# Mirzakhani's Proof of the Witten Conjecture



Teichmuller space: $\mathcal{T}_g$, $\quad X$ any closed smooth genus $g$ surface

$\mathcal{M}_{hyp}(X)$ – space of hyperbolic metrics on $X$

$\mathrm{Diff}^+(X)$ acts on $\mathcal{M}_{hyp}(X)$ as follows, if

$f \in \mathrm{Diff}^+(X)$ and $g \in \mathcal{M}_{hyp}(X)$

$f^*g$ is also a hyperbolic metric on $X$.

$$\mathcal{T}_g = \mathcal{M}_{hyp}(X) / \mathrm{Diff}^0(X),$$

$\star$ $\quad \underbrace{\mathrm{Diff}^0(X)}_{\text{connected comp of Id.}} \subseteq \mathrm{Diff}^+(X)$

$$\mathcal{M}_g = \mathcal{M}_{hyp}(X) / \mathrm{Diff}^0(X).$$

$$\Gamma_g = \frac{\mathrm{Diff}^+(X)}{\mathrm{Diff}^0(X)}$$ – mapping class group of $X$.

$$\boxed{\mathcal{M}_g \cong \mathcal{T}_g / \Gamma_g}.$$

$\Gamma_g$ is a discrete group acting properly discontinuously on $\mathcal{T}_g$ with finite isotropy groups.

$\star$ gives a complex structure on $\cancel{X}$ $\mathcal{T}_g$, ~~that we~~ such that $\mathcal{T}_g$ can be identified with an open ball in $\mathbb{C}^{3g-3}$.

There is a diffeomorphism

$$\mathcal{T}_g \xrightarrow{(l,\tau)} \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \qquad (l_1,\dots,l_{3g-3},\tau_1,\dots,\tau_{3g-3})$$

as we saw in Divakaran's talk. These are the Fenchel-Nielsen coordinates.

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Similarly we have $\mathcal{T}_{g,n}$ and $\mathcal{M}_{g,n}$ when $X_{g,n}$ is a genus $g$ surface with $n$-punctures.

Here we must have finite ~~#~~ volume complete hyperbolic metrics on $X_{g,n}$.

$b=(b_1,\dots,b_n)$

$\mathcal{T}_{g,n}(b)$ = Teichmuller space of compact hyp surfaces with $n$ boundary circles (totally geodesic) of length $(b_1,\dots,b_n)$

$$\mathcal{M}_{g,n}(b) = \mathcal{T}_{g,n}(b)/\mathrm{Mod}_{g,n}$$

If $b=(0,\dots,0)$ then

$$\mathcal{M}_{g,n}(0) \cong \mathcal{M}_{g,n}$$

(Instead of boundary we have cusps in this case)

$$\mathcal{T}_{g,n}(b) \cong \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$$

$$(l_1,\dots,l_{3g-3+n},\tau_1,\dots,\tau_{3g-3+n})$$

$$\omega_{WP} = \sum_{i=1}^{3g-3+n} dl_i \wedge d\tau_i$$

If $b=0$, then $\mathcal{M}_{g,n}$ has a complex structure and $\omega_{WP}$ comes from a Kähler metric.

There is a compactification $\overline{M}_{g,n}(b)$ attaching Surfaces with lengths of geodesics 0.

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$\omega_{WP}$ extends to the boundary of $\overline{M}_{g,n}(b)$ to give a symplectic form on the compactification.

## Idea of Mirzakhani's proof:

Mirzakhani computes the volume of $\overline{M}_{g,n}(b)$ in two different ways.

$\cancel{V_{g,n}(b) = \sum}$

$V_{g,n}(b) = \mathrm{Vol}\left(\overline{M}_{g,n}(b)\right)$ is a polynomial in $b = (b_1, \dots, b_n)$ of degree $3g-3+n$.

$$V_{g,n}(b) = \sum_{|\alpha| \leq 3g-3+n} C_g(\alpha)\, b_1^{\alpha_1} \cdots b_n^{\alpha_n}$$

Where $C_g(\alpha) > 0$ and lies in $\pi^{2N-2|\alpha|} \cdot \mathbb{Q}$.

$N = 3g-3+n$

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Mirzakhani gives two ways of computing these polynomials.

1. She gives a recursive method of calculating these from ~~$V_g$~~ $V_{h,k}(b)$ $h \leq g$, $k \leq n$

(2) She shows that

$$2^{|\alpha|}\alpha!\,(3g-3+n-|\alpha|)!\,C_g(\alpha) = \int_{\overline{M}_{g,n}} \psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n}\,\omega_{WP}^{N-|\alpha|}$$

$N = 3g-3+n$, $\alpha! = \alpha_1!\cdots\alpha_n!$, $|\alpha| = \alpha_1+\cdots+\alpha_n$

$\omega_{WP}$ - Weil Peterson symplectic form.

