Counting Genus 1 curves on del-Pezzo Surfaces

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Complex Manifolds

A complex manifold X of dimension n is a (hausdorff & second countable) topological space along with an open cover

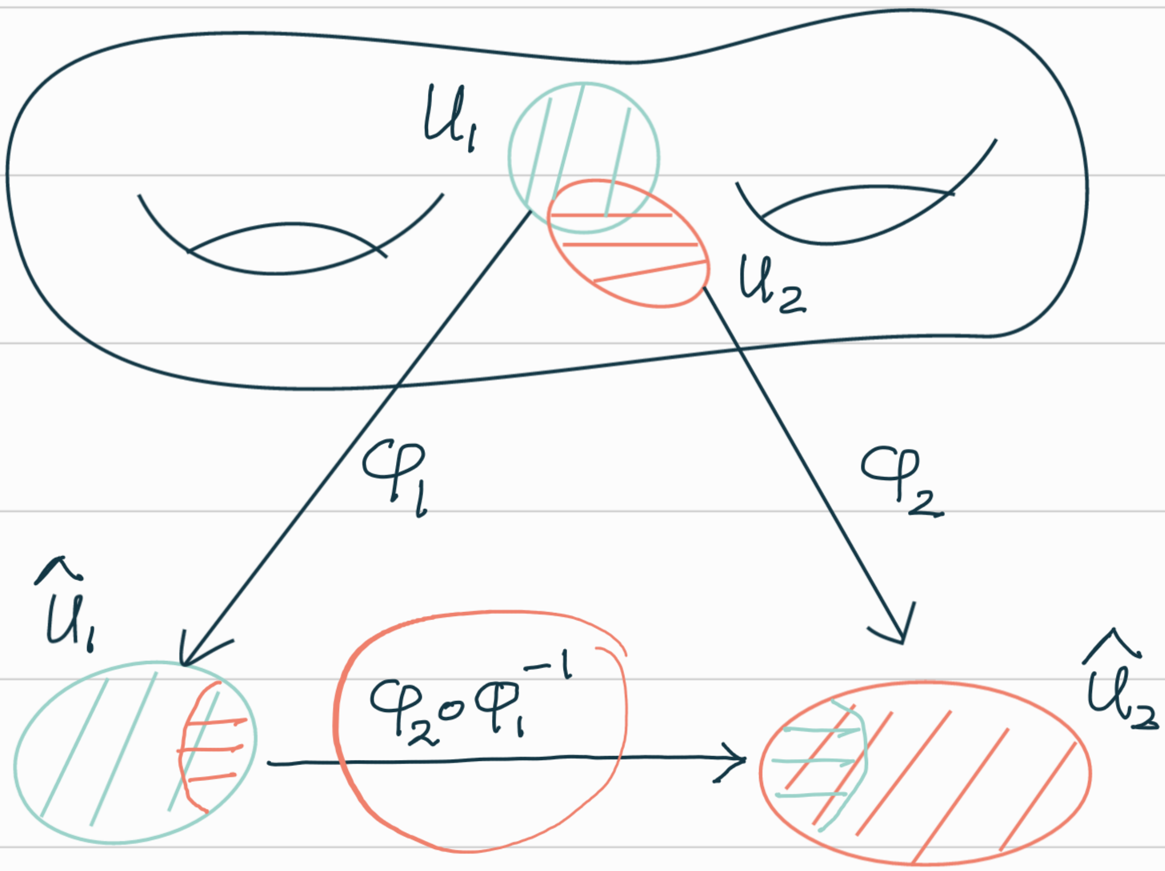
$$X = \bigcup U_i$$

and homeomorphisms

$$\varphi_i : U_i \longrightarrow \hat{U}_i, \quad \hat{U}_i \subseteq \mathbb{C}^n \text{ open}$$

such that

$\varphi_i \circ \varphi_j^{-1}$ is holomorphic.



