# RANKIN-COHEN BRACKETS ON SIEGEL MODULAR FORMS AND SPECIAL VALUES OF CERTAIN DIRICHLET SERIES

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Abstract. Given a fixed Siegel cusp form of genus two, we consider a family of linear maps between the spaces of Siegel cusp forms of genus two by using the Rankin-Cohen brackets and then we compute the adjoint maps of these linear maps with respect to the Petersson scalar product. The Fourier coefficients of the Siegel cusp forms of genus two constructed using this method involve special values of certain Dirichlet series of Rankin type associated to Siegel cusp forms. This is a generalization of the work due to W. Kohnen (Math. Z., 207, 657-660 [1991]) and S. D. Herrero (Ramanujan J., 36, 529-536 [2015]) in case of elliptic modular forms to the case of Siegel cusp forms which is also considered earlier by M. H. Lee (Complex Var. Theory Appl. 31, 97-103 [1996]) for a special case.

## 1. Introduction

Using the existence of adjoint linear maps and properties of Poincaré series in [11] W. Kohnen constructed certain linear maps between spaces of modular forms with the property that the Fourier coefficients of the image of a form involve special values of certain Dirichlet series attached to these forms. In fact, Kohnen constructed the adjoint map with respect to the usual Petersson scalar product of the product map by a fixed cusp form. This result has been generalized by several authors to other automorphic forms (see the list [4, 5, 15, 16, 21, 23]). In particular, M. H. Lee [15] has analogous results for Siegel modular forms, where he uses $C^\infty$ Siegel modular forms and the holomorphic projection operator developed by A. A. Panchishkin in [18].

There are many interesting connections between differential operators and modular forms and many interesting results have been found. In [19, 20], Rankin gave a general description of the differential operators which send modular forms to modular forms. In [6], H. Cohen constructed bilinear operators and obtained elliptic modular forms

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with interesting Fourier coefficients. In [24, 25], Zagier studied the algebraic properties of these bilinear operators and called them Rankin–Cohen brackets.

The work of Kohnen in [11] has been generalized by S. D. Herrero recently in [8], where the author constructed the adjoint map using the Rankin-Cohen brackets by a fixed cusp form instead of the product map. Recently, we extended the work of S. D. Herrero to the case of Jacobi forms [10] which generalise the result of H. Sakata in [21].

Rankin-Cohen brackets for Siegel modular forms of genus two were studied in [3] explicitly and existence of recursion formula in [7] for general genus. In this article we generalise the work of S. D. Herrero to the case of Siegel modular forms of genus two. We follow the same exposition as given in [10].

## 2. Preliminaries on Sigel Modular forms

Let $\mathcal{H}_2 := \{Z = X + iY \in M_{2\times 2}(\mathbb{C}) \mid Z^t = Z,\ Y > 0\}$ be the Siegel upper half-plane. Let $\Gamma_2$ be the symplectic group $Sp_4(\mathbb{Z})$ of genus 2 defined as

$$\Gamma_2 := \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \mid A, B, C, D \in M_{2\times 2}(\mathbb{Z}), AB^t = BA^t, CD^t = DC^t, AD^t - BC^t = I_2 \right\}.$$

The group $\Gamma_2$ acts on $\mathcal{H}_2$ via

$$\left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

Let $k$ be a fixed positive integer. If $M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \Gamma_2$ and $F$ be a complex valued function on $\mathcal{H}_2$, then define

$$F|_k M := det(CZ + D)^{-k} F(M \cdot Z).$$

A Siegel modular form of weight $k$ and genus 2 is a holomorphic function $F : \mathcal{H}_2 \to \mathbb{C}$ satisfying $F|_k M = F,\ \forall M \in \Gamma_2$ and having a Fourier expansion of the form

$$F(Z) = \sum_{T \geq 0} A(T) e^{2\pi i (tr(TZ))}, \tag{1}$$

where the summation runs over positive semidefinite half-integral $(i.e., 2t_{ij},\ t_{ii} \in \mathbb{Z})$ $2 \times 2$ matrices $T$. We denote the space of Siegel modular forms of weight $k$ and genus 2 on $\Gamma_2$ by $M_k(\Gamma_2)$. Further, we say $F$ is a cusp form if and only if the summation in (1) runs over positive definite half-integral matrices $T$. We denote the space of all Siegel cusp forms by $S_k(\Gamma_2)$. One has the following estimate for the Fourier coefficients of Siegel cusp forms of genus 2.