Thus

$$\langle \tau_{\alpha_1}\cdots\tau_{\alpha_n}\rangle = 2^{3g-3+n}\,\alpha_1!\cdots\alpha_n!\,C_g(\alpha).$$

This gives an algorithm for computing the $\tau$-numbers in terms of $V_{g,n}(b)$ [Top degree coefficients]

The recursive methods of calculating $V_{g,n}(b)$ then shows that the $\tau$-numbers satisfy the Virasoro constraints proving Witten conjecture.

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References: ① Weil-Petersson volumes & Intersection theory of the Moduli Space of Curves – Mirzakhani (2007) JAMS.

② Simple geodesics and Weil-Petersson volumes of moduli of bordered Riemann Surfaces – Mirzakhani (2003) ~~Invencionaes~~ Inventiones.

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The main symplectic geometry tool used is Symplectic Reduction.

~~If $T^n$ the n-torus acts on $(M, \omega)$ freely by symplecto-morphisms.~~

~~symplectic manifold~~

~~A moment map $M$: $M \longrightarrow \mathbb{R}^n$ is obtained as follows:~~

~~$$dM(a, X) = \omega(\xi^a, X)$$~~

~~$\xi^a$ is the vector field associated to $a \in \mathbb{R}^n$ whose flow is given by the action of $\exp(a) \in T^n$.~~

~~$a \in \mathbb{R}^n$, $p \in M$~~

$\cancel{m_a(p)}$

$\cancel{m(p)(a) = dm}$

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$(M, \omega)$ – Symplectic manifold, $G$ – cpt Lie group with Lie algebra $\mathfrak{g}$,

$G \curvearrowright M$ by symplectomorphisms then a moment map of that action is a map

$$\mu : M \longrightarrow \mathfrak{g}^* \text{ such that}$$

$$d\mu(\xi)(X) = \omega(\xi^\#, X).$$

$\xi^\#$ is the v.f. on $M$ corresponding to $\xi \in \mathfrak{g}$

~~$\mu$ is the~~

$\mu(\xi): M \to \mathbb{R}$ is just a ~~the~~ Hamiltonian function for the vector field associated with $\xi$, $\xi^\#$

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$\xi \in \mathfrak{g}$ gives a vf on $M$ as follows.

[ Let $\xi^\#$ be the vector field whose flow is given by the action of $\exp(t\xi)$.

$\mu : M \longrightarrow \mathfrak{g}^*$ so for $\xi \in \mathfrak{g}$ $\mu_\xi : M \to \mathbb{R}$.

$$d\mu_\xi(X) = \omega(\xi^\#, X).$$

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$\mu$ constant along flow lines of $\xi$, hence $\mu$ $G$-invariant. ~~$\mu(\exp(t\xi)p)$~~

Assume $G$ acts freely, $\mu$ is a proper submersion then $\mu^{-1}(a)$ has a $G$ action and the quotient $M_a$ has a natural symplectic form called the reduced form.

principal $G$-bundle.

$$\begin{tikzcd} G \arrow[r] & \mu^{-1}(a) \arrow[r, hook] \arrow[d, "/G"] & M \\ & M_a & \end{tikzcd}$$

if $\dim M = 2n$

$\dim G = k$

$\mu^{-1}(a)$ has dim $2n-k$

& $M_a$ has dim $2n-2k$.

Specialising to $G = S^1$, $\mathfrak{g} = \mathbb{R}$.

Then $M_a$ has dim $2n-2$, we want to know how is $M_a$ related to $M_0$ for $|a| < \varepsilon$

~~A circle bundle is the same as a $\mathbb{C}$- line bundle so we can define a class $\varphi \in \Lambda^2 \mu^{-1}(0)$~~

a ~~2 form $\varphi$ on $M$ $\mu^{-1}(0)$ such that~~

$$\cancel{[\varphi] = c_1(S) \in}$$

$$\begin{array}{c} S^1 \hookrightarrow \mu^{-1}(0) \\ \downarrow \\ M_0 \end{array}$$

is a circle bundle along with a connection $\alpha$ obtained from the moment map.

A connection on $\mu^{-1}(0)$ gives a 2-form $\varphi$ called the curvature form such that

$$[\varphi] = c_1(\mu^{-1}(0)),$$

any circle bundle gives a chern class just like a complex line bundle.

