Affine Stratification Number and Moduli Space of Curves

Midwest Algebraic Geometry Graduate Conference

Chitrabhanu Chaudhuri

Northwestern University

November 4, 2012

Organization

1. Definitions and Motivation
2. Results
3. Techniques

Affine Stratification Number

An **affine stratification** of a scheme X is a finite decomposition

$$X = \bigsqcup_{k=0}^{n} \bigsqcup_{i} Y_{k,i} \qquad \text{where} \quad \overline{Y}_{k,i} \backslash Y_{k,i} \subset \bigcup_{k'>k,j} Y_{k',j}$$

and $Y_{k,i}$ are locally closed and affine. The number n, is the length of the stratification and the **minimum** over all affine stratifications will be called the **affine stratification number** of X, denoted as $\operatorname{asn} X$.

Theorem (Roth, Vakil)

If $\operatorname{asn} X = m$ then there exists an affine stratification $X = \bigsqcup_{k=0}^{m} Z_k$, such that

- $\overline{Z}_k = \cup_{k' \geq k} Z_{k'}$*, and* Z_k *dense open affine in* $\overline{Z}_k$
- $\overline{Z}_k$ *is pure co-dimension 1 in* Z_{k-1}.

We call this the **affine cell decomposition** of X.

Equidimensional case

If X is equi-dimensional, then an affine stratification of X will be

$$X = \bigsqcup_{k=0}^{n} Y_k$$

where Y_i is locally closed and codimension i in X, further

$$\overline{Y}_i \backslash Y_i \subset \bigsqcup_{k=i+1}^{n} Y_k$$

n is the length of the stratification and the length of the shortest affine stratification is the affine stratification number of X (denoted $\operatorname{asn} X$).

Properties

Intuitively $\operatorname{asn} X$ measures how far the scheme X is from being affine and how close it is to being proper.

Let $a = \operatorname{asn} X$ and $d = \dim X$. We have the following bounds.

1. $a \leq d$, and equality holds if X has a proper d-dmensional component.
2. If $\mathcal{F}$ is a quasi-coherent sheaf on X, then $H^i(X, \mathcal{F}) = 0$ for $i > \operatorname{asn} X$.
3. If Z is a proper closed subscheme, then $\dim Z \leq a$.
4. If $\mathcal{F}$ is a constructible sheaf then $H^i(X, F) = 0$ for $i > \operatorname{asn} X + \dim X$.

As a consequence of (4) when the base field is $\mathbb{C}$, note that the betti cohomology vanishes above degree $a + d$ (although the real dimension of X is $2d$).

Conjecture

When base field is $\mathbb{C}$, X has the homotopy type of a finite CW-complex of dimension at most $\operatorname{asn} X + \dim X$.

Moduli of Curves

Let $M_{g,n}$ be the moduli space of algebraic curves of genus g and n marked points (over $\mathbb{C}$). (I would like to consider the course moduli spaces but one can also think of them as Deligne-Mumford stacks). Let $\overline{M}_{g,n}$ be it's Deligne-Mumford compactification. This space parametrizes stable curves.

$M_{g,n}$ is a dense open subset of $\overline{M}_{g,n}$, and the complement is a divisor with normal crossings.

Conjecture (Looijenga)

$$\text{asn}\, M_g \leq g - 2 \qquad \textit{when} \quad g \geq 2$$

He proves the conjecture for $g \leq 5$. From this conjecture it follows that

$$\text{asn}\, M_{g,n} \leq g - 1 \quad \text{for} \quad n \geq 1$$

A Filtration

There is a filtration on $\overline{M}_{g,n}$ given by the number of rational components of a curve.

Let $M_{g,n}^{\leqslant k}$ be the open set in $\overline{M}_{g,n}$ parametrizing stable curves with at most k rational components (components of geometric genus 0). There is a generalization of the conjecture in the previous slide by Roth and Vakil.

