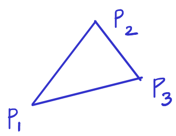
NISER Outreach talk
21 Jan 2022

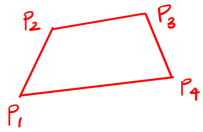
Platonic Solids

POLYGONS:

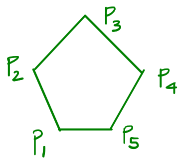
A polygon is a geometric object in the plane consisting of some points $P_1, \ldots, P_n$ called the vertices, and line segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n, P_nP_1$ called edges. No three vertices are lie on the same line.



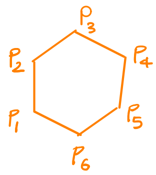
triangle
3-gon



quadrilateral
4-gon

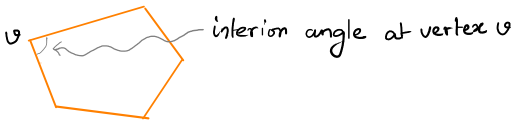


pentagon

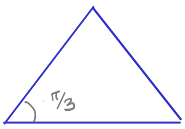


hexagon

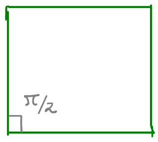
The angle between the two edges meeting at a vertex is called the interior angle at that vertex.



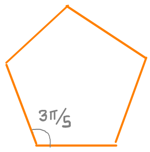
A polygon is called regular if all its edges have the same length. This also means all the interior angles are equal.



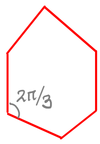
Equilateral △



Square

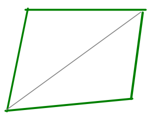


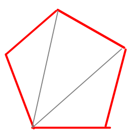
regular pentagon

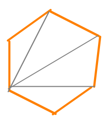


regular hexagon

The sum of all interior angles of an n-gon is $(n-2)\pi$. This can be seen by cutting the polygon into triangles

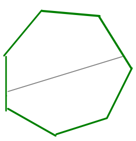




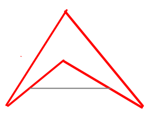


Hence for a regular n-gon any interior angle is $\left(\frac{n-2}{2}\right)\pi$.

A convex polygon is a polygon such that if we pick any two points on the polygon the line segment joining them stays inside the polygon.



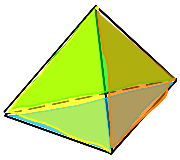
Convex ✓



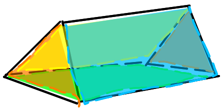
not convex ✗

POLYHEDRA:

A polyhedron is a 3-dimesional solid consisting of polygons joined at their sides.

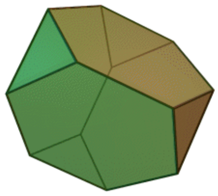


Tetrahedron



Prism

Truncated icosahedron



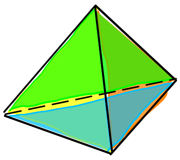
Truncated tetrahedron

The polygons making up the polyhedron are called its faces.

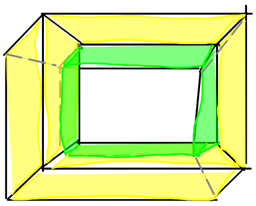
The edges are the edges of the polygons used to form the polyhedron.

The vertices are the vertices of the polygons.

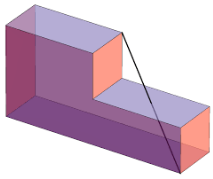
A convex polyhedron is a polyhedron such that for any two points on it, the line segment joining them is inside the polyhedron.



