CONGRUENCES VIA MODULAR FORMS

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Abstract. We prove two congruences for the coefficients of power series expansions in t of modular forms where t is a modular function. As a result, we settle two recent conjectures of Chan, Cooper and Sica. Additionally, we provide tables of congruences for numbers which appear in similar power series expansions and in the study of integral solutions of Apéry-like differential equations.

1. Introduction

In [7], Chan, Cooper and Sica investigate sequences of integers that satisfy congruence properties similar to those of the Apéry numbers associated with the irrationality of $\zeta(3)$. They also conjecture seven congruences and supercongruences for coefficients of power series expansions in t of modular forms where t is a modular function. The term supercongruences appeared in [3] and was the subject of the Ph.D. thesis of Coster [10]. It originally referred to families of congruences that are stronger than ones suggested by formal group theory, but now includes individual congruences (see [1]). Let

$$f(z)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}q^{m^2+mn+6n^2},\quad t_1=t_1(z)=\frac{\eta(z)\eta(23z)}{f(z)}$$

and

$$F(z)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}q^{2m^2+mn+3n^2},\quad t_2=t_2(z)=\frac{\eta(z)\eta(23z)}{F(z)}$$

where $\eta(z)$ is the Dedekind eta-function, $q:=e^{2\pi iz}$ and $z\in\mathbb{H}$. Write

$$f=f(z)=\sum_{n=0}^{\infty}f_nt_1^n\quad\text{and}\quad F=F(z)=\sum_{n=0}^{\infty}F_nt_2^n.$$

In [7], Chan, Cooper and Sica make the following

Conjecture 1.1. *If p is a prime with $\left(\frac{p}{23}\right)=1$ and $n\geq 1$, then*

$$f_{np}\equiv f_n\pmod p$$

and

$$F_{np}\equiv F_n\pmod p.$$

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The first few terms in the sequence $\{f_n\}_{n\geq 0}$ are

$$1,\ 2,\ 6,\ 26,\ 142,\ 876,\ 5790,\ 40020,\ 285582,\ \ldots$$

while for $\{F_n\}_{n\geq 0}$, we have

$$1,\ 0,\ 2,\ 6,\ 30,\ 144,\ 758,\ 4080,\ 22702,\ 128832,\ \ldots.$$

Closed forms for f_n and F_n were not given in [7] and thus a combinatorial approach to Conjecture 1.1 is not yet available. The purpose of this note is to prove this conjecture via modular forms. We have the following.

Theorem 1.2. *If p is a prime with $\left(\frac{p}{23}\right)=1$ and n, $r\geq 1$ are integers, then*

$$f_{np^r}\equiv f_{np^{r-1}} \pmod{p^r} \tag{1}$$

and

$$F_{np^r}\equiv F_{np^{r-1}} \pmod{p^r}. \tag{2}$$

In Section 2, we recall some preliminaries on power series expansions and Eisenstein series with characters and then prove Theorem 1.2. In Section 3, we provide tables of congruences for numbers appearing in other power series expansions found in [7] and in the study of integral solutions of Apéry-like differential equations (see [2], [5], [18]) and mention conjectural supercongruences. Finally, we note that two other conjectural congruences from [7] which involve $f_{2,n}$ and $f_{3,n}$ (see Section 3) have recently been proven in [8]. The remaining three conjectures in [7] are still open.

2. Proof of Theorem 1.2

We first recall a recent result of Jarvis and Verrill (see Proposition 4.2 in [12] or Proposition 3 in [4]). This result is quite useful as it allows one to deduce congruence properties of coefficients in a power series expansion from those of another expansion.

