SUPERCONGRUENCES FOR APÉRY-LIKE NUMBERS

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Abstract. It is known that the numbers which occur in Apéry's proof of the irrationality of $\zeta(2)$ have many interesting congruence properties while the associated generating function satisfies a second order differential equation. We prove supercongruences for a generalization of numbers which arise in Beukers' and Zagier's study of integral solutions of Apéry-like differential equations.

1. Introduction

In the course of his work on proving the irrationality of $\zeta(2)$, Apéry introduced, for an integer $n \geq 0$, the following sequence of numbers [1], [15]

$$B(n) := \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j}.$$

Several authors have subsequently investigated many interesting congruence properties for $B(n)$ and its generalizations. For example, Beukers [4] employed "brute force methods" to prove

$$B(mp^r - 1) \equiv B(mp^{r-1} - 1) \pmod{p^{3r}} \tag{1}$$

for any prime $p > 3$ and integers m, $r \geq 1$. In [14], Stienstra and Beukers related the $B(n)$'s to the p-th Fourier coefficient of a modular form. If we define

$$\eta^6(4z) =: \sum_{n=1}^{\infty} a(n) q^n$$

where

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind eta function, $q := e^{2\pi i z}$ and $z \in \mathbb{H}$, then they proved using the formal Brauer group of some elliptic K3-surfaces that for all odd primes p and any m, $r \in \mathbb{N}$ with m odd, we have

$$B\Big(\frac{mp^r - 1}{2}\Big) - a(p) B\Big(\frac{mp^{r-1} - 1}{2}\Big) + (-1)^{\frac{p-1}{2}} p^2 B\Big(\frac{mp^{r-2} - 1}{2}\Big) \equiv 0 \pmod{p^r}. \tag{2}$$

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Congruence (1) is but one example of a general phenomena called *Supercongruences*. This term refers to the fact that congruences of this type are stronger than the ones suggested by formal group theory. It appeared in [4] and was the subject of the Ph.D. thesis of Coster [8]. In fact, Coster studied the generalized Apéry numbers (see, for example, Theorem 4.3.1 in [8])

$$u(n, A, B, \epsilon) := \sum_{j=0}^{n} \binom{n}{j}^A \binom{n+j}{j}^B \epsilon^j$$

where A, $B \in \mathbb{N}$, $\epsilon = \pm 1$ and proved that

$$u(mp^r, A, B, \epsilon) \equiv u(mp^{r-1}, A, B, \epsilon) \pmod{p^{3r}} \tag{3}$$

if $A \geq 3$ and

$$u(mp^r - 1, A, B, \epsilon) \equiv u(mp^{r-1} - 1, A, B, \epsilon) \pmod{p^{3r}}$$

if $B \geq 3$. Other examples of supercongruences have been observed in the context of number theory (see Chapter 11 in [11]), quantum theory [10], and algebraic geometry [12]. Currently, there is no systematic explanation for such congruences. Perhaps, as mentioned in [14], they are related to formal Chow groups.

It is known that the $B(n)$'s satisfy the recurrence relation

$$(n+1)^2 B(n+1) = (11n^2 + 11n + 3)B(n) + n^2 B(n-1)$$

for $n \geq 1$. This implies that the generating function

$$\mathcal{B}(t) = \sum_{n=0}^{\infty} B(n) t^n$$

satisfies the differential equation

$$L\mathcal{B}(t) = 0$$

where

$$L = t(t^2 + 11t - 1)\frac{d^2}{dt^2} + (3t^2 + 22t - 1)\frac{d}{dt} + t - 3.$$

In [5], Beukers considers the differential equation

$$((t^3 + at^2 + bt)F'(t))' + (t - \lambda)F(t) = 0 \tag{4}$$

where a, b and λ are rational parameters and asks for which values of these parameters this equation has a solution in $\mathbb{Z}[[t]]$. This equation has a unique solution which is regular at the origin with $F(0) = 0$ given by

