ON THE NUMBER OF REPRESENTATIONS OF CERTAIN QUADRATIC FORMS IN 20 AND 24 VARIABLES

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Abstract. In this paper, we find the number of representations of certain quadratic forms in 20 and 24 variables. We get this as an application of the evaluation of certain triple convolution sums of the divisor functions. Further, by comparing our formulas with that of Lomadze, we get expressions of certain cusp forms in terms of some finite sums involving the solution set of the quadratic form representation.

1. Introduction

For positive integers a, b, s, t, define the convolution sum $W_{a,b}^{s,t}(n)$ by

$$W_{a,b}^{s,t}(n) := \sum_{\substack{l,m \in \mathbb{N} \\ al+bm=n}} \sigma_s(l)\sigma_t(m). \tag{1}$$

When $s = t = 1$, it is denoted by $W_{a,b}(n)$, and $W_{a,1}(n) = W_{1,a}(n)$ is denoted by $W_a(n)$. These type of sums were evaluated as early as the 19th century. For example, the sum $W_1(n)$ was evaluated by M. Besge, J. W. L. Glaisher and S. Ramanujan [2, 4, 14]. Some of the convlotution sums of the above type have been obtained by several authors (see for example [5, 15, 12, 17] and also the works of K. S. Williams and his co-authors ([16] and the references therein)).

We now define the triple convolution sums of the divisor functions by

$$W_{a,b,c}^{r,s,t}(n) := \sum_{\substack{l,m,p \in \mathbb{N} \\ al+bm+cp=n}} \sigma_r(l)\sigma_s(m)\sigma_t(p), \tag{2}$$

where $a, b, c, r, s, t \in \mathbb{N}$. We write $W_{a,b,c}^{1,1,1}(n) = W_{a,b,c}(n)$ for $a, b, c \in \mathbb{N}$. In [1], Alaca et al. evaluated the convolution sums $W_{1,2,2}(n), W_{1,1,2}(n)$ and $W_{1,2,4}(n)$ by expressing the product of Eisentein series in terms of their derivatives. In [7, p.11], Kim et. al have treated the convolution sum $W_{1,1,1}(n)$ and as an application, they prove that certain q-series satisfy a particular diffential equation. Using the theory of modular forms and quasimodular forms, in this article, we evaluate the convolution sums $W_{a,b,c}^{1,3,3}(n)$, where $(a, b, c) \in \{(1, 1, 1), (1, 1, 3), (1, 3, 3), (3, 1, 1), (3, 3, 1)\}$ and $W_{a,b,c}^{3,3,3}(n)$, where $(a, b, c) \in \{(1, 1, 1), (1, 1, 3), (1, 3, 3)\}$. As an application, we find formulas for the number of representations of the quadratic form

$$F_k : x_1^2 + x_1x_2 + x_2^2 + \ldots + x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2,$$

when $k = 10, 12$. Let

$$s_{2k}(n) = \text{card}\left\{(x_1, x_2, \cdots, x_{2k}) \in \mathbb{Z}^{2k} : F_k(x_1, x_2, \cdots, x_{2k}) = n\right\}$$

Date: September 11, 2015.

2010 *Mathematics Subject Classification.* Primary 11A25, 11E25; Secondary 11E20, 11F11.

Key words and phrases. convolution sums of the divisor functions, representation numbers of quadratic forms, modular forms of one variable.

be the number of representations of a positive integer n by the quadratic form F_k. For $k = 2, 4, 6, 8$ formulas for s_{2k} are known due to the works of J. Liouville [9], J. G. Huard et. al. [5], O. X. M. Yao and E. X. W. Xia [17] and the authors [13]. In [10], G. A. Lomadze gave formulas for $s_{2k}(n)$ for $2 \leq n \leq 17$, which involves the divisor functions and certain finite sums which involve the solution set of the representation of same quadratic forms of lower variables. However, the other formulas mentioned above are in terms of divisor functions and Fourier coefficients of certain cusp forms. Like in the works of [17] and [13], by comparing the formulas of Lomadze with our results, we also obtain identities connecting the Fourier coefficients of certain cusp forms in terms of finite sums (see Corollary 2.5).