Proposition 2.1. *Let t be a power series*

$$t=\frac{1}{m}\sum_{n=1}^{\infty}a_nu^{n/v},$$

convergent in a neighborhood of $u=0$, with m,v positive integers, $a_n\in\mathbb{Z}$ and $a_1=1$. Suppose that in some neighborhood of $u=0$ we have an equality of convergent power series given by

$$\sum_{n=1}^{\infty}b_nt^{n-1}\,dt=\sum_{n=1}^{\infty}c_nu^{n-1}\,du, \tag{3}$$

for some integers b_n and c_n, $n\geq 1$. Assume p is a prime not dividing m or v. If

$$b_{np^r}\equiv b_{np^{r-1}} \pmod{p^r},$$

then

$$c_{np^r}\equiv c_{np^{r-1}} \pmod{p^r}.$$

Remark 2.2. J. Stienstra has kindly pointed out that one can use formal group theory (see the appendix of [16]) to extend Proposition 2.1 to the case where each of the sums in (3) starts with $n = 0$. Also, since $a_1 = 1$, the converse of Proposition 2.1 is true.

We now discuss the notion of Eisenstein series with characters. For further details, see Chapter 5 of [15]. Let $M_k(\Gamma_0(N), \epsilon)$ be the space of modular forms of weight k on $\Gamma_0(N)$ with character ϵ. Suppose χ and ψ are primitive Dirichlet characters with conductors L and R, respectively. Let

$$E_{k,\chi,\psi}(q) := c_0 + \sum_{n=1}^{\infty}\Big(\sum_{d|n}\psi(d)\chi(n/d)d^{k-1}\Big)q^n \tag{4}$$

where

$$c_0 = \begin{cases} -\dfrac{B_{k,\psi}}{2k} & \text{if} \quad L = 1, \\ 0 & \text{if} \quad L > 1 \end{cases}$$

and $B_{k,\psi}$ is the generalized Bernoulli number associated to ψ. If t is a positive integer and $k \geq 3$ is an integer such that $\chi(-1)\psi(-1) = (-1)^k$, then $E_{k,\chi,\psi}(q^t)$ is in $M_k(\Gamma_0(RLt), \chi\psi)$. Moreover, given N and ϵ, the series $E_{k,\chi,\psi}(q^t)$ such that $RLt \mid N$ and $\chi\psi = \epsilon$ form a basis for the Eisenstein subspace $E_k(\Gamma_0(N), \epsilon)$ of $M_k(\Gamma_0(N), \epsilon)$.

Proof of Theorem 1.2. Let χ be the character $\left(\frac{\cdot}{23}\right)$ and ψ be the trivial character 1. We first note that

$$E_{3,1,\chi}(q) =: \sum_{n=0}^{\infty} e_n q^n$$

and

$$E_{3,\chi,1}(q) =: \sum_{n=0}^{\infty} a_n q^n$$

form a basis for the space $E_3\Big(\Gamma_0(23), \left(\frac{\cdot}{23}\right)\Big)$. By Lemma 0.3 in [17] and a finite computation, we have

$$f\frac{q\frac{dt_1}{dq}}{t_1} = F\frac{q\frac{dt_2}{dq}}{t_2} = -\frac{1}{24}E_{3,1,\chi}(q) - \frac{23}{24}E_{3,\chi,1}(q)$$

and so

$$f\frac{dt_1}{t_1} = F\frac{dt_2}{t_2} = \left[-\frac{1}{24}E_{3,1,\chi}(q) - \frac{23}{24}E_{3,\chi,1}(q)\right]\frac{dq}{q}. \tag{5}$$

By (4), we have

$$a_{np^r} - \left(\frac{p}{23}\right) a_{np^{r-1}} = \sum_{d'|n} \left(\frac{n/d'}{23}\right) (d'p^r)^2 \tag{6}$$

and

$$e_{np^r} - e_{np^{r-1}} = \sum_{d'|n} \left(\frac{d'p^r}{23}\right) (d'p^r)^2. \tag{7}$$