<u>Example:</u> Complex projective spaces

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\sim}$$

where $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$

for any $\lambda \in \mathbb{C} \setminus \{0\}$.

The open sets

$$U_i = \{[z_0, \dots, z_n] \in \mathbb{C}P^n \mid z_i \neq 0\}$$

covers $\mathbb{C}P^n$ and $\varphi_i : U_i \longrightarrow \mathbb{C}^n$

$$\varphi_i([z_0, \dots, z_n]) = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right)$$

$\varphi_i^{-1}(z_1, \dots, z_n)$
$= (z_1, \dots, z_{i-1}, 1, z_i, \dots, z_n)$

are homeomorphisms.

☆ $\mathbb{C}P^1$ is topologically a sphere (one point compactification of $\mathbb{C}$).

$U_0 \cong \mathbb{C}$, $\quad \mathbb{C}P^1 \setminus U_0$

Smooth Algebraic Curves

A 1-dimensional compact complex manifold is called an algebraic curve.

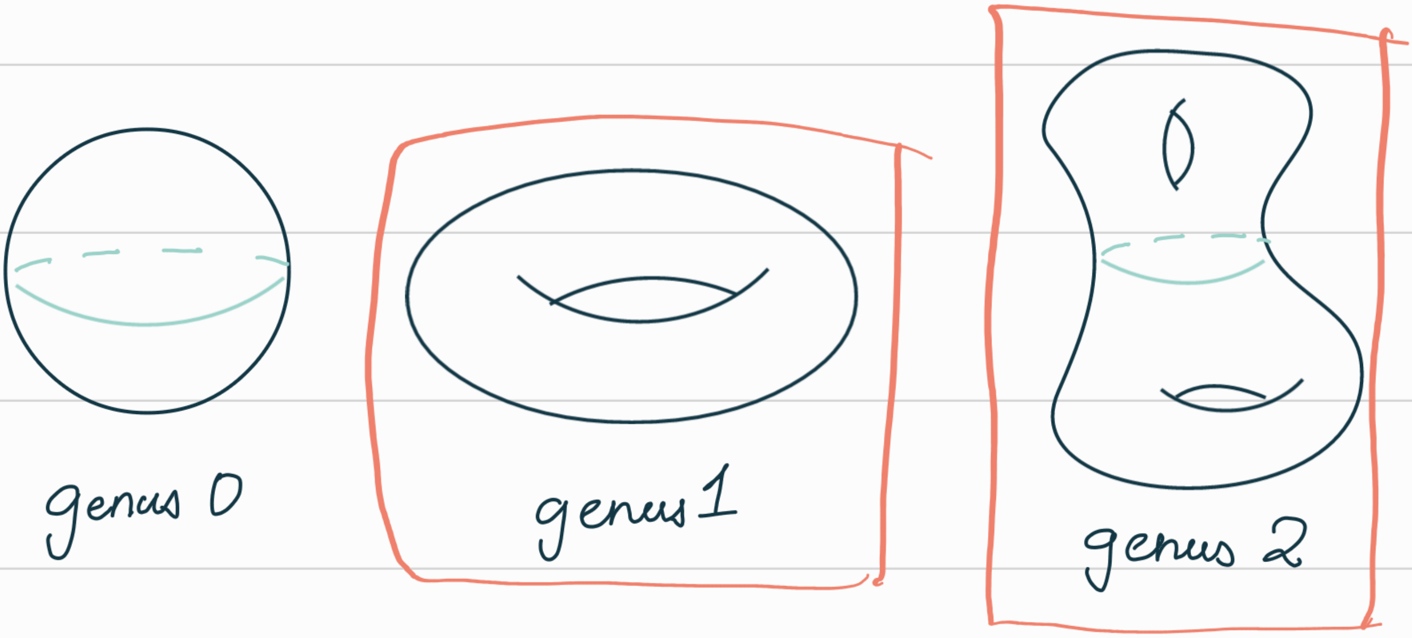
Such a curve X is also a compact orientable surface, hence homeomorphic to

- ☆ The sphere S^2, in which case
  $X \cong \mathbb{C}P^1$, OR.
- ☆ X is homeomorphic to a Σ_g the connected sum of g tori.

