A short introduction to some mathematical contributions of Maryam Mirzakhani

May 15, 2021

Short Bio

Mirzakhani was born on 12th May 1977, in Tehran, Iran.

In the 1995 International Mathematical Olympiad, she became the first Iranian student to achieve a perfect score and to win two gold medals.

She obtained her BSc in mathematics in 1999 from the Sharif University of Technology, Tehran.

She did her graduate studies at Harvard University, under the supervision of Fields medallist Curtis T. McMullen and obtained her PhD in 2004.

Short Bio

Her PhD thesis was published in 3 parts in 3 top journals of Mathematics, Annals of Mathematics, Inventiones Mathematicae and Journal of the American Mathematical Society.

Mirzakhani was awarded the Fields Medal in 2014 for "her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces".

She died of breast cancer on 14 July 2017 at the age of 40.

Mathematical contributions of Maryam Mirzakhani

Mirzakhani was mainly interested in Hyperbolic surfaces their families. The central object that appears throughout her work spread over some 20 papers is the Moduli of hyperbolic surfaces with a fixed genus g and n punctures denoted by $\mathcal{M}_{g,n}$. Her work can be very broadly divided into three parts.

- Volume calculation of $\mathcal{M}_{g,n}$ and related spaces, leading the an asymptotic count of simple closed geodesics on an individual Riemann surface as well as a new proof of Witten conjecture.
- Teichmüller dynamics on $\mathcal{M}_{g,n}$: Here she proved a long-standing conjecture of William Thurston showing Thurston's earthquake flow on $\mathcal{M}_{g,n}$ is ergodic.
- Together with Alex Eskin and Amir Mohammadi, Mirzakhani was able to show that complex geodesics in $\mathcal{M}_{g,n}$ are algebraic subvarieties.

In this talk I shall restrict to the first part.

Hyperbolic plane

The hyperbolic plane $\mathbb{D}$ is the open unit disk in $\mathbb{C}$

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

endowed with the metric

$$\mathrm{d}_{\mathrm{hyp}} = \frac{dx^2 + dy^2}{(1 - |z|^2)^2}.$$

This is a metric of constant curvature -1. To see this one can embed a small part of hyperbolic plane isometrically in $\mathbb{R}^3$ and show that the Gaussian curvature is -1.
The unit sphere in $\mathbb{R}^3$ has constant curvature $+1$.

Hyperbolic distance

Recall in Euclidean plane if $\gamma : [0, 1] \to \mathbb{R}^2$ is a curve, and $\gamma(t) = (x(t), y(t))$ then its length is

$$\ell(\gamma) = \int_0^1 |\gamma'(t)| dt = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

In the hyperbolic plane we measure lengths of curves differently. If

$$\gamma : [0, 1] \to D, \quad \gamma(t) = (x(t), y(t))$$

then length of γ is

$$\ell(\gamma) = \int_0^1 \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} dt = \int_0^1 \frac{2\sqrt{(x'(t))^2 + (y'(t))^2}}{1 - x^2(t) - y^2(t)} dt.$$

Geodesics

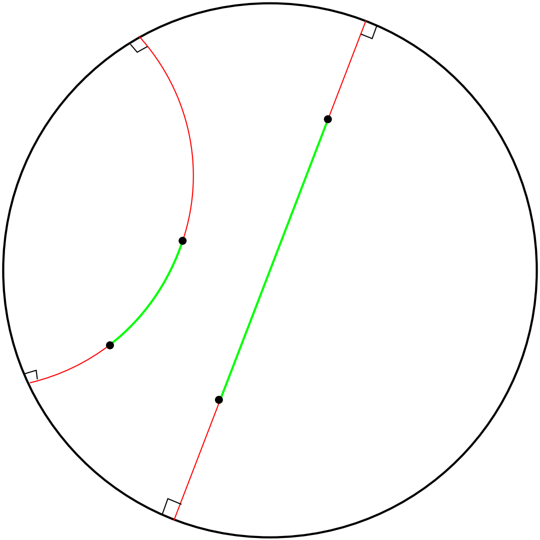
It turns out with this length measure also called hyperbolic metric, the shortest curve between any two points is the unique circle passing through those points and meeting the boundary at right angles.

These curves of minimal length are called geodesics.