2. Preliminaries and Statement of the Results

Let $M_k(N)$ be the space of modular forms of weight k for the congruence subgroup $\Gamma_0(N)$ and $S_k(N)$ be the subspace of cusp forms of weight k for the congruence subgroup $\Gamma_0(N)$. For $k \geq 4$, let E_k denote the normalized Eisenstein series of weight k in $M_k(1)$ given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where $q = e^{2i\pi z}$ and B_k is the k-th Bernoulli number defined by $\displaystyle\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m$. The first few Eisenstein series are given as follows:

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,\ E_6(z) = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n,\ E_8(z) = 1 + 480 \sum_{n \geq 1} \sigma_7(n) q^n, \\ E_{10}(z) &= 1 - 264 \sum_{n \geq 1} \sigma_9(n) q^n,\ E_{12}(z) = 1 + \frac{65520}{691} \sum_{n \geq 1} \sigma_{11}(n) q^n. \end{aligned} \tag{3}$$

The following identity is well-known from the fact that $E_8 = E_4^2$:

$$W_{1,1}^{3,3}(n) = \frac{1}{120} \sigma_7(n) - \frac{1}{120} \sigma_3(n). \tag{4}$$

In order to evaluate the convolutions sums $W_{a,b,c}^{1,s,t}(n)$, we use the structure theorem on quasimodular forms of weight k and depth $\leq k/2$. For details on basics of modular forms and quasimodular forms, we refer the reader to [3, 6, 11]. The Eisenstein series E_2, which is a quasimodular form of weight 2, depth 1 on $SL_2(\mathbb{Z})$ is given by

$$E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma(n) e^{2\pi i n z}$$

and this fundamental quasimodular form will be used in our results. The space of quasimodular forms of weight k, depth $\leq k/2$ on $\Gamma_0(N)$ is denoted by $\tilde{M}_k^{\leq k/2}(N)$. We need the following structure theorem (see [6, 11]). For an even integer k with $k \geq 2$, we have

$$\tilde{M}_k^{\leq k/2}(N) = \bigoplus_{j=0}^{k/2-1} D^j M_{k-2j}(N) \oplus \mathbb{C} D^{k/2-1} E_2, \tag{5}$$

where the differential operator D is defined by $D := \frac{1}{2\pi i} \frac{d}{dz}$. Using this one can express each quasimodular form of weight k and depth $\leq k/2$ as a linear combination of j-th derivatives of modular forms of weight $k - 2j$ on $\Gamma_0(N)$, $0 \leq j \leq k/2 - 1$ and the $(k/2 - 1)$-th derivate of the quasimodular form E_2.

We need the following newforms for our results. Let $\Delta(z) = \sum_{n\geq 1}\tau(n)q^n = \eta^{24}(z)$ be the well-known unique normalized cusp form of weight 12, level 1, studied by Ramanujan. Here $\eta(z)$ is the Dedekind eta function given by

$$\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n).$$

Let $\{\Delta_{k,N,j} : 1 \leq j \leq d\}$ be the basis (of dimension d) of normalized newforms of weight k, level N, having Fourier expansion

$$\Delta_{k,N,j}(z) = \sum_{n\geq 1}\tau_{k,N,j}(n)q^n.$$

If $d = 1$, then we write the function as $\Delta_{k,N}$ and its Fourier coefficients as $\tau_{k,N}(n)$.

The following are the main theorems of this section.