The *genus* of X is

0 if X is homeomorphic to S^2.

g if X is homeomorphic to Σ_g.



Fact: Any smooth algebraic curve can be embedded in $\mathbb{CP}^n$ for some n.

$$g(X) = \frac{1}{2} \dim H_1(X, \mathbb{Z}).$$

$$H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g(X)}$$

Examples:

We are concerned with curves in $\mathbb{C}P^2$. Such a curve is the zero set of a homogeneous polynomial $f \in \mathbb{C}[x_0, x_1, x_2]$.

$$X = V(f) = \{ [z_0, z_1, z_2] \in \mathbb{C}P^2 \mid \underbrace{f(z_0, z_1, z_2) = 0}\}.$$

It can be shown that X is smooth if $\underbrace{\left(\frac{\partial f}{\partial z_0}(z), \frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z) \right)}$ does not vanish at any point $z \in X$.

(Implicit Function theorem.)

Let $\boxed{\lambda \in \mathbb{C}}$, and consider the curve

$$\boxed{E_\lambda \subseteq \mathbb{CP}^2,}$$

$$E_\lambda = \{[z_0, z_1, z_2] \mid z_2^2 z_0 = z_1(z_1 - z_0)(z_1 - \lambda z_0)\}$$

on $\underbrace{U_0 \ (z_0 \neq 0)}_{\cong \mathbb{C}^2}$, E_λ is given by

$$E_\lambda \cap U_0 = \{(x,y) \in \mathbb{C}^2 \mid y^2 = x(x-1)(x-\lambda)\}.$$

When $\lambda \neq 0, 1$, E_λ is smooth.