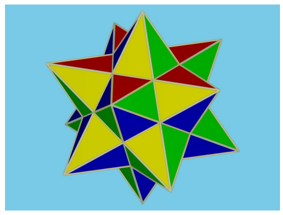
Convex ✓



not convex ✗

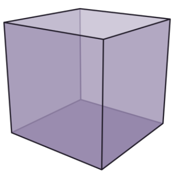


non-convex

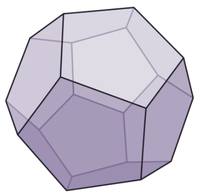


non convex

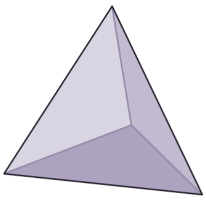
A regular polyhedron is a convex polyhedron all of whose are congruent regular polygons with the same number of faces at each vertex.



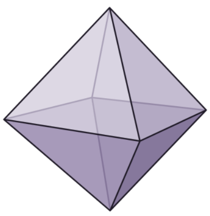
Cube (4,3)
6 faces



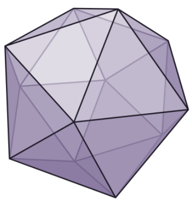
Dodecahedron (5,3)
12 faces



Tetrahedron (3,3)
4 faces



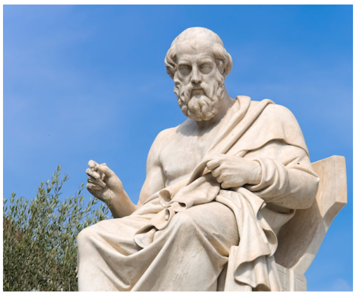
Octahedron (3,4)
8 faces



Icosahedron (3,5)
20 faces

PLATONIC SOLIDS:

The regular polyhedra are also called Platonic solids named after the Greek philosopher Plato who compared them to the five elements, fire, earth, air, water and the universe.



PLATO (424-348 BC)

There are exactly 5 regular polyhedra upto similarity (scaling and rotation), the tetrahedron, cube, octahedron, dodecahedron and the icosahedron.

The type of a regular polyhedron can be determined by a pair of natural numbers (p,q),

p – The number of edges of each face

q – The number of faces meeting at a vertex.

From this pair (p,q) we can determine the number of vertices N_0, number of edges N_1 and the number of faces N_2.

Let us count the pairs (v,e) where v is a vertex lying on the edge e. For each vertex, there are q edges that contain it, thus

$$\#\{(v,e) \mid v \in e\} = \underbrace{q + \dots + q}_{N_0 \text{ times}} = qN_0.$$

Also each edge contains 2 vertices, hence

$$\#\{(v,e) \mid e \ni v\} = \underbrace{2 + \dots + 2}_{N_1 \text{ times}} = 2N_1$$

Thus we get $qN_0 = 2N_1$. Similarly we can count the pairs (e,f) where e is an edge of the face f.

Each face has p edges, so

$$\#\{(e,f) \mid e \text{ is an edge of } f\} = \underbrace{p + \dots + p}_{N_2 \text{ times}} = pN_2.$$

Similarly each edge is shared by 2 faces so

$$\#\{(e,f) \mid e \text{ is an edge of } f\} = \underbrace{2+\dots+2}_{N_1 \text{ times}} = 2N_1 .$$

We thus get the relation

$$\boxed{qN_0 = 2N_1 = pN_2 .}$$

POSSIBLE VALUES OF (p,q) :

To determine the possible values for p and q we shall need an input from geometry.

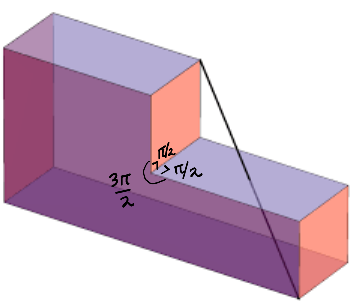
Theorem: For any convex polyhedron the sum of the interior angles of all the faces meeting at a vertex is less than 2π.

This can be intutively seen by cutting along an edges and placing all the faces on the plane.

$\theta < \pi \Rightarrow \theta + \pi/2 + \pi/2 < 2\pi$

An explicit proof of this fact is not very easy & involves ideas from differential geometry.