Theorem 2.1. *Let* $n \in \mathbb{N}$, *then*

$$\begin{aligned}
W^{1,3,3}_{1,1,1}(n) &= \frac{1}{240^2}\big[11\sigma_9(n) + 10(2-3n)\sigma_7(n) - 42\sigma_5(n) + 20(3n-1)\sigma_3(n) + \sigma(n)\big],\\
W^{1,3,3}_{1,1,3}(n) &= \frac{1}{240^2}\big[\frac{91}{671}\sigma_9(n) + \frac{7290}{7381}\sigma_9\left(\frac{n}{3}\right) - \frac{15}{41}n\sigma_7(n) - \frac{1215}{41}n\sigma_7\left(\frac{n}{3}\right) + \frac{10}{41}\sigma_7(n)\\
&\quad + \frac{810}{41}\sigma_7\left(\frac{n}{3}\right) - \frac{276}{13}\sigma_5(n) - \frac{270}{13}\sigma_5\left(\frac{n}{3}\right) + 30n\sigma_3(n) + 30n\sigma_3\left(\frac{n}{3}\right)\\
&\quad - 10\sigma_3(n) + 10\sigma_3\left(\frac{n}{3}\right) + \sigma(n) - \frac{280}{61}\tau_{10,3,1}(n) + 115\tau_{10,3,2}(n)\\
&\quad - \frac{600}{41}n\tau_{8,3}(n) + \frac{400}{41}\tau_{8,3}(n) - \frac{10}{13}\tau_{6,3}(n)\big],\\
W^{1,3,3}_{1,3,3}(n) &= \frac{1}{240^2}\big[\sigma(n) - 20(1-3n)\sigma_3\left(\frac{n}{3}\right) - \frac{6}{13}\sigma_5(n) - \frac{540}{13}\sigma_5\left(\frac{n}{3}\right) - \frac{20}{13}\tau_{6,3}(n)\\
&\quad + 10(2-3n)\sigma_7\left(\frac{n}{3}\right) + \frac{11}{7381}\sigma_9(n) + \frac{7380}{671}\sigma_9\left(\frac{n}{3}\right) + \frac{160}{549}\tau_{10,3,1}(n) + \frac{70}{99}\tau_{10,3,2}(n)\big],\\
W^{1,3,3}_{3,1,1}(n) &= \frac{1}{240^2}\big[\sigma\left(\frac{n}{3}\right) + 20(n-1)\sigma_3(n) - \frac{60}{13}\sigma_5(n) - \frac{486}{13}\sigma_5\left(\frac{n}{3}\right) + \frac{60}{13}\tau_{6,3}(n)\\
&\quad + 10(2-n)\sigma_7(n) + \frac{890}{671}\sigma_9(n) + \frac{6561}{671}\sigma_9\left(\frac{n}{3}\right) + \frac{480}{61}\tau_{10,3,1}(n) - \frac{210}{11}\tau_{10,3,2}(n)\big],\\
W^{1,3,3}_{3,3,1}(n) &= \frac{1}{240^2}\big[\sigma\left(\frac{n}{3}\right) + 10(n-1)\sigma_3\left(\frac{n}{3}\right) + 10(n-1)\sigma_3(n) - \frac{516}{13}\sigma_5\left(\frac{n}{3}\right)\\
&\quad - \frac{30}{13}\sigma_5(n) + \frac{30}{13}\tau_{6,3}(n) + \frac{10-5n}{41}\sigma_7(n) + \frac{810-405n}{41}\sigma_7\left(\frac{n}{3}\right)\\
&\quad + \frac{400}{41}\tau_{8,3}(n) + \frac{10}{671}\sigma_9(n) + \frac{7371}{671}\sigma_9\left(\frac{n}{3}\right) - \frac{280}{183}\tau_{10,3,1}(n)\\
&\quad - \frac{115}{33}\tau_{10,3,2}(n) - \frac{200}{41}n\tau_{8,3}(n)\big].
\end{aligned}$$