If $\boxed{\lambda = 0,}$ note that the equation for E_0 on U_0 becomes

$$\boxed{y^2 = x^2(x-1).}$$

$y^2 = h^2 x^2$

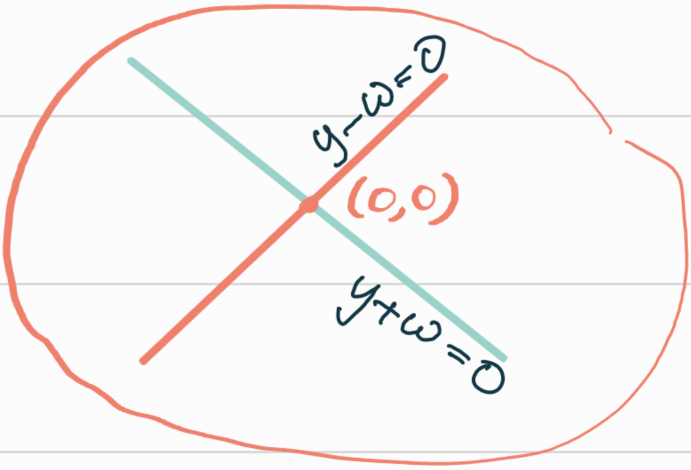
$(y - hx)(y + hx)$

Now near $\boxed{x=0,}$ there is a holomorphic function $h(x)$ such that

$$\boxed{h^2 = (x-1).}$$

Taking $\boxed{w = hx}$ the equation for E_0 near $(0,0)$ becomes

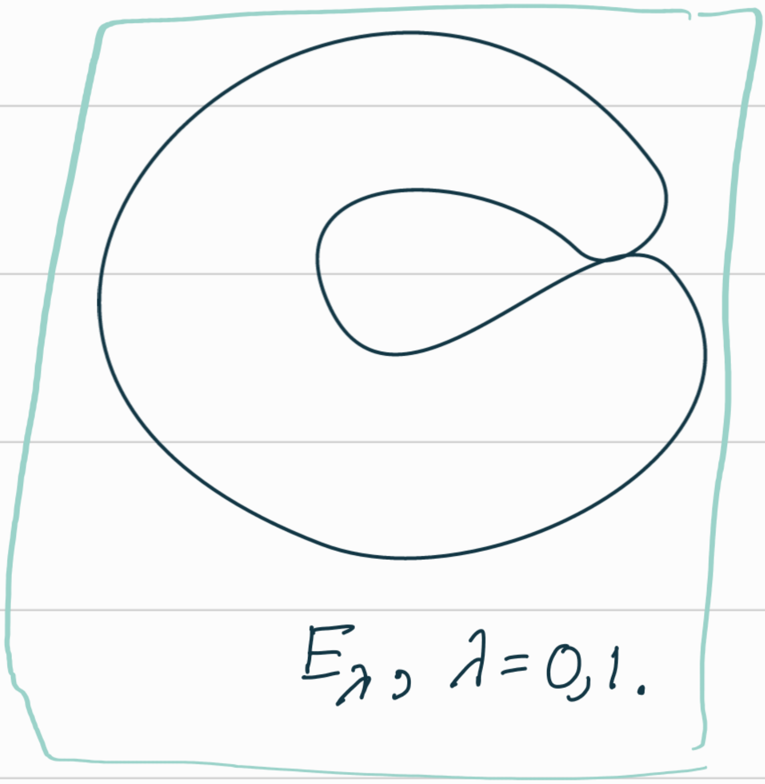
$$\boxed{(y-w)(y+w) = 0.}$$



The point $(0,0)$ is called a node of E_λ.

Topologically, we have the following

$E_\lambda, \lambda \neq 0,1$
genus 1



$E_\lambda, \lambda = 0,1.$

Hence we see that limit of smooth curves may not be smooth.

Curve Counting:

The following was a long standing question in enumerative geometry.

Q. What is the number $\boxed{E_d^{(g)}}$ of degree d, genus g curves in $\mathbb{CP}^2$ passing through $3d-1+g$ generic points?

To justify this question we shall look at small values of g & d.

- When $\underline{g=0}$, $\underline{d=1}$, $E_1^{(0)}$ is the number of lines in $\mathbb{CP}^2$ passing through $\underline{2 \text{ points}}$.

Of course the answer is 1. $2 = 3 \times 2 - 1$

If they were required to pass through fewer than 2 points the anwer would be infinite, where as for more that two points it would be 0.

- For the case $d = 2$, $g = 0$, $E_2^{(0)}$ counts the number of conics in $\mathbb{C}P^2$, through 5 generic points. $3 \times 2 - 1 = 5$

The answer is again 1.

$$g = \frac{(d-1)(d-2)}{2}.$$

To see this, note that a degree two curve is the zero set of a non-zero polynomial of the form

$$\boxed{p(x) = a_1 x_0^2 + a_2 x_1^2 + a_3 x_2^2 + b_1 x_0 x_1 + b_2 x_0 x_2 + b_3 x_1 x_2}$$

Thus the vector space $\boxed{V}$ of such polynomials has $\boxed{\text{dimension } 6.}$

However, we have to exclude the 0 polynomial and of course p and λp give the same curve for any $\lambda \in \mathbb{C}^*$.

Thus the space of degree 2 curves in $\mathbb{C}P^2$

is $\boxed{\mathbb{P}(V) = \dfrac{V \setminus \{0\}}{\sim}}$

where $\boxed{p \sim \lambda p}$ for any $\lambda \in \mathbb{C}^*$.

This space is 5-dimensional.

For a curve to pass through a point $[z_0, z_1, z_2]$ we get a linear equation among the coefficients $\boxed{p(z_0, z_1, z_2) = 0.}$

Thus passing through 5-generic points impose 5 linearly independent linear equations on the set of coefficients and is $\boxed{\mathbb{P}(V)}$ we get just one curve.

Clearly if we had less than 5 points the answer would be infinite, where as for more than 5 points the answer is 0.

