

# A SUPERCONGRUENCE FOR GENERALIZED DOMB NUMBERS

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ABSTRACT. Using techniques due to Coster, we prove a supercongruence for a generalization of the Domb numbers. This extends a recent result of Chan, Cooper and Sica and confirms a conjectural supercongruence for numbers which are coefficients in one of Zagier's seven "sporadic" solutions to second order Apéry-like differential equations.

## 1. INTRODUCTION

It is now well-known that the *Apéry numbers*

$$A(n) := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

play a crucial role in the irrationality proof of  $\zeta(3)$ , satisfy many interesting congruences and are related to modular forms. For example, Gessel [10] showed that

$$(1) \quad A(np) \equiv A(p) \pmod{p^3}$$

for any prime  $p > 3$ , while if

$$F(z) = \frac{\eta^7(2z)\eta^7(3z)}{\eta^5(z)\eta^5(6z)} \quad \text{and} \quad t(z) = \left( \frac{\eta(6z)\eta(z)}{\eta(2z)\eta(3z)} \right)^{12},$$

then by a result of Peters and Stienstra [16], we have

$$F(z) = \sum_{n=0}^{\infty} A(n) t^n(z).$$

Here  $\eta(z)$  is the Dedekind eta-function. Since then, there have been several papers which study arithmetic properties of coefficients of power series expansions in  $t$  of modular forms where  $t$  is a modular function (see [3], [6], [7], [11], [14], [15], [19], [20]).

Our interest is in the sequence of numbers given by

$$D(n) := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}.$$

The first few terms in the sequence of *Domb numbers*  $\{D(n)\}_{n \geq 0}$  are as follows:

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1, 4, 28, 256, 2716, 31504, ...

This ubiquitous sequence (see A002895 of Sloane [18]) not only arises in the theory of third order Apéry-like differential equations [1], odd moments of Bessel functions in quantum field theory [2], uniform random walks in the plane [4], new series for  $1/\pi$  [5], interacting systems on crystal lattices [9] and the enumeration of abelian squares of length  $2n$  over an alphabet with 4 letters [17], but if

$$G(z) = \frac{\eta^4(z)\eta^4(3z)}{\eta^2(2z)\eta^2(6z)} \quad \text{and} \quad s(z) = \left( \frac{\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right)^6,$$

then (see [5])

$$G(z) = \sum_{n=0}^{\infty} (-1)^n D(n) s^n(z).$$

Motivated by (1), Chan, Cooper and Sica [6] recently proved the congruence

$$(2) \quad D(np) \equiv D(p) \pmod{p^3}.$$

The purpose of this short note is to prove a supercongruence for the *generalized Domb numbers*. Recall that the term *supercongruence* refers to congruences that are stronger than those suggested by formal group theory (for recent developments in this area, see [12], [13], [21]). For integers  $A, B$  and  $C \geq 1$ , let

$$(3) \quad D(n, A, B, C) := \sum_{k=0}^n \binom{n}{k}^A \binom{2k}{k}^B \binom{2(n-k)}{n-k}^C.$$

Our main result is the following.

**Theorem 1.1.** *Let  $A, B$  and  $C$  be integers  $\geq 1$  and  $p > 3$  be a prime. For any integers  $m, r \geq 1$ , we have*

$$D(mp^r, A, B, C) \equiv D(mp^{r-1}, A, B, C) \pmod{p^{3r}}$$

if  $A \geq 2$ .

Note that Theorem 1.1 recovers (2) in the case  $A = 2, B = C = 1, r = 1$  and generalizes a numerical observation in Section 3 of [14] (see case (xii) in Table 3). The method of proof for Theorem 1.1 is due to Coster in his influential Ph.D. thesis [8]. Namely, one expresses the summands in (3) as products  $g_{AB}(X, k)$  and  $g_{AB}^*(X, k)$  (see Section 2), then utilizes the combinatorial features of these products. One then writes (3) as two sums, one for which  $p \mid k$  and the other for which  $p \nmid k$ . In the case  $p \nmid k$ , the sum vanishes modulo an appropriate power of  $p$  while for  $p \mid k$ , the sum reduces to the required result. This strategy not only leads to a generalization of (1) (see Theorem 4.3.1 in [8]), but can be used to prove supercongruences for other similar sequences [15]. Additionally, a proof similar to that of Theorem 1.1 can be employed to show