This fact is not true for non-convex polyhedra, for example



$\frac{3\pi}{2} + \pi/2 + \pi/2 = \frac{5\pi}{2} > 2\pi .$

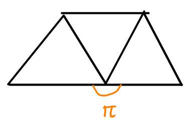
Now lets go case by case:

① $p=3$, that is all faces are equilateral triangles, then the interior angles are $\pi/3$, we see that

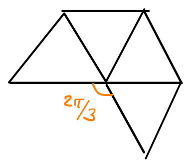
$$q\pi/3 < 2\pi \Rightarrow q<6$$

So possible values of q then are $3, 4, 5$ since $q>2$, atleast 3 faces meet at a vertex.

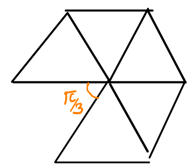
We have the following configurations



$p=3,\ q=3$



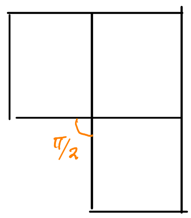
$p=3, q=4$



$p=3, q=5$

② $p=4$, all faces are squares, then interior angles are $\pi/2$ and we have $q\pi/2 < 2\pi \Rightarrow q<4$.

Thus only possible value of q is 3, and the configuration is

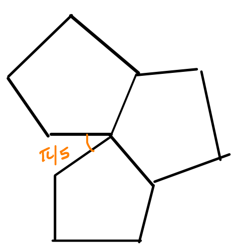


$p=4,\ q=3$

③ $p=5$, then all faces are regular pentagons and interior angles are $\frac{3\pi}{5}$. Thus

$$q \times \frac{3\pi}{5} < 2\pi \Rightarrow q < \frac{10}{3}$$

Thus only value for q is 3 and we have the configuration.



④ $p \geqslant 6$, then, faces are regular p-gons with interior angles $\frac{(p-2)\pi}{p}$. So we get

$$q \times \frac{(p-2)\pi}{p} < 2\pi \Rightarrow q < \frac{2p}{p-2} < \frac{12}{6-2} = 3$$

Thus there are no possibilities for q.

Hence the possible pairs are only (3,3), (3,4), (3,5), (4,3), (5,3).

The pair (p,q) completely determines the polyhedron. More over we see that there is some symmetry among the pairs by switching (p,q) we get another permissible pair.

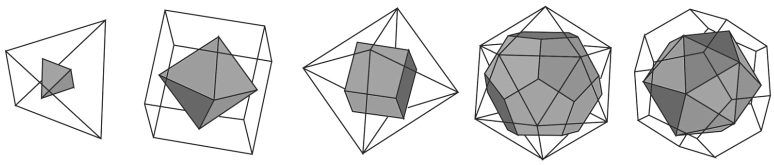
$$(3,3) \longleftrightarrow (3,3)$$

$$(3,4) \longleftrightarrow (4,3)$$

$$(3,5) \longleftrightarrow (5,3)$$

This is demonstrated by duality between the regular polyhedra. The dual of a polyhedron is obtained as follows:

* Start with a polyhedron A
* Take the centers of the faces of A to be the vertices of the dual A^*,
* Take the edges of A^* as the edges joining the centers and perpendicular to edges of A.
* Take the faces of A^* to be the faces bounded by the edges of A^*.



Examples of Platonic Dual-Pairing

Now let us determine the number of faces N_2, edges N_1 and vertices N_0 of a regular polyhedron.

We already know

$$qN_0 = 2N_1 = pN_2$$

So $\quad N_0 = \dfrac{2N_1}{p} \quad$ and $\quad N_2 = \dfrac{2N_1}{q}$.

We shall need an input from topology.

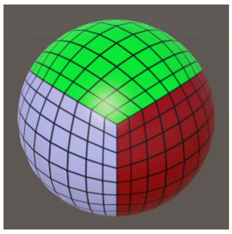
<u>Theorem:</u> For any convex polyhedron $\quad N_0 - N_1 + N_2 = 2$.