- From $d = 3$ onwards, the question becomes interesting.

  For $d = 3$, $g = 0$, the number of degree d genus 0 curves passing through

  $\boxed{3 \times 3 - 1 = 8 \text{ generic points is } 12.}$

How can we expect to solve this question? If we mimic the case of $E_2^{(0)}$ we would expect

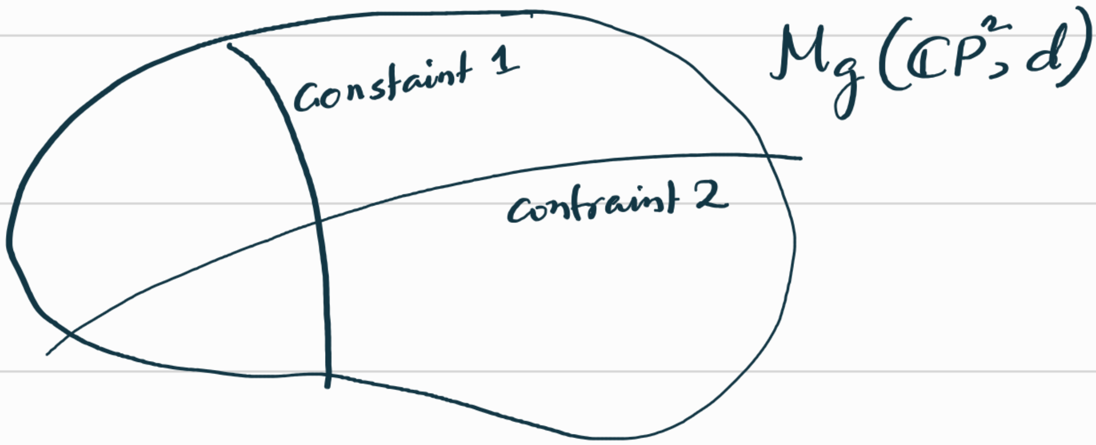
☆ There is a <u>complex manifold $M_g(\mathbb{CP}^2, d)$</u> parametrizing all genus g, degree d curves in $\mathbb{CP}^2$.

☆ All curves which pass through a particular point <u>form a submanifold</u> in $M_g(\mathbb{CP}^2, d)$.

☆ If $\dim M_g(\mathbb{CP}^2, d) = n$, for n generic points we get n distinct submanifolds which istersect transvensely.

☆ $E_d^{(g)}$ is the numbes of points in the intersection of these submanifolds.

Catch: $\boxed{M_g(\mathbb{CP}^2, d)}$ has to be compact for this to work.



Some historical backgroung

☆ Kontsevic-Manin and Ruan-Tian obtained a formula for $E_d^{(0)}$ in 1990's.

☆ Caporaso-Harris computed $E_d^{(g)}$ for any d using completely different methods.

☆ Getzler computed $E_d^{(1)}$ by computing elliptic Gromov-Witten invariants of $\mathbb{CP}^2$.

Moduli of Curves & Stable maps

Stable curves of genus g with n marked points, $(C, p_1, \ldots, p_n)$

(a) $\boxed{C}$ is a algebraic curve of genus g with only nodes a singularities.

(b) $\boxed{p_1, \ldots, p_n} \in C$, distinct smooth points.

(c) Each component of genus 1 has atleast 1 special point and each component of genus 0 has atleast 3 special points.

(This forces $2g-2+n > 0$).