**Theorem 2.1.** [12, 13, 1] *Let $F$ be a Siegel cusp form of weight $k$ and genus 2 with Fourier coefficients $A(T)$. Then*

$$A(T) <<_{F,\epsilon} (det\ T)^{k/2 - 13/36 + \epsilon} \qquad (\epsilon > 0). \tag{2}$$

The Petersson scalar product on $S_k(\Gamma_2)$ is defined as

$$\langle F, G\rangle = \int_{\Gamma_2\backslash\mathcal{H}_2} F(Z)\overline{G(Z)}(det\ Y)^k dZ,$$

where $Z = X + iY$ and $dZ = (det\ Y)^{-3}dXdY$ is an invariant measure under the action of $\Gamma_2$ on $\mathcal{H}_2$. The space $(S_k(\Gamma_2), \langle,\rangle)$ is a finite dimensional Hilbert space. For basic theory on Siegel modular forms, we refer to [2, 17].

### 2.1. **Poincaŕe series.**

**Definition 2.2.** *Let $k \geqslant 6$ be a fixed positive integer and $T$ be a fixed symmetric positive definite half-integral $2\times 2$ matrix. Then the $T$-th Poincaŕe series of weight $k$ and genus 2 is defined as*

$$P_{k,T}(Z) := \sum_{M\in\Delta\backslash\Gamma_2} e^{2\pi i(tr(TZ))}|_k M. \tag{3}$$

*Here $\Delta := \left\{\begin{pmatrix} I_2 & S\\ 0 & I_2\end{pmatrix} \mid S\in M_{2\times 2}(\mathbb{Z}), S^t\ = S\right\}$ is a subgroup of $\Gamma_2$. It is well known that $P_{k,T}\in S_k(\Gamma_2)$.*

The Poincaŕe series has the following property.

**Lemma 2.3.** [14] *Let $F\in S_k(\Gamma_2)$ with Fourier expansion*

$$F(Z) = \sum_{T>0} A(T)e^{2\pi i(tr(TZ))}.$$

*Then*

$$\langle F, P_{k,T}\rangle = c_k(det\ T)^{-k+\frac{3}{2}}A(T), \tag{4}$$

*where*

$$c_k = 2\sqrt{\pi}(4\pi)^{3-2k}\Gamma(k-3/2)\Gamma(k-2).$$

### 2.2. **Rankin-Cohen Brackets for Siegel modular forms of genus 2.**

Let $k, l$ be positive integers and $\nu \geq 0$ be an integer. Let $F$ and $G$ be two complex valued holomorphic functions on $\mathcal{H}_2$. Let $\mathbb{D}$ be the differential operator defined by

$$\mathbb{D} := 4\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'} - \frac{\partial^2}{\partial z^2}, \quad \text{for} \quad Z = \begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix}\in\mathcal{H}_2.$$

Define the $\nu$-th Rankin-Cohen bracket of $F$ and $G$ by

$$[F, G]_\nu := \sum_{r+s+p=\nu} C_{r,s,p}(k,l)\ \mathbb{D}^p(\mathbb{D}^r(F)\mathbb{D}^s(G)), \tag{5}$$

where

$$C_{r,s,p}(k,l) = \frac{(k+\nu-3/2)_{s+p}}{r!}\frac{(l+\nu-3/2)_{r+p}}{s!}\frac{(-(k+l+\nu-3/2))_{r+s}}{p!},$$

and

$$(x)_m = \prod_{0\leqslant i\leqslant m-1} (x-i).$$

*Remark* 2.1. The 0-th Rankin-Cohen bracket is the usual product i.e., $[F,G]_0 = FG$.