For small $a$, $(M_a, \omega_a)$ is symplectomorphic to $(M_0, \omega_0 + a\varphi)$

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If $G = T^n$, $\mathfrak{g} = \mathbb{R}^n$, we have n-circle bundles $c_1, \dots, c_n$ associated to the torus bundle

$$\begin{array}{c} T^n \hookrightarrow \mu^{-1}(0) \\ \downarrow \\ M_0 \end{array}$$

~~These together give the curvature form~~

Hence we ~~gen~~ get $n$ curvature forms $\phi_1, \dots, \phi_n$

**Theorem:** If $\varepsilon > 0$ is sufficiently small & $\mu : M \to \mathbb{R}^n$ moment map for the action on $M$ by $T^n$, then for $|a| < \varepsilon$, $a = (a_1, \dots, a_n)$

$(M_a, \omega_a)$ is symplectomorphic to $(M_0, \omega_0 + a_1\phi_1 + \dots + a_n\phi_n)$

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**Corollary:**

If $m = 2n - 2k = \dim M_a$, $n = \dim M$, ~~$k = \dim G$~~ $T^k \circlearrowright M$.

then

$$\mathrm{Vol}(M_a) = \sum_{|\alpha| \le m} C(\alpha) a^\alpha \qquad \alpha = \alpha_1, \dots, \alpha_n$$

$$C(\alpha) = \frac{1}{\alpha!\,(m - |\alpha|)!} \int_{M_0} \varphi_1^{\alpha_1} \cdots \varphi_n^{\alpha_n} \omega_0^{m - |\alpha|}$$

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proof:

$$\mathrm{Vol}(M_a, \omega_a) = \mathrm{Vol}(M_0, \underbrace{\omega_0 + a_1\varphi_1 + \dots + a_n\varphi_n}).$$

The volume form on $M_0$ is

$$\frac{1}{m!} \Lambda^m (\omega_0 + a_1\varphi_1 + \dots + a_n\varphi_n)$$

$$= \sum_{\substack{\alpha_0, \dots, \alpha_n \ge 0 \\ \alpha_0 + \alpha_1 + \dots + \alpha_n}} \frac{a_1^{\alpha_1} \cdots a_n^{\alpha_n}}{\alpha_0! \cdots \alpha_n!} \varphi_1^{\alpha_1} \cdots \varphi_n^{\alpha_n} \omega_0^{\alpha_0}$$

Let $b = (b_1, \dots, b_n)$ be the following principal $S^1$-bundle

$$\begin{array}{c} S^1 \hookrightarrow S_i(b) \\ \downarrow \\ \mathcal{M}_{g,n}(b) \end{array}$$

$S_i(b) = \{(X, p) \mid X \in \mathcal{M}_{g,n}(b),\ p \in \tilde{\beta}_i\}$, $S^1$ acts on $S_i(b)$

by ~~per~~ translating the point ~~on~~ $p$ on $\beta_i$.

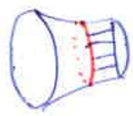
(Orient it so that the tangent to the circle followed by the outward normal agrees with the orientation of $X$ ~~$S_{g,n}$~~)

~~Claim: $c_1$~~

If ~~$b = (0, 0, \dots, 0)$~~ $b_i = 0$ choose a horocycle of length $\frac{1}{4}$ around the cusp these are all distinct.

$$\begin{array}{c} S_i(0) \\ \downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

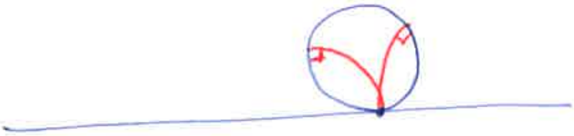
extends to the compactification.



**Theorem:** $c_1(S_i(0)) = c_1(L_i) = \Psi_i$ on $\overline{\mathcal{M}}_{g,n}$.

Idea: At the cusp $\beta_i$ ~~we have flowing along~~ every tangent direction gives a unique point on $\tilde{\beta}_i$

This gives an orientation reversing map to the ~~$S_i$~~ circle $\tilde{\beta}_i$



Hence the circle bundle of $L_i$ and $S_i$ are isomorphic

$L_i|_{q_i}$ is dual to the tangent line ~~on~~ at $q_i$

The spaces

$\widehat{\cancel{M}}_{g,n} =$

Moduli of bordered Riemann surfaces:

$$\widehat{M}_{g,n} = \{(X, p_1, \ldots, p_n) \mid p_i \in \tilde{\beta}_i, X \in \overline{M}_{g,n}(b_1, \ldots, b_n); \; b_i > 0\}.$$

$\widehat{M}_{g,n}$ has dimension $2\times(3g-3+n) + 2n$

$$= \boxed{6g - 6 + 4n}$$

$T^n$ acts on $\widehat{M}_{g,n}$ ~~by strao~~ as follows.