The distance between any two points in the Hyperbolic plane is the length of the shortest curve joining the two points. Hence the length of the unique geodesic between those points.

In general a geodesic on any surface is a curve which can not be perturbed to get a shorter curve

Geodesics



Distance

Geodesic between the origin $O = (0,0)$ and the point $A = (a, 0)$ in D is the straight line OA, parametrized by

$$\gamma : [0,1] \to D, \quad \gamma(t) = (0, at).$$

Hence we can calculate the distance $d(O, A)$ by

$$\begin{aligned} d(O,A) = \ell(\gamma) &= \int_0^1 \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} dt \\ &= \int_0^1 \frac{2|a|}{1 - a^2 t^2} dt = \ln \frac{1 + |a|}{1 - |a|}. \end{aligned}$$

Note that $d(O, A) \to \infty$ as $a \to 1$.

All distances can be calculated using this, since there are isometries of D that take any two points to the origin and a point on the x-axis.

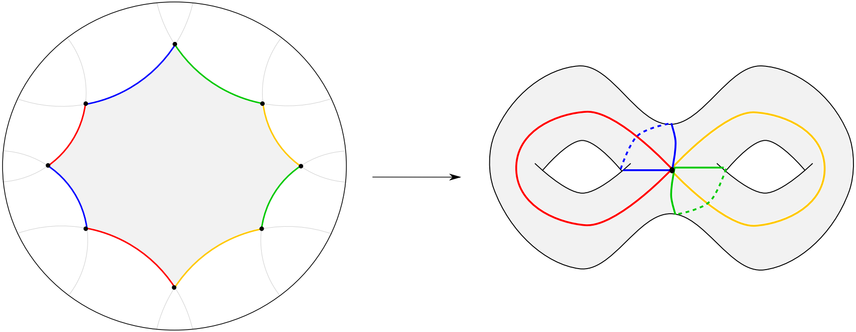
Hyperbolic geometry

This is the starting point of hyperbolic geometry. Some jargon:

- D is a metric space, since we know how to measure distances.
- Distances go off to infinity as we approach the boundary so this is a complete metric space.
- D is a Riemannian manifold of dimension 2, since it is an (open) subset of $\mathbb{R}^2$ and we can measure lengths of curves.
- The geometry of D is a type of non-euclidean geometry since it does not satisfy the parallel postulate of Euclid.

Hyperbolic surfaces

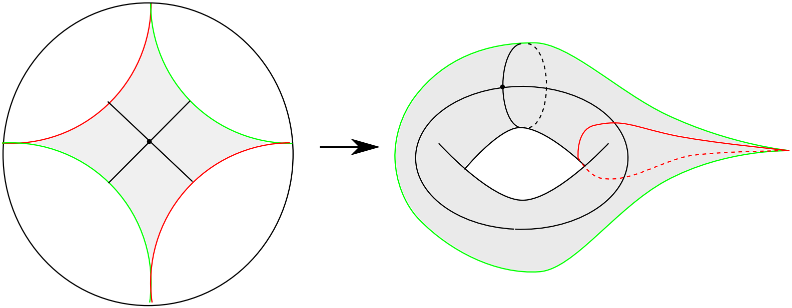
These are surfaces that can be built from geodesic polygons in the hyperbolic plane by identifying sides (quotient space). For example:



is a surface of genus 2.

Punctured torus

Here we have a Geodesic quadrilateral with vertices on the boundary of $\mathbb{D}$ and the quotient is a torus with a puncture.



This surface is has genus 1 and 1 puncture.

Hyperbolic surfaces

Since these surfaces are obtained from the hyperbolic plane, they naturally have a metric: length of a curve is the length of the corresponding curve in the polygon.

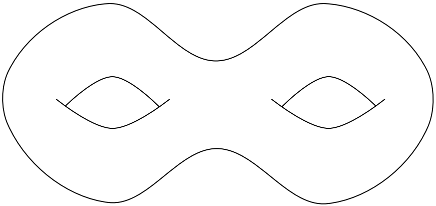
Geodesics are images of the geodesics in the polygon.

As a Riemannian manifold they have constant curvature -1. This makes the surface somewhat rigid.