$$F(t) = \sum_{n=0}^{\infty} u(n) t^n$$

with $u(0) = 1$ and satisfies the recurrence relation

$$b(n+1)^2u(n+1)+(an^2+an-\lambda)u(n)+n^2u(n-1)=0$$

where $n \geq 1$. In [18], Zagier describes a search over a suitably chosen domain of 100 million triples (a, b, λ). He finds 36 triples which yield an integral solution to (4) and classifies seven as "sporadic" cases (for a conjecture concerning the only cases where (4) has an integral solution, see page 354 of [18]). All seven cases (which include $B(n)$) have a binomial sum representation and a geometric origin.

The purpose of this paper is to study congruences, akin to (3), for a generalization of one of the "sporadic" cases which can be expressed in terms of binomial sums and has a parametrization in terms of modular functions. For A, $B \in \mathbb{N}$, let

$$C(n,A,B) := \sum_{k=0}^{n} \binom{n}{k}^A \binom{2k}{k}^B. \tag{5}$$

The first few terms in the sequence of positive integers $\{C(n,2,1)\}_{n\geq 0}$ are as follows:

1, 3, 15, 93, 639, 4653, 35169, 272835, ...

This sequence (see A002893 of Sloane [13]) corresponds to a "sporadic" case of Zagier (see #8 of Table 1 in [18]) and has also appeared in the study of algebraic surfaces (see [3] or Part III of [14]), moments of Bessel functions arising in quantum field theory [2] and cooperative phenomena in crystals [9]. Our main result which is an analogue of (3) is the following.

Theorem 1.1. *Let A, $B \in \mathbb{N}$ and $p > 3$ be a prime. For any m, $r \in \mathbb{N}$, we have*

$$C(mp^r, A, B) \equiv C(mp^{r-1}, A, B) \pmod{p^{3r}}$$

if $A \geq 3$ and

$$C(mp^r, 2, B) \equiv C(mp^{r-1}, 2, B) \pmod{p^{2r}}.$$

As a result of Theorem 1.1, we obtain a three-term congruence which is reminiscent of (2).

Corollary 1.2. *Let A, $B \in \mathbb{N}$ and $p > 3$ be a prime. Let f be a non-cuspidal normalized Hecke eigenform of integer weight $k < 4$ on $\Gamma_0(N)$ with character χ such that*

$$f(z) =: \sum_{n=0}^{\infty} \gamma(n) q^n. \tag{6}$$

Then for any m, $r \in \mathbb{N}$, we have

$$C(mp^r, A, B) - \gamma(p)C(mp^{r-1}, A, B) + \chi(p)p^{k-1}C(mp^{r-2}, A, B) \equiv 0 \pmod{p^{3r+k-4}} \tag{7}$$

if $A \geq 3$. If $k \geq 4$ and $A \geq 3$, this congruence is true modulo p^{3r}. If $A = 2$ and $k < 3$, it is true modulo p^{2r+k-3}. If $A = 2$ and $k \geq 3$, it is true modulo p^{2r}.

The method of proof for Theorem 1.1 is due to Coster (see page 50 of [8]). Namely, the idea is to rewrite the summands in (5) as products $g_{AB}(X, k)$ and $g^*_{AB}(X, k)$ (see Section 2), then exploit the combinatorial properties of these products. One then expresses (5) as two sums, one for which $p \mid k$ and the other for which $p \nmid k$. In the case $p \nmid k$, the sum vanishes modulo an appropriate power of p while for $p \mid k$, the sum reduces to the required result. This approach can be used to prove supercongruences similar to Theorem 1.1 in the remaining "sporadic" cases. Finally, we would like to point out that Theorem 4.2 in [6] follows from Theorem 1.1 by taking $r = 1$ and $B = 0$ if $A \geq 3$ and from Lemma 2.2 below once we use the identity

$$C(n, 2, 0) = \binom{2n}{n}.$$

The paper is organized as follows. In Section 2, we recall some properties of the products $g_{AB}(X, k)$ and $g^*_{AB}(X, k)$. In Section 3, we prove Theorem 1.1 and Corollary 1.2.