$(C, p_1, \dots, p_n) \cong (C', p_1', \dots, p_n')$ if

there is an isomorphism $\varphi: C \to C'$ st

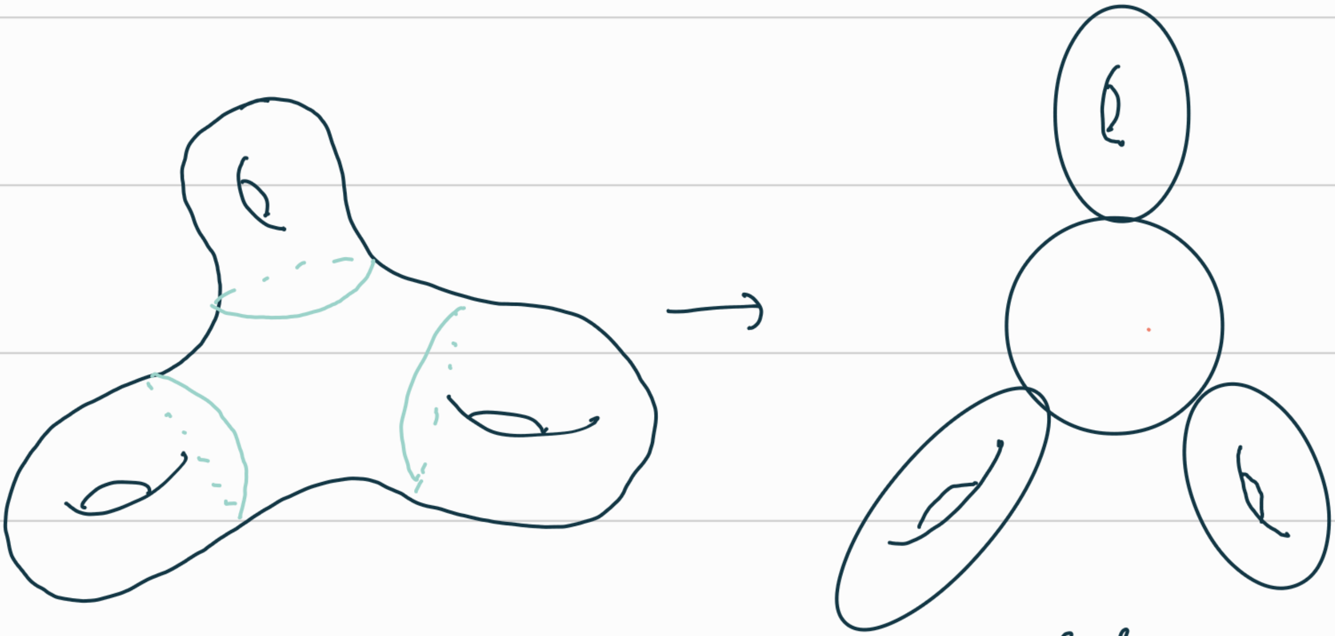
$$\varphi(p_i) = p_i'.$$

$\overline{M}_{g,n}$ = Space of isomorphism classes of stable curves of type (g,n).

☆ $\overline{M}_{g,n}$ compact complex orbifold of dim $3g-3$

☆ $\overline{M}_{g,n} \supseteq M_{g,n}$ as a dense open subspace

$M_{g,n}$ – Space of isomorphism classes of smooth curves of type (g,n)



smooth nodal

Stable curves of genus 3.

Examples:

$(\mathbb{CP}^1, p_1, p_2, p_3)$

$\boxed{\overline{M}_{0,3} = M_{0,3}}$ is a point

(Automorphism group of $\mathbb{CP}^1$, $\boxed{PGL(2,\mathbb{C})}$ is 3 - transitive.)

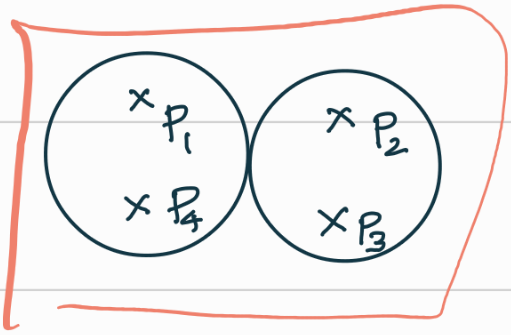
$M_{0,4} \cong \boxed{\mathbb{CP}^1 \setminus \{0,1,\infty\}}$ and $\overline{M}_{0,4} \cong \mathbb{P}^1$.