Theorem 2.2. *Let $n \in \mathbb{N}$, then*

$$\begin{aligned}
W_{1,1,1}^{3,3,3}(n) &= \frac{1}{19200}\sigma_3(n) - \frac{1}{9600}\sigma_7(n) + \frac{91}{13267200}\sigma_{11}(n) + \frac{1}{22112}\tau(n),\\
W_{1,1,3}^{3,3,3}(n) &= \frac{41}{484252800}\sigma_{11}(n) + \frac{6561}{968505600}\sigma_{11}\left(\frac{n}{3}\right) - \frac{7}{196800}\sigma_7(n) - \frac{9}{131200}\sigma_7\left(\frac{n}{3}\right)\\
&\quad + \frac{1}{28800}\sigma_3(n) + \frac{1}{57600}\sigma_3\left(\frac{n}{3}\right) - \frac{133}{720\times 11747}\tau(n) + \frac{145071}{160\times 11747}\tau\left(\frac{n}{3}\right)\\
&\quad - \frac{1}{29520}\tau_{8,3}(n) + \frac{1}{16\times 1241}\tau_{12,3}(n),\\
W_{1,3,3}^{3,3,3}(n) &= \frac{91}{19200\times 691\times 6643}\sigma_{11}(n) + \frac{91\times 6642}{19200\times 691\times 6643}\sigma_{11}\left(\frac{n}{3}\right) - \frac{1}{1180800}\sigma_7(n)\\
&\quad - \frac{61}{590400}\sigma_7\left(\frac{n}{3}\right) + \frac{1}{240^2}\sigma_3(n) + \frac{1}{28800}\sigma_3\left(\frac{n}{3}\right) + \frac{199}{1440\times 11747}\tau(n)\\
&\quad - \frac{1197}{80\times 11747}\tau\left(\frac{n}{3}\right) - \frac{1}{120\times 246}\tau_{8,3}(n) + \frac{1}{144\times 1241}\tau_{12,3}(n).
\end{aligned}$$

We apply the above convolution sums to derive the following theorem.

Theorem 2.3. *The number of representations of a positive integer n by the quadratic form F_{10} is given by*

$$s_{20}(n) = \frac{12}{11}\sigma_9^*(n) + \frac{648}{11}\tau_{10,3,2}(n),$$

where $\sigma_9^*(n) = \sigma_9(n) - 3^5\sigma_9\left(\frac{n}{3}\right)$.

Theorem 2.4. *The number of representations of a positive integer n by the quadratic form F_{12} is given by*

$$s_{24}(n) = \frac{6552}{50443}\sigma_{11}^*(n) + \frac{402624}{11747}\tau(n) + \frac{293512896}{11747}\tau\left(\frac{n}{3}\right) + \frac{46656}{1241}\tau_{12,3}(n),$$

where $\sigma_{11}^*(n) = \sigma_{11}(n) + 3^6\sigma_{11}\left(\frac{n}{3}\right)$.

Corollary 2.5. *Comparing our formulas in Theorem 2.3 and Theorem 2.4 with the formulas (IX) and (XI) in p. 12 of* [10]*, we get the following identities:*

$$\tau_{10,3,2}(n) = \frac{1}{120}\sum_{F_6(x_1,\cdots,x_{12})=n}(42x_1^4 - 27nx_1^2 + n^2), \tag{6}$$

$$\begin{aligned}
&\frac{402624}{11747}\tau(n) + \frac{293512896}{11747}\tau\left(\frac{n}{3}\right) + \frac{46656}{1241}\tau_{12,3}(n)\\
&= \frac{291096}{1765505}\sum_{F_8(x_1,\cdots,x_{16})=n}(135x_1^4 - 54nx_1^2 + 2n^2) + \frac{864}{50443}\sum_{F_6(x_1,\cdots,x_{12})=n}(162x_1^6 - 162nx_1^4 + 36n^2x_1^2 - n^3)\\
&\quad + \frac{30}{50443}\sum_{F_4(x_1,\cdots,x_8)=n}(1215x_1^8 - 2268nx_1^6 + 1260n^2x_1^4 - 210n^3x_1^2 + 5n^4).
\end{aligned} \tag{7}$$

Remark 2.1. It would be interesting to get individual expressions for the cusp forms appearing in (7), which will give an explicit expression for the Ramanujan Tau function.