Letting $u = q$ in (5) implies that

$$\sum_{n=0}^{\infty} f_n t_1^{n-1}\, dt_1 = \sum_{n=0}^{\infty} F_n t_2^{n-1}\, dt_2 = \left[-\frac{1}{24} E_{3,1,\chi}(u) - \frac{23}{24} E_{3,\chi,1}(u)\right] \frac{du}{u}. \tag{8}$$

If we take $v = 1$, $m = 1$, $b_n = f_n$, F_n, respectively, and $c_n = -\frac{1}{24} e_n - \frac{23}{24} a_n$, then by (6) and (7), we have

$$c_{np^r} \equiv c_{np^{r-1}} \pmod{p^r} \tag{9}$$

for primes $p \geq 3$ such that $\left(\frac{p}{23}\right) = 1$ and $r \geq 1$ and for $p = 2$ and $r \geq 3$. An application of Remark 2.2 then implies (1) and (2). To verify (9) for $p = 2$ and $r = 1$ or 2, we first note that (4) implies $e_n = a_n$ if $\left(\frac{n}{23}\right) = 1$ and $e_n = -a_n$ if $\left(\frac{n}{23}\right) = -1$. The case where $\left(\frac{n}{23}\right) = 0$ can be reduced to one of previous two cases since $a_{23n} = 23^2 a_n$ and $e_{23n} = e_n$ for all n. The result then follows upon a routine check that (9) holds in all of these cases.

□

3. Tables

Using the methods in Section 2, we have proven congruences of the form

$$A(np^r) \equiv A(np^{r-1}) \pmod{p^r} \tag{10}$$

for all of the numbers $A(n)$ which appear in Tables 1, 2 and 3. For brevity, we only give the relevant modular function t and modular forms $f(t)$ and $M := f(t)\frac{q\frac{dt}{dq}}{t}$. The coefficients of M in Tables 1, 2 and 3 can be computed using (4), Chapter 4, Section 32 in [11] (for example, see page 85, equation (32.71) for (vi)), [13] or [15]. Given positive integers s_1, s_2, ..., s_k and integers r_1, r_2, ..., r_k, we write

$$s_1^{r_1} s_2^{r_2} \dots s_k^{r_k}$$

for the eta-quotient

$$\eta(s_1 z)^{r_1} \eta(s_2 z)^{r_2} \cdots \eta(s_k z)^{r_k}.$$

Table 1 consists of numbers $f_{i,n}$, $i = 2$, 3, 5, 7 and 11 which are coefficients in the power series expansion in t of the modular forms (see [7])

$$f_2 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+n^2}, \quad f_3 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}, \quad f_5 = \frac{1^5}{5^1},$$

$$f_7 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} \quad \text{and} \quad f_{11} = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+3n^2}.$$

We write $\chi_s := \left(\frac{\cdot}{s}\right)$ for $s = 3, 5, 7, 11$ and $\chi_{-4} := \left(\frac{-4}{\cdot}\right)$. The analogue of Theorem 1.2 is true for primes p satisfying $\chi_{-4}(p) = 1$ in (i), $\chi_3(p) = 1$ in (ii), $\chi_7(p) = 1$ in (iv), $\chi_{11}(p) = 1$ in (v). It is true for all primes in (iii). The only known closed forms are

$$f_{2,n} = \left(\frac{8^n \left(\frac{1}{4}\right)_n}{n!}\right)^2$$

and

$$f_{3,n} = \frac{108^n \left(\frac{1}{6}\right)_n \left(\frac{1}{3}\right)_n}{(n!)^2}.$$

Table 2 lists numbers which arise in Beukers' [5] and Zagier's [18] study of integral solutions of second order Apéry-like differential equations. The choices of t and the parameterizations of f can be found in [17] and [18]. In case (ix), congruence (10) with $A(n)$ replaced by $2^n A(n)$ has been proven in [12].