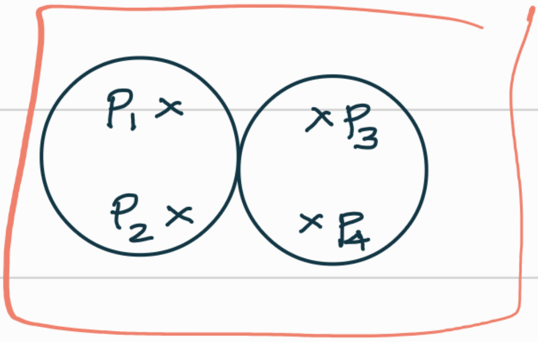
$\boxed{(\mathbb{CP}^1, p_1, p_2, p_3, p_4)} \cong (\mathbb{CP}^1; 0, 1, \infty, \lambda)$

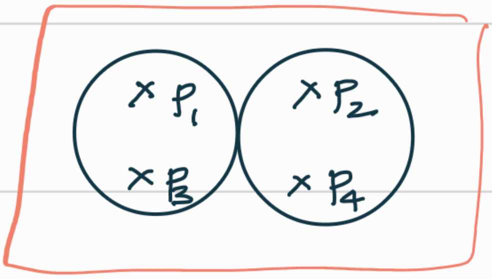
$$\boxed{\lambda = \frac{(p_4 - p_1)(p_3 - p_2)}{(p_4 - p_3)(p_1 - p_2)}}$$

(cross ratio.)

We have the following nodal curves in the limit







$\boxed{\overline{M}_{1,1} \cong \mathbb{CP}^1/S_3}$ where the action of S_3 is not free.

$M_{1,1} \cong \mathbb{H}/SL(2,\mathbb{Z})$

We are interested in curve in some complex manifold X, for that we need the following more general notion.

Let X be a projective manifold,

$\beta \in H_2(X, \mathbb{Z})$ then a stable map

of type (g, n, β), $(C, p_1, \ldots, p_n, \mu)$

(a) C is a genus g nodal curve

(b) $p_1, \ldots, p_n$ distinct smooth marked

points on C.

(c) $\mu : C \longrightarrow X$, $\mu_* [C] = \beta$.

(d) Stability condition: finite automorphism

group.

$$(C, p_1, \dots, p_n, \mu) \cong (C', p_1', \dots, p_n', \mu')$$

$$\begin{tikzcd} C \arrow[rr, "\varphi"] \arrow[dr, "\mu"'] & & C' \arrow[dl, "\mu'"] \\ & X & \end{tikzcd}$$

$\varphi(p_i) = p_i'$.

$\Rightarrow \mu'(p_i') = \mu(p_i)$

$H_2(X)$

$\boxed{\overline{M}_{g,n}(X, \underline{\underline{\beta}})}$ - Space of isomorphism classes of stable maps of type (g, n, β).

This is a proper (compact) algebraic stack, on which we can do intersection theory.

We have evaluation maps

$$ev_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow X$$

$$ev_i([C, p_1, \ldots, p_n, \mu]) = \mu(p_i).$$

Back to Counting problem:

Suppose X is a projective manifold

$\beta \in H_2(X, \mathbb{Z})$, $\quad x_1, \ldots, x_n \in X$.

Q. How many genus g curves are there in X, that represent β and pass through $x_1, \ldots, x_n$.

Let us assume $\quad n = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}(X, \beta)$

then we can expect that the above number is equal to

$$\#\{[C, p_1, \ldots, p_n, \mu] \in \mathcal{M}_{g,n}(X, \beta) \mid \mu(p_i) = x_i\}$$

If $Y_i = ev_i^{-1}(x_i)$ then that number should in principle be equal to

$$\#(Y_1 \cap \ldots \cap Y_n).$$

Therefore, the moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ converts the counting problem to an intersection theory problem.

