline{M}_{0,n}$.

Using those techniques we can decompose the vector spaces into irreducible representations of S_{2g+2}.

The differential is S_{2g+2} equivariant, and counting dimensions of the isotypic components for the standard representation, of ${}_gF_1^{-g+1,2g-1}$ and ${}_gF_1^{-g+2,2g-1}$ we infer that ${}_gF_2^{-g+1,2g-1} \neq 0$. The computations work only up to genus 5.

Tables

	$p=-2$	$p=-1$	$p=0$
$q=5$	4	1	0

Table: ${}_3F_1^{p,q}$, multiplicities of $s_{7,1}$

	$p=-3$	$p=-2$	$p=-1$	$p=0$
$q=7$	11	6	1	0

Table: ${}_4F_1^{p,q}$, multiplicities of $s_{9,1}$

	$p=-4$	$p=-3$	$p=-2$	$p=-1$	$p=0$
$q=9$	37	36	10	1	0

Table: ${}_5F_1^{p,q}$, multiplicities of $s_{11,1}$

A bit of Operads

Recall that $H_\bullet(M_{0,n})$ suitably suspended forms a cyclic operad called the **Gravity** operad (Grav).

The **Hyper-commutative** (Hycom) operad is given by

$$\mathrm{Hycom}(n) = H_\bullet(\overline{M}_{0,n})$$

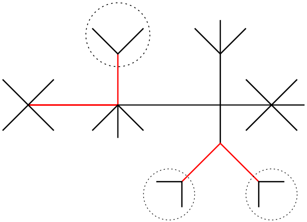
Due to Getzler, we know that Grav and Hycom are Koszul-dual to each other. He shows that Grav is homotopic to the cobar operad of Hycom.

We use the spectral sequence ${}_nE^{\bullet,\bullet}$ to give another proof of the Koszul duality. We show that Hycom is homotopic to the cobar operad of Grav.

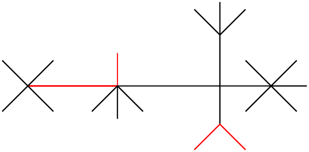
$$\mathrm{Hycom} \simeq \mathbf{B}\mathrm{Grav}$$

Almost a Colored Operad

There is a natural coloring given by the parity of legs and edges of dual graphs of type (0,2k). For a **good** tree T, we define the stripped tree, by stripping off, all the external vertexes which have ramification 2.



Good Tree



Stripped Tree

It is clear that these stripped trees form a set of colored trees, so there is a colored operad lurking around which we make use of in our computations.

An $\mathbb{S}$-module is a sequence of vector spaces with S_n action. Given an $\mathbb{S}$-module we can get a free colored operad by summing over trees.

The Poincaré characteristic of an $\mathbb{S}$-module is a symmetric function which records the symmetric group action.

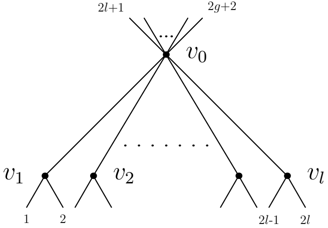
Getzler and Kapranov give a method to obtain the characteristic of the free operad generated from an $\mathbb{S}$-module $\mathcal{V}$, from the characteristic of $\mathcal{V}$, using what they call the Legendre transform of a symmetric function.

If we add up the rows of the spectral sequence ${}_gF_1^{\bullet,\bullet}$, we get what is almost a free colored $\mathbb{S}$-module obtained from a colored $\mathbb{S}$-module closely related to the operad Grav.

We generalize the formula of Getzler and Kapranov to colored operads and use it to study the vector spaces ${}_gF_1^{p,q}$ as S_{2g+2} modules.

Second Proof

This proof came out of carefully analyzing the results of the computations in the first proof and works for all genus. Consider the following trees $T_{l,g}$.



Clearly

$$\mathrm{Aut}(T_{l,g}) \cong S_{2g-2l+1} \times (S_l \wr S_2) \subset S_{2g+1}.$$

Let

$$W_{l,g} = H_c^{2g-1-l}(M_{T_{l,g}}) \qquad \text{for } l = 0, \ldots, g+1.$$

and

$$V_{l,g} = (W_{l,g})^{\mathrm{Aut}(T_{l,g})}$$

Second proof cont.

We have the following commutative diagram,

$$
\begin{array}{ccccc}
{}_{2g+2}E_1^{-g,2g-1} & \xrightarrow{d_1} & {}_{2g+2}E_1^{-g+1,2g-1} & \xrightarrow{d_1} & {}_{2g+2}E_1^{-g+2,2g-1} \\
\uparrow & & \uparrow & & \uparrow \\
V_{g,g} & \xrightarrow{d_1} & V_{g-1,g} & \xrightarrow{d_1} & V_{g-2,g} \\
 & & \downarrow & & \downarrow \\
 & & {}_{g}F_1^{-g+1,2g-1} & \xrightarrow{d_1} & {}_{g}F_1^{-g+2,2g-1}
\end{array}
$$

The proof then concludes by showing that $\dim V_{g,g} = 1$ and $d_1 : V_{g,g} \to V_{g-1,g}$ is non-zero, hence the kernel of $d_1 : V_{g-1,g} \to V_{g-2,g}$ is non-trivial.

Here the main ingredient in the proof is the fact that the top graded component of the gravity operad is the suspension of the Lie operad.

$$\mathrm{Grav}^{\mathsf{top}} \cong \Lambda\mathrm{Lie}$$

hence it can be deduced that

$$W_{g,l} \cong \Lambda\mathrm{Lie}(T_{g,l})$$

also the differential d_1 is the differential in the cobar operad of $\Lambda\mathrm{Lie}$.

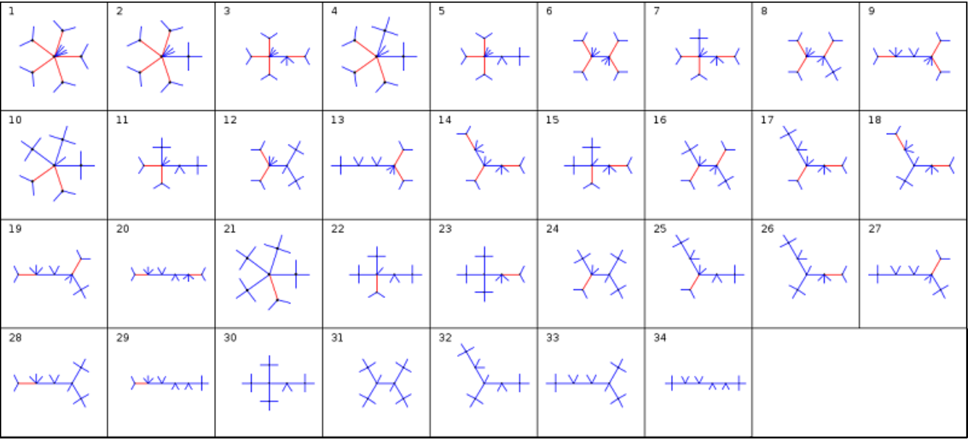
To explicitly calculate the differential $d_1 : V_{g,g} \to V_{g,g-1}$ we use a basis of the free Lie algebra called the Lyndon basis.

To know more about all of this please read my thesis.

Further Research

- Investigate similar questions about trigonal or tetragonal locus, and other Brill-Noether loci.
- See whether there are similar operadic interpretation of the (co)homology as in the hyper-elliptic case.

Thank You !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