3. PROOFS

For the proofs of our theorems, we need the newforms $\Delta_{k,N}(z)$, $(k,N) \in \{(6,3),(8,3),(12,3)\}$, $\Delta_{10,3,1}(z)$, $\Delta_{10,3,2}(z)$. Below we give their expression in terms of Eisenstein series and eta products. We have used the L-functions and modular forms database [8] to get these expressions. (The expression for $\Delta_{8,3}(z)$ appeared in [13, Eq.(10)].)

$$\begin{aligned}
\Delta_{6,3}(z) &= \eta^6(z)\eta^6(3z),\\
\Delta_{8,3}(z) &= \eta^{12}(z)\eta^4(3z) + 81\eta^6(z)\eta^4(3z)\eta^6(9z) + 18\eta^9(z)\eta^4(3z)\eta^3(9z),\\
\Delta_{10,3,1}(z) &= \frac{-1}{8}E_4(z)\Delta_{6,3}(z) + \frac{9}{8}E_4(3z)\Delta_{6,3}(z),\\
\Delta_{10,3,2}(z) &= \frac{1}{10}E_4(z)\Delta_{6,3}(z) + \frac{9}{10}E_4(3z)\Delta_{6,3}(z),\\
\Delta_{12,3}(z) &= \frac{98}{81}\Delta(z) - 3402\Delta(3z) - \frac{17}{81}E_6(z)\Delta_{6,3}(z).
\end{aligned}$$

3.1. **Proof of Theorem 2.1.** We need the following convolution sums (see [13, 17]).

Proposition 3.1. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}
W_{1,3}^{3,3}(n) &= -\frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{9840}\sigma_7(n) + \frac{81}{9840}\sigma_7\left(\frac{n}{3}\right) + \frac{1}{246}\tau_{8,3}(n),\\
W_{1,3}^{1,1}(n) &= \frac{7}{80}\sigma_5(n) - \frac{1}{8}n\sigma_3(n) + \frac{1}{24}\sigma_3(n) - \frac{1}{240}\sigma\left(n\right),\\
W_{3,1}^{1,3}(n) &= \frac{1}{104}\sigma_5(n) - \frac{81}{1040}\sigma_5\left(\frac{n}{3}\right) + \frac{1-n}{24}\sigma_3(n) - \frac{1}{240}\sigma\left(\frac{n}{3}\right) - \frac{1}{104}\tau_{6,3}(n),\\
W_{1,3}^{1,3}(n) &= \frac{1}{1040}\sigma_5(n) + \frac{9}{104}\sigma_5\left(\frac{n}{3}\right) + \frac{1-3n}{24}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{240}\sigma\left(n\right) + \frac{1}{312}\tau_{6,3}(n).
\end{aligned}$$

The vector space $M_{10}(3)$ is of dimension 4 with a basis $\{E_{10}(z), E_{10}(3z), \Delta_{10,3,1}(z), \Delta_{10,3,2}(z)\}$, the vector space $M_8(3)$ is of dimension 3 with a basis $\{E_8(z), E_8(3z), \Delta_{8,3}(z)\}$, the vector space $M_6(3)$ is of dimension 3 with a basis $\{E_6(z), E_6(3z), \Delta_{6,3}(z)\}$, and the space $M_4(1)$ has dimension 2 with a basis $\{E_4(z), E_4(3z)\}$. Now using the structure theorem of quasimodular forms and using the above basis, we get the following.

$$\begin{aligned}
E_2(z)E_4^2(z) &= E_{10}(z) + \frac{3}{2}DE_8(z),\\
E_2(z)E_4(z)E_4(3z) &= \frac{91}{7381}E_{10}(z) + \frac{7290}{7381}E_{10}(3z) + \frac{6720}{61}\Delta_{10,3,1}(z)\\
&\quad -\frac{2760}{11}\Delta_{10,3,2}(z) + \frac{3}{164}DE_8(z) + \frac{243}{164}DE_8(3z) + \frac{14400}{41}D\Delta_{8,3}(z),\\
E_2(z)E_4^2(3z) &= \frac{1}{7381}E_{10}(z) + \frac{7380}{7381}E_{10}(3z) - \frac{1280}{183}\Delta_{10,3,1}(z)\\
&\quad -\frac{560}{33}\Delta_{10,3,2}(z) + \frac{3}{2}DE_8(3z),\\
E_2(3z)E_4^2(z) &= \frac{820}{7381}E_{10}(z) + \frac{6561}{7381}E_{10}(3z) - \frac{11520}{61}\Delta_{10,3,1}(z)\\
&\quad +\frac{5040}{11}\Delta_{10,3,2}(z) + \frac{1}{2}DE_8(z),
\end{aligned}$$

$$\begin{aligned} E_2(3z)E_4(z)E_4(3z) &= \frac{10}{7381}E_{10}(z) + \frac{7371}{7381}E_{10}(3z) + \frac{2240}{61}\Delta_{10,3,1}(z) \\ &\quad + \frac{920}{11}\Delta_{10,3,2}(z) + \frac{1}{164}DE_8(z) + \frac{81}{164}DE_8(3z) + \frac{4800}{41}D\Delta_{8,3}(z). \end{aligned}$$