Table 3 contains numbers listed in [2] as part of a discussion on third order Apéry-like differential equations. Here

$$L_1(z) := -\frac{7}{240}E_4(z) + \frac{1}{60}E_4(2z) - \frac{3}{80}E_4(3z) + \frac{21}{20}E_4(6z),$$

$$L_2(z) := \frac{1}{120}E_4(z) - \frac{2}{15}E_4(2z) - \frac{3}{40}E_4(3z) + \frac{6}{5}E_4(6z),$$

and

$$L_3(z) := \frac{1}{240}E_4(z) - \frac{1}{60}E_4(2z) - \frac{27}{80}E_4(3z) + \frac{27}{20}E_4(6z)$$

where $E_4(z)$ is the usual weight 4 Eisenstein series on $SL_2(\mathbb{Z})$. The choices of t and the parameterizations of f can be found in [6], [9] and [14].

Finally, we have numerically observed extensions of (10) modulo p^{2r} (subject to the above conditions for p odd) in (i), (ii), (iv), (v), (vii), (viii), (ix) and (x) and modulo p^{3r} for (iii), (xii) and (xiii). Here $p \geq 5$ for (xii). Coster [10] has proven an extension of (10) modulo p^{3r} for (vi) and (xi). It might of interest to see if combinatorial techniques can be applied to some of these conjectural extensions.

TABLE 1

$A(n)$	t	M
(i) $f_{2,n}$	$\dfrac{2^{12}}{f_2^6}$	$-4E_{3,1,\chi_{-4}}(q)-16E_{3,\chi_{-4},1}(q)$
(ii) $f_{3,n}$	$\dfrac{1^6 3^6}{f_3^6}$	$-9E_{3,1,\chi_3}(q)-27E_{3,\chi_3,1}(q)$
(iii) $f_{5,n}$	$\dfrac{5^6}{1^6}$	$E_{4,1,\chi_5}(q)$
(iv) $f_{7,n}$	$\dfrac{1^3 7^3}{f_7^3}$	$-\frac{7}{8}E_{3,1,\chi_7}(q)-\frac{49}{8}E_{3,\chi_7,1}(q)$
(v) $f_{11,n}$	$\dfrac{1^2 11^2}{f_{11}^2}$	$-\frac{1}{3}E_{3,1,\chi_{11}}(q)-\frac{11}{3}E_{3,\chi_{11},1}(q)$

TABLE 2

$A(n)$	t	$f(t)$	M
(vi) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^3$	$\dfrac{1^3 6^9}{2^3 3^9}$	$\dfrac{2^1 3^6}{1^2 6^3}$	$\dfrac{1^1 2^4 3^5}{6^4}$
(vii) $\displaystyle\sum_{k=0}^{\lfloor \frac{n}{3}\rfloor}(-1)^k 3^{n-3k}\binom{n}{3k}\binom{3k}{k}\binom{2k}{k}$	$-\dfrac{9^3}{1^3}$	$\dfrac{1^3}{3^1}$	$\dfrac{3^9}{9^3}$
(viii) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$	$\dfrac{1^4 6^8}{2^8 3^4}$	$\dfrac{2^6 3^1}{1^3 6^2}$	$\dfrac{1^1 2^4 3^5}{6^4}$
(ix) $\displaystyle\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}4^{n-2k}\binom{n}{2k}\binom{2k}{k}^2$	$\dfrac{1^4 4^2 8^4}{2^{10}}$	$\dfrac{2^{10}}{1^4 4^4}$	$\dfrac{2^4 4^6}{8^4}$
(x) $\displaystyle\sum_{k=0}^{n}\sum_{l=0}^{k}(-1)^k 8^{n-k}\binom{n}{k}\binom{k}{l}^3$	$\dfrac{1^5 3^1 4^5 6^2 12^1}{2^{14}}$	$\dfrac{2^{15} 3^2 12^2}{1^6 4^6 6^5}$	$\dfrac{2^7 6^{11}}{1^1 3^5 4^1 12^5}$

