EVALUATION OF THE CONVOLUTION SUMS $\sum_{l+15m=n}\sigma(l)\sigma(m)$ AND $\sum_{3l+5m=n}\sigma(l)\sigma(m)$ AND AN APPLICATION

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Dedicated to Srinivasa Ramanujan on the occasion of his 125th Birth Anniversary

Abstract. We evaluate the convolution sums $\sum_{l,m\in\mathbb{N},l+15m=n}\sigma(l)\sigma(m)$ and $\sum_{l,m\in\mathbb{N},3l+5m=n}\sigma(l)\sigma(m)$ for all $n\in\mathbb{N}$ using the theory of quasi-modular forms and use these convolution sums to determine the number of representations of a positive integer n by the form

$x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+5(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)$.

We also determine the number of representations of positive integers by the quadratic form

$$x_1^2+x_2^2+x_3^2+x_4^2+6(x_5^2+x_6^2+x_7^2+x_8^2),$$

by using the convolution sums obtained earlier by Alaca, Alaca and Williams [6, 3].

1. Introduction

Following [33, 29], for $n, N\in\mathbb{N}$, we define $W_N(n)$ as follows.

$$W_N(n)=\sum_{m<n/N}\sigma(m)\sigma(n-Nm), \tag{1}$$

where $\sigma_r(n)$ is the sum of the r-th powers of the divisors of n. We write $\sigma_1(n)=\sigma(n)$. Also, following [1], we define $W_{a,b}(n)$ for $a,b\in\mathbb{N}$ by

$$W_{a,b}(n):=\sum_{\substack{l,m\\ al+bm=n}}\sigma(l)\sigma(m). \tag{2}$$

Note that $W_{1,N}(n)=W_{N,1}(n)=W_N(n)$. These type of sums were evaluated as early as the 19th century. For example, the sum $W_1(n)$ was evaluated by Besge, Glaisher and Ramanujan [9, 15, 28].

The convolution sums $W_N(n)$ (for $1\le N\le 24$ with a few exceptions) and $W_{a,b}(n)$ for $(a,b)\in\{(2,3),\ (3,4),(3,8),(2,9)\}$ have been evaluated by using either elementary methods or analytic methods (which use ideas of Ramanujan) or algebraic methods (using quasimodular forms) (cf. [9, 15, 28, 24, 20, 21, 27, 18, 1, 2, 6, 3, 4, 5, 12, 13, 22, 33, 34, 29]). Evaluation of

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these convolution sums has been applied to find the number of representations of integers by certain quadratic forms (cf. [18, 1, 2, 6, 3, 4, 33, 34]). In [29], Royer used the theory of quasimodular forms, especially the structure of the space of quasimodular forms (see Eq. (6) below), to evaluate the convolution sums $W_N(n)$ for $1 \leq N \leq 14$, except for $N = 12$. For a list of evaluation of the convolution sums $W_N(n)$, we refer the reader to Table 1 in [29]. In this article, following the method of Royer, we evaluate the convolution sums $W_{15}(n)$ and $W_{3,5}(n)$ by using the theory of quasimodular forms. The evaluation of these convolution sums is then used to determine the number of representations of a positive integer by a certain quadratic form. More precisely, we use these convolution sums to determine the number of representations of integers by the quadratic form $x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 5(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)$. We also give a formula for the number of representations of integers by the quadratic forms $x_1^2 + x_2^2 + x_3^2 + x_4^2 + k(x_5^2 + x_6^2 + x_7^2 + x_8^2)$, $k = 3, 6$, by using the convolution sums $W_3(n)$, $W_6(n)$, $W_{12}(n)$, $W_{24}(n)$, $W_{2,3}(n)$ and $W_{3,4}(n)$ evaluated by K. S. Williams and his co-authors [1, 6, 3, 18]. The formula for $k = 3$ was obtained by Alaca-Williams [7], where the terms corresponding to the cusp forms are different from our formula. The referee has informed us that the evaluation when $k = 6$ has been carried out in a similar manner by Köklüce.

2. Evaluation of $W_{a,b}(n)$ and some applications

2.1. **Evaluation of $W_{15}(n)$ and $W_{3,5}(n)$.** In this section, following Royer [29], we evaluate the convolution sums $W_{15}(n)$ and $W_{3,5}(n)$ by using the theory of quasimodular forms. As an application, we use these convolution sums together with the convolution sum $W_5(n)$ derived by Lemire and Williams [22] to obtain a formula for the number of representations of a positive integer n by the quadratic form Q given by:

$$Q : x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 5(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2). \quad (3)$$

Let

$$\begin{aligned} \Delta_{4,5}(z) = [\Delta(z)\Delta(5z)]^{1/6} \; = \; \eta^4(z)\eta^4(5z) \; = \; q\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{5n})^4 \\ = \sum_{n\geq 1}\tau_{4,5}(n)q^n \end{aligned} \quad (4)$$

be the normalized newform of weight 4 on $\Gamma_0(5)$ (see [29]), where $q = e^{2\pi iz}$. In the above, $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta-function and the Ramanujan function $\Delta(z) = \eta^{24}(z)$ is the normalized cusp form of weight 12 on the full modular group $SL_2(\mathbb{Z})$. The following theorem was proved by Lemire and Williams [22]:

Theorem 2.1.