2. Preliminaries

We first recall the definition of two products and one sum and list some of their main properties. For more details, see Chapter 4 of [8]. For A, $B \in \mathbb{N}$ and integers k, $j \geq 1$ and X, we define

$$g_{AB}(X, k) = \prod_{i=1}^{k} \left(1 - \frac{X}{i}\right)^A \left(1 + \frac{X}{i}\right)^B,$$

$$g^*_{AB}(X, k) = \prod_{\substack{i=1 \\ p \nmid i}}^{k} \left(1 - \frac{X}{i}\right)^A \left(1 + \frac{X}{i}\right)^B,$$

and for a fixed prime $p > 3$

$$S_j(k) = \sum_{\substack{i=1 \\ p \nmid i}}^{k} \frac{1}{i^j}.$$

The following proposition provides some of the main properties of $g_{AB}(X, k)$, $g^*_{AB}(X, k)$ and $S_j(k)$. We note that parts (3) and (5) are straightforward to prove while (1), (2) and (4) require a short argument (see parts (i) and (ii) of Lemma 4.2.1 and parts (i), (ii) and (iv) of Lemma 4.2.5 in [8]).

Proposition 2.1. *For any A, B, $m \in \mathbb{N}$, $X \in \mathbb{Z}$ and integers $k, r \geq 1$, we have*

(1) $S_j(mp^r) \equiv 0 \pmod{p^r}$ *for* $j \not\equiv 0 \pmod{p-1}$,
(2) $S_{2j-1}(mp^r) \equiv 0 \pmod{p^{2r}}$ *for* $j \not\equiv 0 \pmod{\frac{p-1}{2}}$,
(3) $g_{AB}(pX, k) = g^*_{AB}(pX, k) g_{AB}(X, \lfloor \frac{k}{p} \rfloor)$,
(4) $g^*_{AB}(X, k) \equiv 1 + (B - A)S_1(k)X + \frac{1}{2}\Big((A - B)^2 S_1(k)^2 - (A + B)S_2(k)\Big)X^2 \pmod{X^3}$,
(5) $\binom{n}{k}^A \binom{n+k}{k}^B = (-1)^{Ak} \left(\frac{n}{n-k}\right)^A g_{AB}(n, k)$.

By (1), taking $j = 1$ in (2) and (4) of Proposition 2.1, we have

$$g^*_{AB}(mp^r, np^s) \equiv 1 \pmod{p^{r+2s}} \tag{8}$$

for any non-negative integers m, n, r and s with $s \le r$. We now require a reduction result for one of the binomial coefficients occurring in $C(n, A, B)$.

Lemma 2.2. *For a prime $p > 3$ and integers $m \ge 0$, $r \ge 1$, we have*

$$\binom{2mp^r}{mp^r} \equiv \binom{2mp^{r-1}}{mp^{r-1}} \pmod{p^{3r}}.$$

Proof. If $p \mid k$, then

$$\begin{aligned}\binom{mp^r}{k} &= \binom{mp^{r-1}}{\frac{k}{p}} \prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{k} \Big(\frac{mp^r-\lambda}{\lambda}\Big) \\ &= \binom{mp^{r-1}}{\frac{k}{p}} (-1)^{k-\lfloor \frac{k}{p} \rfloor} \prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{k} \Big(1-\frac{mp^r}{\lambda}\Big) \\ &= \binom{mp^{r-1}}{\frac{k}{p}} g_{10}^{*}(mp^r, k).\end{aligned}$$

This implies

$$\binom{2mp^r}{mp^r} = \binom{2mp^{r-1}}{mp^{r-1}} g_{10}^{*}(2mp^r, mp^r).$$

By (8), we have $g_{10}^{*}(2mp^r, mp^r) \equiv 1 \pmod{p^{3r}}$ and the result follows. □