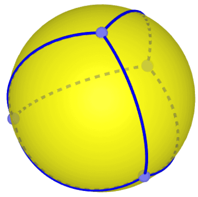
This theorem is due to the Swiss mathematician Leonhard Euler and is related to something called the Euler Characteristic.

The Euler Characteristic of the sphere is 2. Any convex polyhedron can be drawn on the sphere. From that we can say

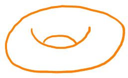
$N_0 - N_1 + N_2$ for a polyhedron inscibed on a sphere

$= $ Euler characteristic of th sphere $= 2$.

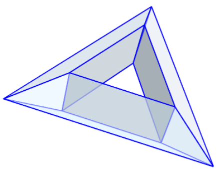




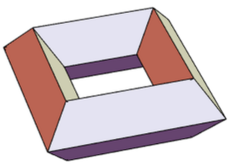
This is not true in general. For example Euler characteristic of a torus is 0

Euler Characteristic () = 0

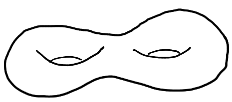
Hence for a polyhedron on a torus $N_0 - N_1 + N_2 = 0$.

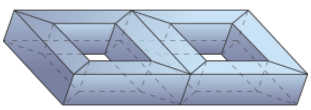


$N_0 - N_1 + N_2 = 9 - 18 + 9 = 0$



$N_0 - N_1 + N_2 = 16 - 32 + 16 = 0$

Similarly Euler characteristic of the double torus  is -2, so



$N_0 - N_1 + N_2 = 28 - 60 + 30 = -2$.

Now we have two relations among N_0, N_1 & N_2 for regular polyhedra

$$N_0 - N_1 + N_2 = 2 \qquad — ①$$

$$q N_0 = 2N_1 = p N_2 \qquad — ②$$

From these we get

$$\frac{2N_1}{q} - N_1 + \frac{2N_1}{p} = 2$$

After some manipulation this gives us

$$N_1 \left(\frac{2p+2q-pq}{pq}\right) = 2$$

$$\Rightarrow \qquad N_1 = \frac{2pq}{4-(p-2)(q-2)}$$

Thus $\quad N_0 = \dfrac{4p}{4-(p-2)(q-2)}$, $\quad N_1 = \dfrac{2pq}{4-(p-2)(q-2)}$, $\quad N_2 = \dfrac{4q}{4-(p-2)(q-2)}$.

Thus we can now list all the regular polyhedra

	Type	Faces	Edges	Vertices
Tetrahedron	(3, 3)	4	6	4
Cube	(4, 3)	6	12	8
Octahedron	(3, 4)	8	12	6
Dodecahedron	(5, 3)	12	30	20
Icosahedron	(3, 5)	20	30	12

DESCARTE'S THEOREM:

At any vertex v of a polyhedron, let the angular defect be

$$\delta_v = 2\pi - \sum \theta_i$$

Sum of interior angles of faces meeting at v.

Thus

$$\sum \delta_v = 2\pi N_0 - \sum_v \sum_i \theta_i$$

Sum over interior angles of all faces.

$$\sum_f (E_f - 2)\pi$$ where E_f is the number of edges of the face f.

$$\sum_f (E_f - 2)\pi = \pi \sum_f E_f - 2\pi N_2 = 2\pi N_1 - 2\pi N_2$$

Thus

$$\sum_v \delta_v = 2\pi (N_0 - N_1 + N_2).$$

For a convex polyhedron $N_0 - N_1 + N_2 = 2$ hence we get Descarte's Theorem.

Theorem: For a convex polyhedron $\sum_v \delta_v = 4\pi$.

If the polyhedron is regular all vertices have the same angular defect δ. So

$$\sum_v \delta = N_0 \delta = 4\pi \Rightarrow \delta = 4\pi/N_0 > 0.$$

——— X ———