Conjecture (Roth and Vakil)

$$\operatorname{asn} M_{g,n}^{\leqslant k} \leq g - 1 + k \quad \textit{for} \quad g > 0, k \geq 0$$

This is an easy to state but hard to answer question. No significant progress has been made towards this conjecture except for checking explicit cases for small g and k.
So I investigated similar question but for locus of curves of low gonality.
The easiest case is to start with hyper-elliptic curves which have gonality 2.

Hyperelliptic Locus

Let $H_g \subset M_g$ be the hyperelliptic locus, that is the subspace parametrizing isomorphism classes of hyperelliptic curves. A hyper-elliptic curve of genus g is a double cover of $\mathbb{P}^1$, ramified over $2g+2$ points. Let $\overline{H}_g$ be the closure of H_g in $\overline{M}_g$.

There is a filtration on $\overline{H}_g$, induced by the filtration on $\overline{M}_g$. And we ask what is asn $H_g^{\leqslant k}$.

We have the following partial answer

Lemma (C)

$$\text{asn}\, H_g^{\leqslant k} \leq g - 1 + k \quad \textit{for all } g, k$$

The inequality is shown by demonstrating an explicit affine cell decomposition of length $g - 1 + k$.

Sharpness

For k large the bound is not effective. For example when $k \geq g$, then

$$\operatorname{asn} H_g^{\leqslant k} \leq 2g - 1 \leq g - 1 + k$$

since $\dim \overline{H}_g = 2g - 1$, so dimension gives a better bound.

Question: How effective is the bound when k is small with respect to g?

Bit of experimentation with small g and k, shows that the bound is sharp for small k. The following result shows the sharpness in case $k = 0$,

Theorem (C)

$$\operatorname{asn} H_g^{\leqslant 0} = g - 1 \quad \textit{for} \quad g \geq 2$$

Reduction to question on $\overline{M}_{0,n}$

The Hurwitz space, $\mathcal{H}ur_{g,d}$, parametrizes genus g, d-sheeted simply branched covers of $\mathbb{P}^1$ (with a numbering of branch points). By Riemann-Hurwitz, the number of branch points is $2g + 2d - 2$. The HUrwitz space can be constructed as a finite cover

$$q : \mathcal{H}ur_{g,d} \to M_{0,2g+2d-2}$$

There is also a natural map

$$\pi : \mathcal{H}ur_{g,d} \to M_g$$

In particular $\mathcal{H}ur_{g,2} \cong M_{0,2g+2}$ and there's a finite map $\mathcal{H}ur_{g,2} \to H_g$. So we get a map $\pi : M_{0,2g+2} \to H_g \subset M_g$ that extends to the compactifications.

$$\pi : \overline{M}_{0,2g+2} \to \overline{H}_g$$

Admisible covers

Harris and Mumford introduced a compactification of the Hurwitz space using admissible covers. So that now there is a finite map

$$q : \overline{\mathcal{H}ur}_{g,d} \to \overline{M}_{0,2g+2d-2}$$

An **admissible cover** of genus g is a semi-stable curve, that is a double cover of a stable genus 0 curve with $2g + 2d - 2$ marked points. So that

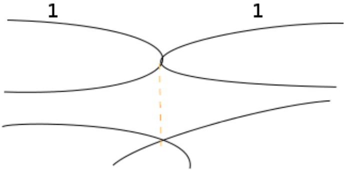
- There is simple branching over the marked points.
- At nodes the branching can be more complicated but the ramification along the two components meeting at the node have to be the same.

The map π extends to the compactification, but stabilization involved.