$$D(mp^r, 1, 1, 1) \equiv D(mp^{r-1}, 1, 1, 1) \pmod{p^{2r}},$$

thereby confirming another conjectural supercongruence in Section 3 of [14] (see case (ix) in Table 2). The details are left to the interested reader. The numbers  $D(n, 1, 1, 1)$  are coefficients in one of Zagier's seven "sporadic" solutions (see #10 in Table 1 of [20] or the modular parameterization given by Case E in Table 3 of [20]) to a general family of second order Apéry-like differential equations. Our hope is that the present note will inspire others to further explore the techniques in [8]. In Section 2, we recall the relevant properties of the products  $g_{AB}(X, k)$  and  $g_{AB}^*(X, k)$  and then prove Theorem 1.1.

## 2. PROOF OF THEOREM 1.1

We first recall the definition of two products and one sum and list some of their main properties. For more details, see Chapter 4 of [8]. For integers  $A, B \geq 0$ ,  $k, j \geq 1$  and  $X$  and for a fixed prime  $p > 3$ , we define

$$g_{AB}(X, k) = \prod_{i=1}^k \left(1 - \frac{X}{i}\right)^A \left(1 + \frac{X}{i}\right)^B,$$

$$g_{AB}^*(X, k) = \prod_{\substack{i=1 \\ p \nmid i}}^k \left(1 - \frac{X}{i}\right)^A \left(1 + \frac{X}{i}\right)^B,$$

and

$$S_j(k) = \sum_{\substack{i=1 \\ p \nmid i}}^k \frac{1}{i^j}.$$

The following proposition (see Lemmas 4.2.1 and 4.2.5 in [8]) provides some of the main properties of  $g_{AB}(X, k)$ ,  $g_{AB}^*(X, k)$  and  $S_j(k)$ .

**Proposition 2.1.** *For any integers  $A, B \geq 0$ ,  $X \in \mathbb{Z}$  and integers  $m, k, r \geq 1$ , we have*

- (i)  $S_j(mp^r) \equiv 0 \pmod{p^r}$  for  $j \not\equiv 0 \pmod{p-1}$ ,
- (ii)  $S_{2j-1}(mp^r) \equiv 0 \pmod{p^{2r}}$  for  $j \not\equiv 0 \pmod{\frac{p-1}{2}}$ ,
- (iii)  $g_{AB}(pX, k) = g_{AB}^*(pX, k)g_{AB}(X, \lfloor \frac{k}{p} \rfloor)$ ,
- (iv)  $g_{AB}^*(X, k) \equiv 1 + (B-A)S_1(k)X + \frac{1}{2}((A-B)^2S_1(k)^2 - (A+B)S_2(k))X^2 \pmod{X^3}$ ,
- (v)  $\binom{n}{k}^A \binom{n+k}{k}^B = (-1)^{Ak} \left(\frac{n}{n-k}\right)^A g_{AB}(n, k)$ .

We now prove Theorem 1.1.

*Proof of Theorem 1.1.* We first note that it suffices to prove the result with  $p \nmid n$ ,  $p \nmid m$  where  $m, n \geq 1$  are integers and  $p > 3$  is a prime. We now assume that  $A \geq 2$  and  $B \geq C \geq 1$ . Recall that for integers  $m, n, r \geq 1$  with  $p \nmid n$ ,  $p \nmid m$  and  $s \geq 0$  with  $s \leq r$ , we have

$$(4) \quad \text{ord}_p \left( \frac{mp^r}{np^s} \right)^A = A(r-s).$$

Also, by Lemma 2.2 in [15], we have for a prime  $p > 3$  and integers  $m \geq 0$ ,  $r \geq 1$

$$(5) \quad \binom{2mp^r}{mp^r} \equiv \binom{2mp^{r-1}}{mp^{r-1}} \pmod{p^{3r}}.$$