3. Proofs of Theorem 1.1 and Corollary 1.2

We are now in a position to prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. For integers m, n, $r \ge 1$ and $s \ge 0$ with $s \le r$, we have

$$ord_p \binom{mp^r}{np^s}^{A} \ge A(r-s). \tag{9}$$

Furthermore, by (3) and (5) of Proposition 2.1, we have, for $s \ge 1$,

$$\begin{aligned}\binom{mp^r}{np^s}^{A}\binom{2np^s}{np^s}^{B} &= (-1)^{Anp^s}\Big(\frac{mp^r}{mp^r-np^s}\Big)^{A} g_{A0}(mp^{r-1}, np^{s-1}) g_{A0}^{*}(mp^r, np^s) \binom{2np^s}{np^s}^{B} \\ &= \binom{mp^{r-1}}{np^{s-1}}^{A} g_{A0}^{*}(mp^r, np^s)\binom{2np^s}{np^s}^{B}.\end{aligned}$$

By (8) and Lemma 2.2, we have

$$g_{A0}^{*}(mp^r, np^s) \equiv 1 \pmod{p^{r+2s}}$$

and

$$\binom{2np^s}{np^s} \equiv \binom{2np^{s-1}}{np^{s-1}} \pmod{p^{3s}}.$$

Thus

$$\binom{mp^r}{np^s}^A \binom{2np^s}{np^s}^B \equiv \binom{mp^{r-1}}{np^{s-1}}^A \binom{2np^{s-1}}{np^{s-1}}^B \pmod{p^{min(A(r-s)+r+2s, A(r-s)+3s)}}.$$

As $r \geq s$, we have $min(A(r-s)+r+2s, A(r-s)+3s) = A(r-s)+3s$. If $A \geq 3$, then

$$\binom{mp^r}{np^s}^A \binom{2np^s}{np^s}^B \equiv \binom{mp^{r-1}}{np^{s-1}}^A \binom{2np^{s-1}}{np^{s-1}}^B \pmod{p^{3r}}. \tag{10}$$

If $A = 2$, then $A(r-s)+3s \geq 2r$ and so

$$\binom{mp^r}{np^s}^A \binom{2np^s}{np^s}^B \equiv \binom{mp^{r-1}}{np^{s-1}}^A \binom{2np^{s-1}}{np^{s-1}}^B \pmod{p^{2r}}. \tag{11}$$

We now split $C(mp^r, A, B)$ into two sums, namely

$$C(mp^r, A, B) = \sum_{\substack{k=0\\ p\nmid k}}^{mp^r} \binom{mp^r}{k}^A \binom{2k}{k}^B + \sum_{\substack{k=0\\ p\mid k}}^{mp^r} \binom{mp^r}{k}^A \binom{2k}{k}^B.$$

If $A \geq 3$, then the first sum vanishes modulo p^{3r} using (9) and the result follows by (10). A similar argument is true for $A = 2$ via (9) and (11).

□

Proof of Corollary 1.2. If $r \geq 2$, we have by Theorem 1.1

$$C(mp^r, A, B) \equiv C(mp^{r-1}, A, B) \pmod{p^{3r}}$$

if $A \geq 3$. Thus for $r \geq 3$

$$\chi(p)p^{k-1}C(mp^{r-1}, A, B) \equiv \chi(p)p^{k-1}C(mp^{r-2}, A, B) \pmod{p^{3r+k-4}}. \tag{12}$$

The modular form (6) has the property that $\gamma(p) = 1 + \chi(p)p^{k-1}$. This fact combined with (12) implies (7) for $A \geq 3$ and $k < 4$. Similarly, we have the result for the other cases.