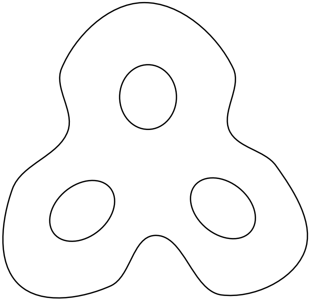
Note that a compact hyperbolic surface can not be embedded isometrically in $\mathbb{R}^3$. Any compact surface in $\mathbb{R}^3$ has a point of positive curvature.

Genus

The genus of a closed surface is just the number of holes it has. A surface is hyperbolic if it has genus at least 2.



Genus 2



Genus 3

Moduli Space

There is a nice enough topological space $\mathcal{M}_{g,n}$ parametrizing all possible hyperbolic surfaces of genus g with n punctures.

Points of $\mathcal{M}_{g,n}$ correspond to isometry classes of hyperbolic surfaces.

$\mathcal{M}_{g,n}$ is called the moduli space of genus g hyperbolic surfaces with n punctures.

This space is almost a manifold, but not quite. It is an orbifold of dimension $6g - 6 + 2n$, the quotient of a manifold by the action (not free) of a finite group.

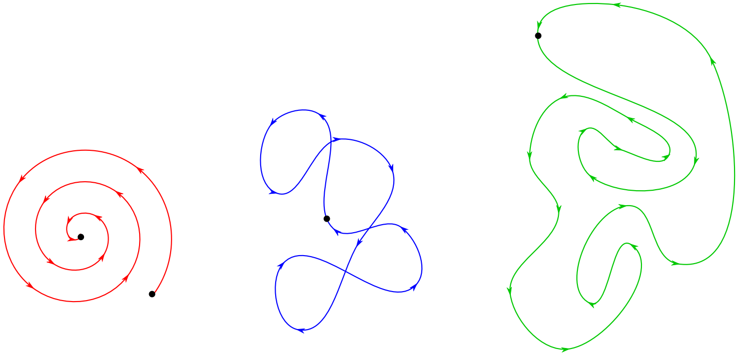
Simple closed geodesic

Let X be a closed hyperbolic surface.
A path $\gamma : [0, 1] \to X$ is a simple closed geodesic if:

- $\gamma([0, 1])$ is a geodesic, for s, t close by $\gamma([s, t])$ is the shortest path between $\gamma(s)$ and $\gamma(t)$.
- $\gamma(0) = \gamma(1)$,
- $\gamma(s) \neq \gamma(t)$ if $0 \leq s < t < 1$.

In words γ has the same starting and ending points, which is also a geodesic and which does not cross itself.

Curves



Red curve is not closed, blue curve is closed but not simple, green curve is simple and closed.

Number of geodesics of bounded length

Any closed curve can be slightly perturbed to get a closed geodesic.

In fact on hyperbolic surfaces, there is a unique geodesic in each free homotopy class of closed curves.

It was known that the number $c_X(L)$ of closed geodesics on X of length at most L has the asymptotic expression

$$c_X(L) \sim \frac{e^L}{L}.$$

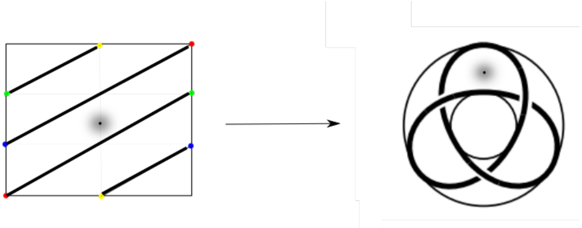
Not much was known about the number of simple closed geodesics in general.

Simple closed Geodesics

There are infinitely many simple closed geodesics on any hyperbolic surface X of genus > 0.

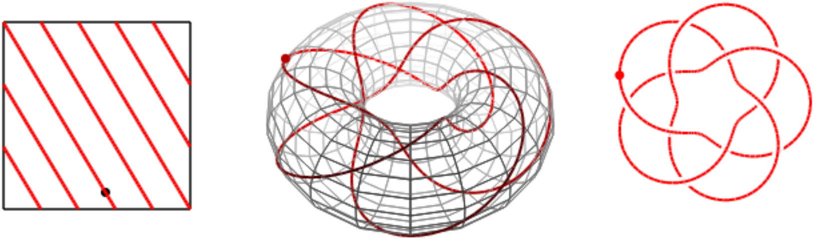
For example on the punctured torus any (m, n) torus knot is a simple closed curve and the unique geodesic in its free homotopy class is a simple closed geodesic.