Now, taking  $j = 2$  in (i),  $j = 1$  in (ii) and  $X = mp^r$ ,  $k = np^s$  in (iv) of Proposition 2.1, we have

$$(6) \quad g_{AB}^*(mp^r, np^s) \equiv 1 \pmod{p^{r+2s}}$$

for any non-negative integers  $m$ ,  $n$ ,  $r$  and  $s$  with  $s \leq r$ . Letting  $n = mp^r$ ,  $k = np^s$ ,  $A \rightarrow A + 2C$ ,  $B = 0$  in (v) and  $X = mp^{r-1}$ ,  $k = np^s$  in (iii) of Proposition 2.1, we have, for  $s \geq 1$ ,

$$(7) \quad \begin{aligned} \binom{mp^r}{np^s}^{A+2C} &= (-1)^{(A+2C)np^s} \left( \frac{mp^r}{mp^r - np^s} \right)^{A+2C} g_{(A+2C)0}(mp^r, np^s) \\ &= (-1)^{Anp^{s-1}} \left( \frac{mp^{r-1}}{mp^{r-1} - np^{s-1}} \right)^{A+2C} g_{(A+2C)0}^*(mp^r, np^s) g_{(A+2C)0}(mp^{r-1}, np^{s-1}) \\ &= \binom{mp^{r-1}}{np^{s-1}}^{A+2C} g_{(A+2C)0}^*(mp^r, np^s). \end{aligned}$$

In the last step of (7), we have applied (v) of Proposition 2.1 with  $n = mp^{r-1}$ ,  $k = np^{s-1}$ ,  $A \rightarrow A + 2C$  and  $B = 0$ . Thus,

$$(8) \quad \begin{aligned} \binom{mp^r}{np^s}^{A+2C} \binom{2np^s}{np^s}^{B-C} \binom{2mp^{r-1}}{2np^{s-1}}^C \\ = \binom{mp^{r-1}}{np^{s-1}}^{A+2C} g_{(A+2C)0}^*(mp^r, np^s) \binom{2np^s}{np^s}^{B-C} \binom{2mp^{r-1}}{2np^{s-1}}^C. \end{aligned}$$

Similarly, letting  $n = 2mp^r$ ,  $k = 2np^s$ ,  $A = C$ ,  $B = 0$  in (v) and  $X = 2mp^{r-1}$ ,  $k = 2np^s$  in (iii) of Proposition 2.1, we have

$$(9) \quad \begin{aligned} \binom{2mp^r}{2np^s}^C &= (-1)^{2Cnp^s} \left( \frac{2mp^r}{2mp^r - 2np^s} \right)^C g_{C0}(2mp^r, 2np^s) \\ &= \left( \frac{2mp^{r-1}}{2mp^{r-1} - 2np^{s-1}} \right)^C g_{C0}^*(2mp^r, 2np^s) g_{C0}(2mp^{r-1}, 2np^{s-1}) \\ &= \binom{2mp^{r-1}}{2np^{s-1}}^C g_{C0}^*(2mp^r, 2np^s). \end{aligned}$$

In the last step of (9), we have taken  $n = 2mp^{r-1}$ ,  $k = 2np^{s-1}$ ,  $A = C$  and  $B = 0$  in (v) of Proposition 2.1. By (5) and (6), we have

$$(10) \quad \binom{2np^s}{np^s}^{B-C} \equiv \binom{2np^{s-1}}{np^{s-1}}^{B-C} \pmod{p^{3s}}$$

and

$$(11) \quad g_{(A+2C)_0}^*(mp^r, np^s) \equiv g_{C_0}^*(2mp^r, 2np^s) \equiv 1 \pmod{p^{r+2s}}.$$

For  $r \geq s$ ,  $A \geq 2$  and  $C \geq 1$ , we now claim that

$$(12) \quad \frac{\binom{mp^r}{np^s}^{A+2C} \binom{2np^s}{np^s}^{B-C}}{\binom{2mp^r}{2np^s}^C} \equiv \frac{\binom{mp^{r-1}}{np^{s-1}}^{A+2C} \binom{2np^{s-1}}{np^{s-1}}^{B-C}}{\binom{2mp^{r-1}}{2np^{s-1}}^C} \pmod{p^{3r}}.$$