By comparing the n-th Fourier coefficinets and using the convolution sums $W_{1,1}^{3,3}, W_{1,3}^{3,3}, W_{1,1}^{1,3}$, $W_{1,3}^{1,3}$ from Proposition 3.1 we get the required triple convolution sums.

3.2. **Proof of Theorem 2.2.** The vector space $M_{12}(1)$ has dimension 2 with a basis $\{E_{12}(z), \Delta(z)\}$, where $\Delta(z)$ is the unique normalized newform of weight 12 and level 1. Now $E_4^3(z) \in M_{12}(1)$ and writing as linear combination of basis, we have

$$E_4^3(z) = E_{12}(z) + \frac{432000}{691}\Delta(z).$$

The dimension of the space $M_{12}(3)$ is 5 having a basis $\{E_{12}(z), E_{12}(3z), \Delta(z), \Delta(3z), \Delta_{12,3}(z)\}$, where $\Delta_{12,3}(z)$ is the unique normalized newform of weight 12 and level 3.
Now $E_4^2(z)E_4(3z), E_4(z)E_4^2(3z) \in M_{12}(3)$. Writing as linear combination of the above basis, we get

$$E_4^2(z)E_4(3z) = \frac{82}{6643}E_{12}(z) + \frac{6561}{6643}E_{12}(3z) - \frac{2553600}{11747}\Delta(z) + \frac{12534134400}{11747}\Delta(3z) + \frac{86400}{1241}\Delta_{12,3}(z)$$

and

$$E_4(z)E_4^2(3z) = \frac{1}{6643}E_{12}(z) + \frac{6642}{6643}E_{12}(3z) + \frac{1910400}{11747}\Delta(z) - \frac{206841600}{11747}\Delta(3z) + \frac{96000}{1241}\Delta_{12,3}(z).$$

By comparing the n-th Fourier coefficinets and using convolution sums $W_{1,1}^{3,3}$ from (4) and $W_{1,3}^{3,3}$ from Proposition 3.1 we get the required convolution sums.

3.3. **Proof of Theorem 2.3.** Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in N$ we know that (see [5], [10])

$$s_4(n) = 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right), \tag{8}$$

and

$$s_8(n) = 24\sigma_3(n) + 216\sigma_3\left(\frac{n}{3}\right). \tag{9}$$

Then $s_{20}(n)$ is given by

$$\begin{aligned} s_{20}(n) &= \sum_{\substack{a,b,c\in\mathbb{N}_0\\ a+b+c=n}} \Big(\sum_{F_2(x_1,\cdots,x_4)=a} 1\Big)\Big(\sum_{F_4(x_5,\cdots,x_{12})=b} 1\Big)\Big(\sum_{F_4(x_{13},\cdots,x_{20})=c} 1\Big) \\ &= s_4(n) + 2s_8(n) + \sum_{\substack{a,b\in\mathbb{N}\\ a+b=n}} s_8(a)s_8(b) + 2\sum_{\substack{a,b\in\mathbb{N}\\ a+b=n}} s_4(a)s_8(b) + \sum_{\substack{a,b,c\in\mathbb{N}\\ a+b+c=n}} s_4(a)s_8(b)r_8(c) \\ &= 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 48\sigma_3(n) + 432\sigma_3\left(\frac{n}{3}\right) + 24^2W_{1,1}^{3,3} + 48\times 216W_{1,3}^{3,3} + 216^2W_{1,1}^{3,3}\left(\frac{n}{3}\right) \\ &\quad + 24^2W_{1,1}^{1,3} + 9\times 24^2W_{1,3}^{1,3} - 48\times 36W_{3,1}^{1,3} - 36\times 216W_{1,1}^{1,3}\left(\frac{n}{3}\right) + 12\times 24^2W_{1,1,1}^{1,3,3} \\ &\quad + 9\times 24^3W_{1,1,3}^{1,3,3} + 12\times 216^2W_{1,3,3}^{1,3,3} - 36\times 24^2W_{3,1,1}^{1,3,3} - 72^3W_{3,1,3}^{1,3,3} - 1296^2W_{1,1,1}^{1,3,3}\left(\frac{n}{3}\right). \end{aligned}$$