The (m, n) torus knot is the image of a line in $\mathbb{R}^2$ with slope m/n for integers m and n passing through the origin, under the quotient map $\mathbb{R}^2 \to T$.



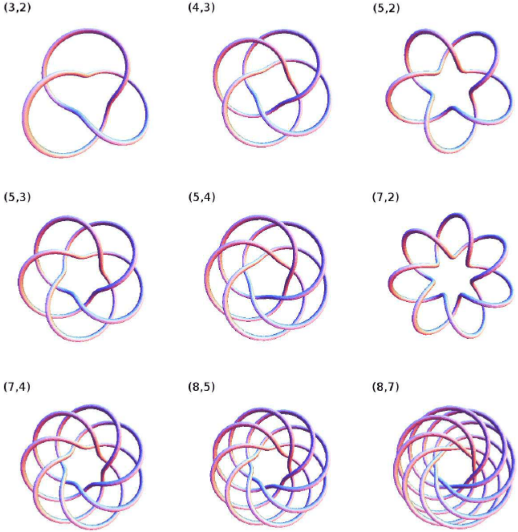
A (2,3) torus knot also called the Trefoil knot.

$(5, 3)$ Torus Knot



Renato Paes Leme, `https://observablehq.com/@renatoppl/torus-knots`

Torus knots



Manuel Arrayás and José L. Trueba, `https://arxiv.org/abs/1106.1122`

Counting simple closed Geodesics

Let us now fix a closed hyperbolic surface of genus g with n punctures, $X \in \mathcal{M}_{g,n}$.

Let $s_X(L)$ be the number of simple closed geodesics in X whose length is at most L. Then Mirzakhani proves that asymptotically

$$s_x(L) \sim \eta(X) L^{6g-6}$$

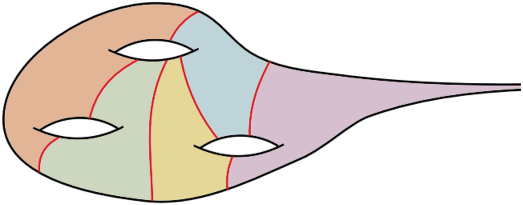
where $\eta(X)$ is a constant depending on the surface X.

Moreover $\eta : \mathcal{M}_{g,n} \to \mathbb{R}_+$ is a continuous function.

Pants decomposition

One of the main ingredients of the proof of the asymptotic formula for $s_X(L)$ is Mirzakhani's recursive formula for the volume of $\mathcal{M}_{g,n}$.

Given a surface of genus g with n punctures it can be cut along $3g-3+n$ simple closed geodesics, to get $2g-2+n$ pairs of pants:



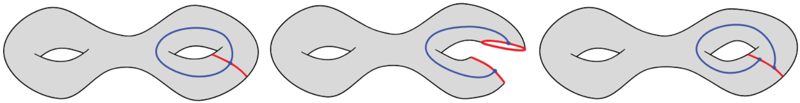
Similarly we can glue $2g-2+n$ pairs of pants along pairs of boundary geodesics to get a surface of type (g,n).

Fenchel-Nielsen coordinates

To glue any two boundary geodesics of two pairs of pants, there are two parameters involved

- the length of the boundary geodesics being glued,
- the twist parameter.

Different length and twist parameters give rise to different surfaces.



This rise to Fenchel-Nielson coordinates on (a cover of) $\mathcal{M}_{g,n}$, $(l_1, \ldots, l_{3g-3+n}, \tau_1, \ldots, \tau_{3g-3+n})$, $l_i > 0$ and $\tau_i \in [0, l_i)$.

Weil-Peterson volume

The volume of Mg, n can be measured in terms of the Fenchel-Nielson coordinates

$$\mathrm{Vol}(\mathcal{M}_{g,n}) = \int \cdots \int 1 \; dl_1 \ldots dl_{3g-3+n} d\tau_1 \ldots d\tau_{3g-3+n}.$$

Mirzakhani gave a recursive formula for these volumes.

Using this and a generalisation of McShane identity she proved her formula for $s_X(L)$.

In another direction she also proves the Witten conjecture by showing that the volume can also be expressed in terms of Chern classes of line bundles on $\mathcal{M}_{g,n}$.