In an ideal world if $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is smooth & proper, we get this number using cup product in the cohomology ring of the moduli space.

$$\left(ev_1^*[x_1] \smile \cdots \smile ev_n^*[x_n]\right) \frown \left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right] \in \mathbb{Q}.$$

But in reality $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is not smooth. Infact it is reducible & not equidimensional.

However there is still a reasonable class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in H^{\exp \dim}(\overline{\mathcal{M}}_{g,n}(X,\beta), \mathbb{Q})$$

which plays the role of the fundamental class.

The expected dim turns out to be

$$3g-3+\dim H^0(C,\mu^*TX)-\dim H^1(C,\mu^*TX)$$

$$=-K_X\cdot\beta+(\dim X-3)(1-g).$$

Gromov-Witten Invariants:

X proj manifold $\beta\in H_2(X,\mathbb{Z})$,

$\boxed{\gamma_1,\dots,\gamma_n}\in H^*(X,\mathbb{Z})$ then

$$\boxed{\langle\gamma_1,\dots,\gamma_n\rangle_{g,\beta}}=\left[\boxed{\prod ev_i^*(\gamma_i)}\cap[\overline{M}_{g,n}(X,\beta)]^{vir}\right]$$

$$\boxed{\in\mathbb{Q}.}$$

Note that $\langle\gamma_1,\dots,\gamma_n\rangle_{g,\beta}=0$

unless $\sum\dim\gamma_i=\exp\dim\overline{M}_{g,n}(X,\beta)$.

If we take γ_i to be classes of points, then we denote the number by $N_\beta^{(g)}$.

For $\mathbb{CP}^2$, $H_2(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$ so $\beta \in \mathbb{Z}$ is just the degree of the curve and we have

$$N_\beta^{(g)}(\mathbb{CP}^2) = E_\beta^{(g)}(\mathbb{CP}^2).$$

In general it may happen that

$$N_\beta^{(g)}(X) \neq E_\beta^{(g)}(X).$$

We say that the Gromov-Witten invariants are enumerative if the equality holds.

del-Pezzo Surfaces:

A del-Pezzo surface X is a projective complex manifold of dim 2 with ample anticanonical bundle $\Lambda^2 TX$.

Equivalently a del-Pezzo surface can be

(a) $\boxed{\mathbb{CP}^2,}$ (b) $\boxed{\mathbb{CP}^1 \times \mathbb{CP}^1}$ or

(c) $\boxed{\mathbb{CP}^2 \text{ blown up at upto } 8 \text{ points.}}$

These are also called Fano surfaces.

The $\boxed{\text{ampleness of } \Lambda^2 TX}$ makes these surfaces special, for instance

Theorem (Vakil): The Gromov-Witten invariants of del-Pezzo surfaces are enumerative.

In joint work with Nilkantha Das we found recursive formula for $\boxed{E^{(1)}_{\beta}(X)}$ for a del-Pezzo surface X.

We broadly follow the techniques of Getzler, which can be summarised as follows:

(a) Getzler obtained a relation among

S^4 invariant classes in $H^4(\overline{M}_{1,4})$,

say R.

(b) There is a forgetful map

$$\pi : \overline{M}_{1,4+n}(X, \beta) \longrightarrow \overline{M}_{1,4}$$

$$[C, p_1, \ldots, p_{4+n}, \mu] \longmapsto (C, p_1, \ldots, p_4).$$

Of coure $\pi^* R = 0$.

(c) If $\mu \in H^4(X, \mathbb{Z})$ is the class of a point,

$\xi_X = c_1(TX)$, $k_\beta = \xi_X \cdot \beta$, then

$$(\pi^* R \cdot Z)[M_{1,k_\beta+2}(X,\beta)^{vir}] = 0$$

(d) We can also compute the left hand side using properties of Gromov-Witten invariants, which gives the recursion relation of $N_\beta^{(1)}$ in terms of $N_{\beta_1}^{(1)}$, and $N_{\beta_2}^{(0)}$ where $\beta_1 < \beta$.