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{132}\sigma_3\left(\frac{n}{5}\right) + \frac{5-6n}{120}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{5}\right) - \frac{1}{130}\tau_{4,5}(n). \tag{5}$$

In order to evaluate $W_{15}(n)$ and $W_{3,5}(n)$, we use the structure theorem on quasimodular forms of weight k and depth $\leq k/2$. Let $k \geq 2$ and $N \geq 1$ be natural numbers. Let $M_k(\Gamma_0(N))$ denote the $\mathbb{C}$-vector space of modular forms of weight k on the congruence subgroup $\Gamma_0(N)$. For details on modular forms of integral weight we refer the reader to [30, 31, 10]. We now define quasimodular forms. A complex valued holomorphic function f defined on the upper half-plane $\mathcal{H}$ is called a quasimodular form of weight k, depth s (s is a non-negative integer), if there exist holomorphic functions $f_0, f_1, \dots f_s$ on $\mathcal{H}$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{i=0}^{s} f_i(z)\left(\frac{c}{cz+d}\right)^i,$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and such that f_s is holomorphic at the cusps and not identically vanishing. It is a fact that the depth of a quasimodular form of weight k is less than or equal to $k/2$. For details on quasimodular forms we refer to [19, 25, 10]. The Eisenstein series E_2, which is a quasimodular form of weight 2, depth 1 on $SL_2(\mathbb{Z})$ is given by

$$E_2(z) = 1 - 24\sum_{n\geq 1}\sigma(n)e^{2\pi i n z}$$

and this fundamental quasimodular form will be used in our results. The space of quasimodular forms of weight k, depth $\leq k/2$ on $\Gamma_0(N)$ is denoted by $\tilde{M}_k^{\leq k/2}(\Gamma_0(N))$. We need the following structure theorem (see [19, 25]). For an even integer k with $k \geq 2$, we have

$$\tilde{M}_k^{\leq k/2}(\Gamma_0(N)) = \bigoplus_{j=0}^{k/2-1} D^j M_{k-2j}(\Gamma_0(N)) \oplus \mathbb{C}D^{k/2-1}E_2, \tag{6}$$

where the differential operator D is defined by $D := \frac{1}{2\pi i}\frac{d}{dz}$. Using this one can express each quasimodular form of weight k and depth $\leq k/2$ as a linear combination of j-th derivatives of modular forms of weight $k-2j$ on $\Gamma_0(N)$, $0 \leq j \leq k/2-1$ and the $(k/2-1)$-th derivate of the quasimodular form E_2.

We need the following newforms of weights 2 and 4 on $\Gamma_0(15)$ in order to use the structure of the space $\tilde{M}_4^{\leq 2}(\Gamma_0(15))$ to prove our theorem. These newforms are either eta-products or eta-quotients or linear combinations of these. Define the functions $\Delta_{4,15;j}(z)$, $j=1,2$ and $\Delta_{2,15}(z)$ as follows.

$$\begin{aligned}
\Delta_{4,15;1}(z) &:= \Delta_{4,5}(z) + 9\Delta_{4,5}(3z) + 5\Delta_{2,15}(z)^2 + 2\frac{\eta(z)^5\eta(15z)^5}{\eta(3z)\eta(5z)} \\
&= \sum_{n\geq 1} \tau_{4,15;1}(n)q^n, \qquad (7)\\
\Delta_{4,15;2}(z) &:= \Delta_{4,5}(z) + 9\Delta_{4,5}(3z) + 7\Delta_{2,15}(z)^2 = \sum_{n\geq 1} \tau_{4,15;2}(n)q^n,
\end{aligned}$$

$$\Delta_{2,15}(z) = \eta(z)\eta(3z)\eta(5z)\eta(15z), \qquad (8)$$

where $\Delta_{4,5}(z)$ is the newform given by (4). In [14, 16], conditions are given in order to determine the modularity of an eta-quotient (with weight, level and character). Using these conditions, it follows that the functions $\Delta_{4,15;j}(z)$, $j=1,2$ are cusp forms of weight 4 on $\Gamma_0(15)$ and $\Delta_{2,15}(z)$ is a cusp form of weight 2 on $\Gamma_0(15)$. We now show that these cusp forms are newforms in the respective spaces of cusp forms. A theorem of J. Sturm [32] states that the Fourier coefficients upto $\frac{k}{12} \times i_N$ determines a modular form of weight k on $\Gamma_0(N)$, where i_N denotes the index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$. The first few Fourier coefficients of newforms of given weight and level are obtained using the database of L-functions, modular forms, and related objects (see [23]). Comparing the Fourier coefficients obtained from the database with the Fourier coefficients of the cusp forms defined in (7), (8), we conclude that the forms $\Delta_{4,15;j}(z)$, $j=1,2$ and $\Delta_{2,15}(z)$ are newforms.