To see this, we first note that by (9) and (11), (10) and (11), we have

$$(13) \quad \binom{2mp^{r-1}}{2np^{s-1}}^C = \binom{2mp^r}{2np^s}^C - \gamma p^{r+2s} \binom{2mp^{r-1}}{2np^{s-1}}^C,$$

$$(14) \quad \binom{2np^s}{np^s}^{B-C} = \binom{2np^{s-1}}{np^{s-1}}^{B-C} + \alpha p^{3s}$$

and

$$(15) \quad g_{(A+2C)_0}^*(mp^r, np^s) = 1 + \beta p^{r+2s}$$

for some  $\gamma$ ,  $\alpha$  and  $\beta \in \mathbb{Z}$ . After substituting (13)–(15) into the right hand side of (8) and multiplying, we consider the following seven terms:

- (a)  $p^{r+2s} \binom{mp^{r-1}}{np^{s-1}}^{A+2C} \binom{2np^{s-1}}{np^{s-1}}^{B-C} \binom{2mp^{r-1}}{2np^{s-1}}^C$
- (b)  $p^{3s} \binom{mp^{r-1}}{np^{s-1}}^{A+2C} \binom{2mp^r}{2np^s}^C$
- (c)  $p^{r+5s} \binom{mp^{r-1}}{np^{s-1}}^{A+2C} \binom{2mp^{r-1}}{2np^{s-1}}^C$
- (d)  $p^{r+2s} \binom{2mp^r}{2np^s}^C \binom{2np^{s-1}}{np^{s-1}}^{B-C} \binom{mp^{r-1}}{np^{s-1}}^{A+2C}$
- (e)  $p^{2r+4s} \binom{2np^{s-1}}{np^{s-1}}^{B-C} \binom{mp^{r-1}}{np^{s-1}}^{A+2C} \binom{2mp^{r-1}}{2np^{s-1}}^C$
- (f)  $p^{r+5s} \binom{2mp^r}{2np^s}^C \binom{mp^{r-1}}{np^{s-1}}^{A+2C}$
- (g)  $p^{2r+7s} \binom{2mp^{r-1}}{2np^{s-1}}^C \binom{mp^{r-1}}{np^{s-1}}^{A+2C}$ .

As  $ord_p$  is at least  $3r + C(r - s)$  in each of the cases (a)–(g) above and we have (4), (12) follows. Now, using the identity

$$\binom{a-b}{c-d} \binom{b}{d} = \frac{\binom{a}{c} \binom{c}{d} \binom{a-c}{b-d}}{\binom{a}{b}},$$

we have

$$D(mp^r, A, B, C) = \binom{2mp^r}{mp^r}^C \sum_{k=0}^{mp^r} \frac{\binom{mp^r}{k}^{A+2C} \binom{2k}{k}^{B-C}}{\binom{2mp^r}{2k}^C}.$$

We now split  $D(mp^r, A, B, C)$  into two sums, namely

$$D(mp^r, A, B, C) = \binom{2mp^r}{mp^r}^C \sum_{\substack{k=0 \\ p \nmid k}}^{mp^r} \frac{\binom{mp^r}{k}^{A+2C} \binom{2k}{k}^{B-C}}{\binom{2mp^r}{2k}^C} + \binom{2mp^r}{mp^r}^C \sum_{\substack{k=0 \\ p|k}}^{mp^r} \frac{\binom{mp^r}{k}^{A+2C} \binom{2k}{k}^{B-C}}{\binom{2mp^r}{2k}^C}.$$

Since  $A \geq 2$ ,  $B \geq C \geq 1$ , the first sum vanishes modulo  $p^{3r}$  using (4) and the result then follows from reindexing the second sum and applying (5) and (12). A similar argument holds in the case  $A \geq 2$ ,  $C > B \geq 1$  upon noting that

$$\frac{\binom{mp^r}{k}^{A+2B} \binom{2(mp^r-k)}{mp^r-k}^{C-B}}{\binom{2mp^r}{2k}^B} \equiv 0 \pmod{p^{3r}}$$

if  $p \nmid k$  and

$$\frac{\binom{mp^r}{np^s}^{A+2B} \binom{2(mp^r-np^s)}{mp^r-np^s}^{C-B}}{\binom{2mp^r}{2np^s}^B} \equiv \frac{\binom{mp^{r-1}}{np^{s-1}}^{A+2B} \binom{2(mp^{r-1}-np^{s-1})}{mp^{r-1}-np^{s-1}}^{C-B}}{\binom{2mp^{r-1}}{2np^{s-1}}^B} \pmod{p^{3r}}$$

if  $p \mid k$ . □

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