$$\gamma \in T^n, \quad \gamma = (\gamma_1, \ldots, \gamma_n)$$

$$\gamma \cdot (X; p_1, \ldots, p_n) = (X, \gamma_1 p_1, \ldots, \gamma_n p_n)$$

~~Mirzakhani shows that there is a map~~

There is the obvious map

$$\widehat{M}_{g,n} \longrightarrow \mathbb{R}^n \text{ given by}$$

$l$:

$$(X, p_1, \ldots, p_n) \longmapsto \left(l^2_{\beta_1}(X)/2, \ldots, l^2_{\beta_n}(X)/2\right)$$

$\boxed{6}$ $x = b^2/2$
$\Rightarrow b = \sqrt{2x}$

Clearly ~~#~~

$\cancel{\ell^{-1}(b_1, \ldots)}$

$\cancel{\ell^{-1}(\sqrt{2b_1^2}, \ldots, \sqrt{2b_{2n}}) \longrightarrow}$

$\ell^{-1}(b_1^2/2, \ldots, b_n^2/2)$ is a principal $T^n$ bundle over $\overline{M}_{g,n}(b_1, \ldots, b_n)$.

<u>Theorem:</u> $\widehat{M}_{g,n}$ has a natural $T^n$ invariant symplectic form such that.

(1) $\ell = (\ell^2_{\beta_1}(X)/2, \ldots, \ell^2_{\beta_n}(X)/2)$ is the moment map for the $T^n$ action on $\widehat{M}_{g,n}$.

(2) $s: \ell^{-1}(b_1^2/2, \ldots, b_n^2/2) \longrightarrow \overline{M}_{g,n}(b_1, \ldots, b_n)$

is a symplectomorphism. with the W-P form on the range.

$$\ell^{-1}(0, \ldots, 0) = \mathcal{M}_{g,n}$$

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Now using Symplectic reduction

we get

$$V_{g,n}(b) = \sum_{|\alpha| \leq 3g-3+n} C(\alpha)\, b^{2\alpha}$$

$$C_g(\alpha_1, \ldots, \alpha_n) = \frac{1}{2^{|\alpha|}\alpha_1! \ldots \alpha_n! (3g-3+n-|\alpha|!)} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \ldots \psi_n^{\alpha_n} \omega^{3g-3+n-|\alpha|}.$$

# Asides:

## Hyperbolic metric:

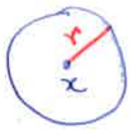
$M$ - 2d Riemannian manifold, $x \in M$,

curvature at $x$, $K(x)$ is given by the following

$$\text{Area}\left(B_r(x)\right) = \pi r^2 - \frac{1}{12\pi} K(x) r^4 + o(r^4)$$

$K(x)$ - Gaussian or Sectional curvature of $M$ at $x$.

The normalisation is done so that the curvature of the unit sphere in $\mathbb{R}^3$ is constant 1.




Hyperbolic surface uniform sectional curvature -1.

Example: $\mathbb{H}$ with the metric

$$\frac{dx^2 + dy^2}{y^2}$$

## Witten Conjecture:

$\tau$ numbers satisfy the Virasoro constraints.

$$\langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle = \int_{\overline{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \qquad d_1 + \ldots + d_n = 3g-3+n.$$

String eqn: $\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n+1} = \sum_{d_i > 0} \langle \tau_{d_1} \cdots \tau_{d_i - 1} \cdots \tau_{d_n} \rangle_{g,n}$

Dilaton eqn:

$$\langle \tau_1 \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n+1} = (2g-2+n) \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$$

$$
\begin{aligned}
L_k:\quad (2k+3)!!\,\langle \tau_k \tau_{d_1} \cdots \tau_{d_n} \rangle
&= \frac{1}{2} \sum_{i+j=k-1} (2i+1)!!\,(2j+1)!! \sum_{I \subseteq \{d_1 \ldots d_n\}} \langle \tau_i \tau_{d_I} \rangle \cdot \langle \tau_j \tau_{d_{I^c}} \rangle \\
&+ \frac{1}{2} \sum_{i+j=k-1} (2i+1)!!\,(2j+1)!!\, \langle \tau_i, \tau_j, \tau_{d_1} \cdots \tau_{d_n} \rangle_{g-1,n+2} \\
&+ \sum_{j=1}^{n} \frac{(2k+2d_j+1)!!}{(2d_j-1)!!} \langle \tau_{d_1}, \ldots, \tau_{d_j+k}, \ldots \tau_{d_n} \rangle_{g,n}
\end{aligned}
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