*Remark* 2.2. One has the following relation

$$[F|_k M, G|_l M]_\nu = [F,G]|_{k+l+2\nu} M,\ \forall M \in \Gamma_2. \tag{6}$$

**Theorem 2.4.** [3, 7] *Let $k$ and $l$ are positive integers and $\nu \geqslant 0$ be an integer. Let $F \in M_k(\Gamma_2)$ and $G \in M_l(\Gamma_2)$. Then*

$$[F,G]_\nu \in M_{k+l+2\nu}(\Gamma_2).$$

*If $\nu > 0$, then*

$$[F,G]_\nu \in S_{k+l+2\nu}(\Gamma_2).$$

2.3. **Certain Dirichlet series associated to Siegel modular forms.** Let $F \in S_k(\Gamma_2)$ and $G \in S_l(\Gamma_2)$ with Fourier coefficients $A(T)$ and $B(T)$ respectively, then for a fixed positive definite $2\times 2$ matrix $S$ and a non-negative integer $m$, the Dirichlet series $L_{F,G;S,m}$ is defined as

$$L_{F,G;S,m}(\sigma) = \sum_{T>0} \frac{det\ (T)^m A(T+S)\overline{B(T)}}{(det\ (T+S))^\sigma}. \tag{7}$$

The above series converges for $Re(\sigma) > \frac{k+l}{2} - m + \frac{5}{18}$. Using the special values of the above series for $m = 0$, M. H. Lee [15] constructed Siegel cusp forms using the adjoint w.r.t. the Petersson scalar product of the product map by a fixed Siegel cusp form, which generalises the work of Kohnen in case of elliptic cusp forms [11]. The series for $m = 0$, also appeared [5] in the study of mixed Siegel modular forms. We use the Rankin-Cohen bracket of Siegel modular forms of genus 2 and generalize the work of M. H. Lee, and hence the work of S. D. Herrero [8] in the case elliptic modular forms. Now we state the main theorem.

## 3. Statement of the Theorem

For a fixed $G \in S_l(\Gamma_2)$ and an integer $\nu \geq 0$, consider the map

$$T_{G,\nu} : S_k(\Gamma_2) \to S_{k+l+2\nu}(\Gamma_2)$$

defined by

$$T_{G,\nu}(F) = [F,G]_\nu.$$

Then $T_{G,\nu}$ is a $\mathbb{C}$-linear map between finite dimensional Hilbert spaces and therefore has an adjoint map $T^*_{G,\nu} : S_{k+l+2\nu}(\Gamma_2) \to S_k(\Gamma_2)$ given by

$$\langle F, T_{G,\nu}(H)\rangle = \langle T^*_{G,\nu}(F), H\rangle,\ \forall F \in S_{k+l+2\nu}(\Gamma_2) \text{ and } H \in S_k(\Gamma_2).$$

Now we compute the Fourier coefficients of $T^*_{G,\nu}(F)$ in terms of the special values of Dirichlet series defined in (7).

**Theorem 3.1.** *Let $k \geqslant 6, l$ be natural numbers and $\nu \geq 0$ be a fixed integer. Let $G \in S_l(\Gamma_2)$ with Fourier expansion*

$$G(Z) = \sum_{T_1 > 0} A(T_1) e^{2\pi i(tr(T_1 Z))}.$$

*Then the image of any cusp form $F \in S_{k+l+2\nu}(\Gamma_2)$ with Fourier expansion*

$$F(Z) = \sum_{T_2 > 0} B(T_2) e^{2\pi i(tr(T_2 Z))},$$

*under $T^*_{G,\nu}$ is given by*

$$T^*_{G,\nu}(F)(Z) = \sum_{T>0} C(T) e^{2\pi i(tr(TZ))},$$

*where*

$$C(T) = \alpha(k,l,\nu) \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^{k+r-3/2}\ L_{F,G;T,s}(k+l+2\nu-(p+3/2)),$$

*with*

$$\alpha(k,l,\nu) = \frac{(-1)^\nu \Gamma_2(k+l+2\nu-\frac{3}{2})}{2\sqrt{\pi}\Gamma(k-\frac{3}{2})\Gamma(k-2)(4\pi)^{2(l+\nu)}},$$

*and the Gamma function $\Gamma_2(\sigma)$ is defined as*

$$\Gamma_2(\sigma) = \int_{\mathbb{Y}} e^{-trY}(det\ Y)^{\sigma-3/2} dY, \quad for\ Re(\sigma) > 3/2, \tag{8}$$

*where $\mathbb{Y} = \{Y \in M_{2\times 2}(\mathbb{C}) \mid Y^t = Y > 0\}$.*

*Remark* 3.1. Using the estimate given in Theorem 2.1, one can show that the above series converges absolutely.