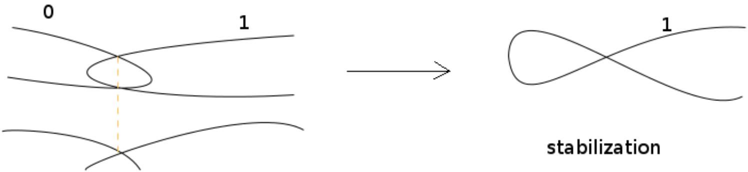
$$\pi : \overline{\mathcal{H}ur}_{g,d} \to \overline{M}_g$$

Examples

Admissible cover



Admissible cover



stabilization

Upper Bound

In our case we have,

$$\pi : \overline{\mathcal{H}ur}_{g,2} \cong \overline{M}_{0,2g+2} \to \overline{H}_g$$

Here π is the quotient map under the action of the symmetric group S_{2g+2} which acts by permuting the fixed points.
Let

$$M_{0,2g+2}^{\leqslant k} = \pi^{-1} H_g^{\leqslant k}$$

An affine stratification of $M_{0,2g+2}^{\leqslant k}$ gives an affine stratification of $H_g^{\leqslant k}$ of the same length. There is a combinatorial criterion to determine which curves are in $M_{0,2g+2}^{\leqslant k}$. It can be seen that the curves in $M_{0,2g+2}^{\leqslant k}$ have at most $g-1+k$ nodes.

The upper bound is then obtained by showing an affine stratification of $M_{0,2g+2}^{\leqslant k}$ of length $g-1+k$.

Dual Graphs of Stable curves

The dual graph of a marked stable graph is obtained by placing a vertex for each component, an edge for each node and a leg for each marked point. Further the vertices are labelled by the geometric genus of the component it represents.

The genus of the curve can be obtained by the formula

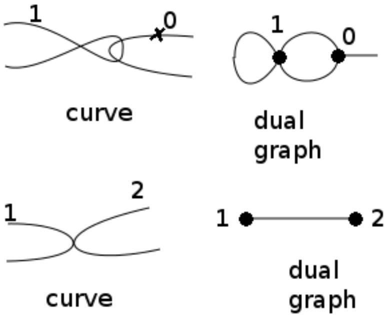
$$g = \sum_{v \in \text{ vertices of } G} g_v + b_1(G)$$

For a graph G, denote by M_G, the locus of curves whose dual graph is G, and by $\overline{M}_G$ its closure. Then

$$\overline{M}_g = \bigcup_{G,\ \text{genus}(G)=g} M_G$$

$\overline{M}_G$ is a subvariety of codimension equal to the number of edges of G. We denote the set of isomorphism classes of dual graphs of genus g, with n marked points by $\Gamma(g, n)$.

Examples of Dual Graphs



Sharpness for $k = 0$

If $\mathcal{L}$ is a local system on X, then $H^i(X, \mathcal{L}) = 0$ for $i > \operatorname{asn} X + \dim X$.

Consider the constant sheaf $\mathbb{C}$ on $M_{0,2g+2}^{\leqslant 0}$. Push forward of the constant sheaf by π gives a local system $\mathcal{L}$ on $H_g^{\leqslant 0}$. Note

$$H^i(H_g^{\leqslant 0}, \mathcal{L}) = H^i(M_{0,2g+2}^{\leqslant 0}, \mathbb{C})$$

Since $\dim H_g^{\leqslant 0} = 2g - 1$ if we can show that

$$H^{3g-2}(H_g^{\leqslant 0}, \mathcal{L}) = H^{3g-2}(M_{0,2g+2}^{\leqslant 0}, \mathbb{C}) \neq 0$$

that would imply $\operatorname{asn} H_g^{\leqslant 0}$ is at least $g - 1$, and show sharpness of the upper bound when $k = 0$.

Lemma (C)

The cohomology group $H^{3g-2}(M_{0,2g+2}^{\leqslant 0}, \mathbb{C})$ is non-zero and has a pure Hodge structure of weight $2(2g-1)$ and $H^k(M_{0,2g+2}^{\leqslant 0}) = 0$ for $k > 3g - 2$.

A Spectral Sequence

For a genus 0 graph G with $2g+2$ marked points, either $M_G \subset M^{\leqslant 0}_{0,2g+2}$, in which case we call the graph **good** or intersection is empty. Good graphs can be determined by a simple combinatorial criterion.