TABLE 3

$A(n)$	t	f	M
(xi) $\displaystyle\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2$	$\dfrac{1^{12}6^{12}}{2^{12}3^{12}}$	$\dfrac{2^7 3^7}{1^5 6^5}$	$L_1(z)$
(xii) $(-1)^n\displaystyle\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$	$\dfrac{2^6 6^6}{1^6 3^6}$	$\dfrac{1^4 3^4}{2^2 6^2}$	$L_2(z)$
(xiii) $(-1)^n\displaystyle\sum_{k=0}^{[n/3]}(-1)^k\frac{3^{n-3k}(3k)!}{(k!)^3}\binom{n}{3k}\binom{n+k}{k}$	$\dfrac{3^4 6^4}{1^4 2^4}$	$\dfrac{1^3 2^3}{3^1 6^1}$	$L_3(z)$

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REFERENCES

[1] S. Ahlgren, K. Ono, *A Gaussian hypergeometric series evaluation and Apery number congruences*, J. Reine Angew. Math. **518**, (2000), 187–212.
[2] G. Almkvist, D. van Straten and W. Zudilin, *Generalizations of Clausen's formula and algebraic transformations of Calabi-Yau differential equations*, Proc. Edinb. Math. Soc. (2), to appear.
[3] F. Beukers, *Some congruences for the Apéry numbers*, J. Number Th. **21** (1985), no. 2, 141–155.
[4] F. Beukers, *Another congruence for the Apéry numbers*, J. Number Th. **25** (1987), no. 2, 201–210.
[5] F. Beukers, *On B. Dwork's accessory parameter problem*, Math. Z. **241** (2002), no. 2, 425–444.
[6] H. Chan, S. Chan and Z. Liu, *Domb's numbers and Ramanujan-Sato type series for* $1/\pi$, Adv. Math. **186** (2004), 396–410.
[7] H. Chan, S. Cooper and F. Sica, *Congruences satisfied by Apéry-like numbers*, Int. J. Number Theory **6** (2010), no. 1, 89–97.
[8] H. Chan, A. Kontogeorgis, C. Krattenthaler and R. Osburn, *Supercongruences satisfied by coefficients of* ${}_2F_1$ *hypergeometric series*, Ann. Sci. Math. Québec **34** (2010), no. 1, 25–36.
[9] H. Chan, H. Verrill, *The Apéry numbers, the Almkvist-Zudilin numbers and new series for* $1/\pi$, Math. Res. Lett. **16** (2009), no. 3, 405–420.
[10] M. Coster, *Supercongruences*, Ph.D. thesis, Universiteit Leiden, 1988.
[11] N. Fine, *Basic hypergeometric series and applications*, American Mathematical Society, Providence, RI, 1988.
[12] F. Jarvis, H. Verrill, *Supercongruences for the Catalan-Larcombe-French numbers*, Ramanujan J. **22** (2010), no. 2, 171–186.
[13] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, Amer. Math. Soc., CBMS Regional Conf. in Math., vol. 102, 2004.

[14] C. Peters, J. Stienstra, *A pencil of $K3$-surfaces related to Apéry's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero*. Arithmetic of complex manifolds (Erlangen, 1988), 110–127, Lecture Notes in Math., 1399, Springer, Berlin, 1989.
[15] W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, **79**. American Mathematical Society, Providence, RI, 2007.
[16] J. Stienstra, F. Beukers, *On the Picard-Fuchs equation and the formal Brauer group of certain elliptic $K3$ surfaces*, Math. Ann. **271** (1985), no. 2, 269–304.
[17] H. Verrill, *Some congruences related to modular forms*, available at `http://www.mpim-bonn.mpg.de/Research/MPIM+Preprint+Series`
[18] D. Zagier, *Integral solutions of Apéry-like recurrence equations*, Group and Symmetries: From Neolithic Scots to John McKay, 349–366, CRM Proc. Lecture Notes, **47**, Amer. Math. Soc., Providence, RI, 2009.

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