*Remark* 3.2. Fixing $G(Z) \in S_l(\Gamma_2)$ and suppose that $S_k(\Gamma_2)$ is the one dimensional space generated by $F(Z)$, then applying the above theorem we get $T^*_{G,\nu}(H)(Z) = \alpha_G F(Z)$ for some constant $\alpha_G$ and for all $H \in S_{k+l+2\nu}(\Gamma_2)$. Now equating the $T$-th Fourier coefficients both the sides, we get a relation among the special values of the associated Dirichlet series $L_{H,G;T,m}$ (7) with the $T$-th Fourier coefficients of $F(Z)$. For example taking $G = \chi_{10}$ the Igusa cusp form of weight 10 (see [9]) and $H = \chi_{10}^2$, then $T^*_{\chi_{10},0}(\chi_{10}^2) = \alpha_{\chi_{10}}\chi_{10}$ for some constant $\alpha_{\chi_{10}}$ and then equating the $T$-th Fourier coefficients on both sides, we get a relation among the special values of the associated Dirichlet series $L_{\chi_{10}^2,\chi_{10};T,0}$ with the $T$-th Fourier coefficients of $\chi_{10}$. Similarly taking $G = \chi_{10}$ and $H = \Upsilon_{20}$ (the Hecke-eigen form of weight 20, see [22]) then $T^*_{\chi_{10},0}(\Upsilon_{20}) = \beta_{\chi_{10}}\chi_{10}$ for some constant $\beta_{\chi_{10}}$ and then equating the $T$-th Fourier coefficients on both sides, we get a relation among the special values of the associated Dirichlet series $L_{\Upsilon_{20},\chi_{10};T,0}$ with the $T$-th Fourier coefficients of $\chi_{10}$.

## 4. Proof of Theorem 3.1

We need the following lemma to proof the main theorem.

**Lemma 4.1.** *Using the same notation as in Theorem 3.1, the sum*

$$\sum_{M\in\Delta\backslash\Gamma_2}\int_{\Gamma_2\backslash\mathcal{H}_2}|F(Z)\overline{[e^{2\pi i(tr(TZ))}\mid_k M,G]_\nu(Z)}\,(det\ Y)^{k+l+2\nu}|dZ$$

*converges.*

*Proof.* One can follow the same method as given in Lemma 1 in [8] or Lemma 4.1 in [10] to get a proof. □

Now we give a proof of Theorem 3.1. Write

$$T^*_{G,\nu}(F)(Z)=\sum_{T>0}C(T)e^{2\pi i(tr(TZ))}.$$

Using Lemma 2.3, we have

$$\langle T^*_{G,\nu}F,P_{k,T}\rangle=c_k(det\ T)^{-k+\frac{3}{2}}C(T),$$

where

$$c_k=2\sqrt{\pi}(4\pi)^{3-2k}\Gamma(k-3/2)\Gamma(k-2).$$

On the other hand, by definition of the adjoint map we have

$$\langle T^*_{G,\nu}F,P_{k,T}\rangle=\langle F,T_{G,\nu}(P_{k,T})\rangle=\langle F,[P_{k,T},G]_\nu\rangle.$$

Hence we get

$$C(T)=\frac{(det\ T)^{k-\frac{3}{2}}}{c_k}\langle F,[P_{k,T},G]_\nu\rangle. \tag{9}$$