We have a spectral sequence in compactly supported cohomology.

$$E_1^{-p,q} = \sum_{\substack{G \in \Gamma(0,2g+2) \\ G \text{ good with } p \text{ edges}}} H_c^{q-p}(M_G)$$

This spectral sequence converges to $H_c^{q-p}(M^{\leqslant 0}_{0,2g+2})$.The spectral sequence has

- a natural mixed Hodge structure
- an action of S_{2g+2}

We use this spectral sequence to show $H_c^{g}(M^{\leqslant 0}_{0,2g+2})$ is non-trivial and use Poincaré duality.

Genus 2 case

	$p=-1$	$p=0$
$q=6$	0	s_6
$q=5$	$2s_6 + 2s_{4,2} + s_{5,1}$	$s_{4,2}$
$q=4$	$s_{3,3} + 3s_{4,2} + 2s_{5,1} + s_{2,2,2} + 3s_{3,2,1} + 2s_{4,1,1}$	$s_{4,1,1} + s_{3,2,1}$
$q=3$	$s_{5,1} + 2s_{3,3} + s_{4,2} + 3s_{3,2,1} + s_{4,1,1} + 2s_{2,2,1,1} + s_{3,1,1,1}$	$s_{3,3} + s_{4,1,1} + s_{2,2,1,1}$

Calculations

From the Spectral equence it turns out that

$$H_c^g(M_{0,2g+2}^{\leqslant 0}) = E_\infty^{-g-1,2g-1}$$

and that

$$E_\infty^{-g-1,2g-1} = E_2^{-g-1,2g-1}$$

To analyze the Spectral Sequence we use the action of the symmetric group S_{2g+2}. Ezra Getzler calculated the S_n equivariant cohomology of $M_{0,n}$ an $\overline{M}_{0,n}$.

Using those techniques we can decompose the vector spaces appearing in the first page of the spectral sequence into irreducible representations of S_{2g+2}. The differential of the spectral sequence is equivariant for the symmetric group action and counting dimensions of the isotypic components we infer that $E_2^{-g-1,2g-1} \neq 0$.

Calculations and Symmetric functions

Using ideas from Modular operds paper by Getzler and Kapranov, we represent characters of the symmetric group using MacDonalds Symmetric functions.

The calculations using symmetric functions were carried out using a Maple package SF (with modifications) by John Stembridge. The lemma can be proven computationally only up to genus 6. To prove it for all genus we notice that for small genera the isotypic component corresponding to the standard representation of S_{2g+2} yields the result.

The differential in the spectral sequence is closely related to the differential of the cobar operad of a certain operad.

Operads

$H_*(M_{0,n})$ form an operad (upto degree shift) called the Gravity operad $\mathcal{G}rav$, and $H_*(\overline{M}_{0,n})$ forms the Hypercommutative operad $\mathcal{H}yComm$.

$$M_{0,n} \subset \overline{M}_{0,n} \text{ a dense open sub-variety}$$

$$\overline{M}_{0,n} \backslash M_{0,N} \text{ normal crossings divisor}$$

This is exactly the set-up for Deligne's spectral sequence using differentials with logarithmic singularities.

Using this spectral sequence *Getzler*, shows that

$$\mathcal{H}yComm \text{ is } \textbf{Koszul-Dual} \text{ to } \mathcal{G}rav$$

Getzler an Kapranov also prove a combinatorial formula relating the S_n character of $\mathcal{G}rav$ with that of $\mathcal{H}yComm$.

There is a spectral sequence in compactly supported cohomology which converges to $H^*(\overline{M}_{0,n})$, and is dual to Deligne's spectral sequence is an appropriate sense.

The spectral sequence we use is analogous to this one but for the quasi-projective variety

$$M_{0,2g+2}^{\leqslant 0}$$

There is a colored operad floating around obtained from the gravity operad. So we generalize the formula of Getzler-Kapranov to colored operads and use that to do the computations.

Further Research

- Investigate similar questions about trigonal or tetragonal locus, and other Brill-Noether loci.
- See whether there are similar operadic interpretation of the (co)homology as in the hyper-elliptic case.

Thank You !