□

Remark 3.1. We would like to mention an alternative approach to a weaker version of Corollary 1.2 (with p^r instead of p^{2r}) in the case $A = 2$ and $B = 1$ as this computation motivated Theorem 1.1. If we consider the modular function for $\Gamma_0(6)$

$$t(z) = \frac{\eta(z)^4\eta(6z)^8}{\eta(2z)^8\eta(3z)^4},$$

then (see Theorems 5.1, 5.2 and 5.3 in [7], Table 2 in [16], Theorem 1 in [17] or Table 3, Case C in [18])

$$f(t) := \sum_{n=0}^{\infty} C(n,2,1)t^n = \frac{\eta(2z)^6\eta(3z)}{\eta(z)^3\eta(6z)^2}.$$

If $\sigma_\chi(n) := \sum_{d|n} \chi(d)d^2$, then

$$E(z) = -\frac{1}{9} + \sum_{n=1}^{\infty} \sigma_{\chi_{-3}}(n)q^n$$

is an Eisenstein series of weight 3 on $\Gamma_0(3)$ with character χ_{-3}. One can check that

$$E(z) + 8E(2z) = -f(q)\frac{q\frac{dt}{dq}}{t}$$

and thus apply Theorem 1.1 in [16].

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References

[1] R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, Astérisque, 61 (1979) 11–13.
[2] D. Bailey, J. Borwein, D. Broadhurst, M. Glasser, Elliptic integral evaluations of Bessel moments and applications, J. Phys. A 41 (2008) 205203 46 pp.
[3] A. Beauville, Les familles stables de courbes elliptiques sur P^1 admettant quatre fibre singulières, C.R. Acad. Sci. Paris Sér. I Math. 294 (1982) 657–660.
[4] F. Beukers, Some congruences for the Apéry numbers, J. Number Th. 21 (1985) 141–155.
[5] F. Beukers, On B. Dwork's accessory parameter problem, Math. Z. 241 (2002) 425–444.
[6] H. Chan, S. Cooper, F. Sica, Congruences satisfied by Apéry-like numbers, Int. J. Number Theory 6 (2010) 89–97.
[7] S. Cooper, Series and iterations for $1/\pi$, Acta Arith. 141 (2010) 33–58.
[8] M. Coster, Supercongruences, Ph.D. thesis, Universiteit Leiden, 1988.
[9] C. Domb, On the theory of cooperative phenomena in crystals, Advances in Phys. 9 (1960) 149–361.
[10] K. Kimoto, Higher Apéry-like numbers arising from special values of the spectral zeta function for the non-commutative harmonic oscillator, preprint available at `http://arxiv.org/abs/0901.0658`
[11] K. Ono, The web of modularity: arithmetic of the coefficients of modular forms and q-series, CBMS Regional Conference Series in Mathematics, 102. American Mathematical Society, Providence, RI, 2004.
[12] F. Rodriguez-Villegas, Hypergeometric families of Calabi-Yau manifolds, in: N. Yui, J. Lewis, (Eds.), Calabi-Yau varieties and mirror symmetry, Fields Institute Communications, 38. American Mathematical Society, Providence, RI, 2003, pp. 223–231.
[13] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, available at `http://www.research.att.com/~njas/sequences/`
[14] J. Stienstra, F. Beukers, On the Picard-Fuchs equation and the formal Brauer group of certain elliptic $K3$-surfaces, Math Ann. 271 (1985) 269–304.
[15] A. van der Poorten, A proof that Euler missed ... Apéry's proof of the irrationality of $\zeta(3)$, An informal report, Math. Intelligencer 1 (1978/79) 195–203.
[16] H. Verrill, Congruences related to modular forms, Int. J. Number Theory 6 (2010) 1367–1390.
[17] Y. Yang, On differential equations satisfied by modular forms, Math. Z. 246 (2004) 1–19.

[18] D. Zagier, Integral solutions of Apéry-like recurrence equations, in: J. Harnad, P. Winternitz, (Eds.), Group and Symmetries: From Neolithic Scots to John McKay, CRM Proceedings & Lecture Notes, 47. American Mathematical Society, Providence, RI, 2009, pp. 349–366.

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