By definition,

$$\begin{aligned}
\langle F,[P_{k,T},G]_\nu\rangle &= \int_{\Gamma_2\backslash\mathcal{H}_2}F(Z)\overline{[P_{k,T},G]_\nu\,(Z)}\,(det\ Y)^{k+l+2\nu}dZ\\
&= \int_{\Gamma_2\backslash\mathcal{H}_2}F(Z)\overline{[\sum_{M\in\Delta\backslash\Gamma_2}e^{2\pi i(tr(TZ))}\mid_k M,G]_\nu(Z)}\,(det\ Y)^{k+l+2\nu}dZ\\
&= \int_{\Gamma_2\backslash\mathcal{H}_2}F(Z)\sum_{M\in\Delta\backslash\Gamma_2}\overline{[e^{2\pi i(tr(TZ))}\mid_k M,G]_\nu(Z)}\,(det\ Y)^{k+l+2\nu}dZ\\
&= \int_{\Gamma_2\backslash\mathcal{H}_2}\sum_{M\in\Delta\backslash\Gamma_2}F(Z)\overline{[e^{2\pi i(tr(TZ))}\mid_k M,G]_\nu(Z)}\,(det\ Y)^{k+l+2\nu}dZ.
\end{aligned}$$

By Lemma 4.1, we can interchange the sum and the integration in $\langle F,[P_{k,T},G]_\nu\rangle$, which gives

$$\langle F,[P_{k,T},G]_\nu\rangle=\sum_{M\in\Delta\backslash\Gamma_2}\int_{\Gamma_2\backslash\mathcal{H}_2}F(Z)\overline{[e^{2\pi i(tr(TZ))}\mid_k M,G]_\nu(Z)}\,(det\ Y)^{k+l+2\nu}dZ.$$

Using the change of variable $Z$ to $M^{-1}\cdot Z$ in each integral and using (6), we get

$$\langle F, [P_{k,T}, G]_\nu\rangle = \sum_{M\in\Delta\backslash\Gamma_2}\int_{M\cdot\Gamma_2\backslash\mathcal{H}_2} F(Z)\overline{[e^{2\pi i(tr(TZ))}, G]_\nu(Z)}\,(det\ Y)^{k+l+2\nu}dZ.$$

Now using the Rankin unfolding argument, we see that the above sum equals to

$$\begin{aligned}
&\int_{\Delta\backslash\mathcal{H}_2} F(Z)\overline{[e^{2\pi i(tr(TZ))}, G]_\nu(Z)}\,(det\ Y))^{k+l+2\nu}dZ\\
&=\int_{\Delta\backslash\mathcal{H}_2} F(Z)\sum_{r+s+p=\nu} C_{r,s,p}(k,l)\overline{\mathbb{D}^p(\mathbb{D}^r(e^{2\pi i(tr(TZ))})\mathbb{D}^s(G(Z)))}\,(det\ Y)^{k+l+2\nu}dZ\\
&=\sum_{r+s+p=\nu} C_{r,s,p}(k,l)\int_{\Delta\backslash\mathcal{H}_2} F(Z)\,\overline{\mathbb{D}^p(\mathbb{D}^r(e^{2\pi i(tr(TZ))})\mathbb{D}^s(G(Z)))}\,(det\ Y)^{k+l+2\nu}dZ. \quad (10)
\end{aligned}$$

The repeated action of the operator $\mathbb{D}$ on $e^{2\pi i(tr(TZ))}$ gives

$$\mathbb{D}^r(e^{2\pi i(tr(TZ))}) \ = \ (4\pi i)^{2r}(det\ T)^r e^{2\pi i(tr(TZ))},$$

and

$$\mathbb{D}^s(G(Z)) = (4\pi i)^{2s}\sum_{T_1>0} A(T_1)(det\ T_1)^s e^{2\pi i(tr(T_1Z))}.$$