Now, we substitute the expressions for the covolution sums using (4) and Theorem 2.1, the required formula follows.

3.4. **Proof of Theorem 2.4.** We proceed as in the case of 20 variables. We have

$$
\begin{aligned}
s_{24}(n) &= \sum_{\substack{a.b,c\in\mathbb{N}_0\\ a+b+c=n}} \Big(\sum_{F_4(x_1,\cdots,x_8)=a} 1\Big)\Big(\sum_{F_4(x_9,\cdots,x_{16})=b} 1\Big)\Big(\sum_{F_4(x_{17},\cdots,x_{24})=c} 1\Big) \\
&= 3s_8(n) + 3\sum_{\substack{a,b\in\mathbb{N}\\ a+b=n}} s_8(a)s_8(b) + \sum_{\substack{a,b,c\in\mathbb{N}\\ a+b+c=n}} s_8(a)s_8(b)s_8(c) \\
&= 72\sigma_3(n) + 648\sigma_3\left(\frac{n}{3}\right) + 3\sum_{\substack{a,b\in\mathbb{N}\\ a+b=n}} \left(24\sigma_3(a) + 216\sigma_3\left(\frac{a}{3}\right)\right)\left(24\sigma_3(b) + 216\sigma_3\left(\frac{b}{3}\right)\right) \\
&\quad + \sum_{\substack{a,b,c\in\mathbb{N}\\ a+b+c=n}} \left(24\sigma_3(a) + 216\sigma_3\left(\frac{a}{3}\right)\right)\left(24\sigma_3(b) + 216\sigma_3\left(\frac{b}{3}\right)\right)\left(24\sigma_3(c) + 216\sigma_3\left(\frac{c}{3}\right)\right) \\
&\quad = 72\sigma_3(n) + 648\sigma_3\left(\frac{n}{3}\right) + 3\times 24^2 W_{1,1}^{3,3}(n) + 54\times 24^2 W_{1,3}^{3,3}(n) + 3^5\times 24^2 W_{1,1}^{3,3}\left(\frac{n}{3}\right) \\
&\quad +24^3\ W_{1,1,1}^{3,3,3}(n) + 3^3\times 24^3\ W_{1,1,3}^{3,3,3}(n) + 3^5\times 24^3\ W_{1,3,3}^{3,3,3}(n) + 216^3\ W_{1,1,1}^{3,3,3}\left(\frac{n}{3}\right).
\end{aligned}
$$

Substituting the convolution sums using (4), Proposition 3.1 and Theorem 2.2, we get the required formula for $s_{24}(n)$.

We give below a table giving the first 15 values of $s_{10}(n)$ and $s_{24}(n)$.

n	$s_{20}(n)$	$s_{24}(n)$
1	60	72
2	1620	2376
3	25980	47592
4	275460	646344
5	2040552	6305904
6	10965780	45821160
7	44559840	255215808
8	145963620	1125009864
9	417830460	4097478600
10	1091417976	12975540336
11	2573551440	37101202848
12	5569628100	96867424872
13	11570383560	232791251760
14	22593025440	526183909056
15	41415305832	1128351033648

Acknowledgements

We have used the open-source mathematics software SAGE (www.sagemath.org) to do our calculations. The second author is partially funded by SERB grant SR/FTP/MS-053/2012. He would like to thank HRI, Allahabad for the warm hospitality where this work has been carried out. Finally, the authors thank the referee for his/her comments and also for pointing out the reference [7].

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