Now replacing $F$ and $G$ by their Fourier expansions in (10), we have

$$\begin{aligned}
\langle F, [P_{k,T}, G]_\nu\rangle \ &= \ (4\pi i)^{2\nu}\sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r\int_{\Delta\backslash\mathcal{H}_2}\left(\sum_{T_2>0} B(T_2)e^{2\pi i(tr(T_2Z))}\right)\\
&\qquad\times\left(\sum_{T_1>0}(det\ T_1)^s(det(T+T_1))^p\overline{A(T_1)}e^{-2\pi i\overline{tr(T+T_1)Z}}\right)(det\ Y)^{k+l+2\nu}dZ\\
&= (4\pi i)^{2\nu}\sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r\sum_{T_2>0}\sum_{T_1>0}(det\ T_1)^s(det(T+T_1))^p\,\overline{A(T_1)}\,B(T_2)\\
&\qquad\times\int_{\Delta\backslash\mathcal{H}_2} e^{2\pi i(tr(T_2Z))}\ e^{-2\pi i\overline{tr(T+T_1)Z}}(det\ Y)^{k+l+2\nu}dZ\\
&= \quad (4\pi i)^{2\nu}\sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r\sum_{T_2>0}\sum_{T_1>0}(det\ T_1)^s(det(T+T_1))^p\,\overline{A(T_1)}\,B(T_2)\\
&\qquad\times\int_{\Delta\backslash\mathcal{H}_2} e^{2\pi i(tr(T_2-(T+T_1))X)))}\ e^{-2\pi(tr(T_2+T+T_1)Y)}(det\ Y)^{k+l+2\nu}\frac{dXdY}{det\ (Y)^3}. \quad (11)
\end{aligned}$$

We know that the set $\mathbb{F} := \{Z = X+iY \in \mathcal{H}_2 \mid X\in\mathbb{X}, Y\in\mathbb{Y}\}$ is a fundamental domain for the action of $\Delta$ on $\mathcal{H}_2$, where

$$\mathbb{X} = \left\{X = \begin{pmatrix} u & x\\ x & u'\end{pmatrix} \mid \frac{-1}{2}\leqslant u\leqslant\frac{1}{2},\ \frac{-1}{2}\leqslant x\leqslant\frac{1}{2},\ \frac{-1}{2}\leqslant u'\leqslant\frac{1}{2}\right\},$$

and

$$\mathbb{Y} = \left\{ Y \in M_{2\times 2}(\mathbb{C}) \mid Y^t = Y > 0 \right\}.$$

Integrating over this fundamental domain, $\langle F, [P_{k,T}, G]_\nu \rangle$ equals

$$(4\pi i)^{2\nu} \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r \sum_{T_2>0} \sum_{T_1>0} (det\ T_1)^s (det(T+T_1))^p\ \overline{A(T_1)}\ B(T_2)$$
$$\times \int_{\mathbb{X}} \int_{\mathbb{Y}} e^{2\pi i(tr(T_2-(T+T_1))X)))}\ e^{-2\pi(tr(T_2+T+T_1)Y)} (det\ Y)^{k+l+2\nu-3} dXdY.$$

Integrating on $\mathbb{X}$ first, $\langle F, [P_{k,T}, G]_\nu \rangle$ equals

$$(4\pi i)^{2\nu} \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r \sum_{T_1>0} (det\ T_1)^s (det(T+T_1))^p\ \overline{A(T_1)}\ B(T+T_1)$$
$$\times \int_{\mathbb{Y}} e^{-4\pi(tr(T+T_1)Y)} (det\ Y)^{k+l+2\nu-3} dY. \tag{12}$$

Now integration over $\mathbb{Y}$ gives

$$\int_{\mathbb{Y}} e^{-4\pi(tr(T+T_1)Y)} (det\ Y)^{k+l+2\nu-3} dY \ = \ \frac{\Gamma_2\left(k+l+2\nu-\frac{3}{2}\right)}{(det\ (4\pi(T+T_1)))^{k+l+2\nu-\frac{3}{2}}}. \tag{13}$$

Substituting the value from (13) in (12), $\langle F, [P_{k,T}, G]_\nu \rangle$ equals

$$(-1)^\nu \frac{\Gamma_2\left(k+l+2\nu-\frac{3}{2}\right)}{(4\pi)^{2(k+l+\nu-\frac{3}{2})}} \sum_{r+s+p=\nu} C_{r,s,p}(k,l)(det\ T)^r \sum_{T_1>0} \frac{(det\ T_1)^s\ \overline{A(T_1)}\ B(T+T_1)}{(det(T+T_1))^{k+l+2\nu-(p+\frac{3}{2})}}. \tag{14}$$

Now substituting $\langle F, [P_{k,T}, G]_\nu \rangle$ from (14) in (9), we get the required expression for $C(T)$ as given in Theorem 3.1.

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