The following are the main theorems of this section.

Theorem 2.2. *Let $n \in \mathbb{N}$, then*

$$\begin{aligned}
W_{15}(n) \;&=\; \frac{1}{624}\sigma_3(n) + \frac{3}{208}\sigma_3\left(\frac{n}{3}\right) + \frac{25}{624}\sigma_3\left(\frac{n}{5}\right) + \frac{75}{208}\sigma_3\left(\frac{n}{15}\right) \\
&\quad + \frac{5-2n}{120}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{15}\right) - \frac{1}{455}\tau_{4,5}(n) - \frac{9}{455}\tau_{4,5}\left(\frac{n}{3}\right) \\
&\quad - \frac{1}{84}\tau_{4,15;1}(n) - \frac{1}{80}\tau_{4,15;2}(n), \\
W_{3,5}(n) \;&=\; \frac{1}{624}\sigma_3(n) + \frac{3}{208}\sigma_3\left(\frac{n}{3}\right) + \frac{25}{624}\sigma_3\left(\frac{n}{5}\right) + \frac{75}{208}\sigma_3\left(\frac{n}{15}\right) \\
&\quad + \frac{5-6n}{120}\sigma\left(\frac{n}{3}\right) + \frac{1-2n}{24}\sigma\left(\frac{n}{5}\right) - \frac{1}{455}\tau_{4,5}(n) - \frac{9}{455}\tau_{4,5}\left(\frac{n}{3}\right) \\
&\quad - \frac{1}{84}\tau_{4,15;1}(n) + \frac{1}{80}\tau_{4,15;2}(n).
\end{aligned}$$

Proof. Let E_k denote the normalized Eisenstein series of weight k on $SL_2(\mathbb{Z})$ (see [30] for details). When $k=4$, the Eisenstein series E_4 has the following Fourier expansion.

$$E_4(z) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n. \qquad (9)$$

Using the structure of $\tilde{M}_4^{\leq 2}(15)$ from (6), we get

$$\tilde{M}_4^{\leq 2}(\Gamma_0(15)) = M_4(\Gamma_0(15)) \oplus DM_2(\Gamma_0(15)) \oplus \mathbb{C}DE_2. \tag{10}$$

Using the dimension formula (see for example [26, 31]), it follows that the vector space $M_4(\Gamma_0(15))$ has dimension 8 (a basis of this vector space contains 4 non-cusp forms and 4 cusp forms) and the vector space $M_2(\Gamma_0(15))$ has dimension 4 (a basis of this space contains 3 non-cusp forms and 1 cusp form). Now, it is easy to see that the set

$$\{E_4(z), E_4(3z), E_4(5z), E_4(15z), \Delta_{4,5}(z), \Delta_{4,5}(3z), \Delta_{4,15;1}(z), \Delta_{4,15;2}(z)\}$$

forms a basis of the space $M_4(\Gamma_0(15))$ and the set

$$\{\Phi_{1,15}(z), \Phi_{5,15}(z), \Phi_{1,3}(z), \Delta_{2,15}(z)\}$$

forms a basis of the space $M_2(\Gamma_0(15))$, where

$$\Phi_{a,b}(z) := \frac{1}{b-a}(bE_2(bz) - aE_2(az)). \tag{11}$$

Consider the quasimodular form $E_2(z)E_2(15z)$ which belongs to $\tilde{M}_4^{\leq 2}(\Gamma_0(15))$. Therefore, using (10) and the bases mentioned above, we have

$$\begin{aligned} E_2(z)E_2(15z) \quad = \quad & \frac{1}{260}E_4(z) + \frac{9}{260}E_4(3z) + \frac{5}{52}E_4(5z) + \frac{45}{52}E_4(15z) \\ & -\frac{576}{455}\Delta_{4,5}(z) - \frac{5184}{455}\Delta_{4,5}(3z) - \frac{48}{7}\Delta_{4,15;1}(z) \\ & -\frac{36}{5}\Delta_{4,15;2}(z) + \frac{28}{5}D\Phi_{1,15}(z) + \frac{4}{5}DE_2(z). \end{aligned}$$

Similarly, considering $E_2(3z)E_2(5z)$, which is a quasimodular form of weight 4, depth 2 and level 15, we get

$$\begin{aligned} E_2(3z)E_2(5z) \quad = \quad & \frac{1}{260}E_4(z) + \frac{9}{260}E_4(3z) + \frac{5}{52}E_4(5z) + \frac{45}{52}E_4(15z) \\ & -\frac{576}{455}\Delta_{4,5}(z) - \frac{5184}{455}\Delta_{4,5}(3z) - \frac{48}{7}\Delta_{4,15;1}(z) + \frac{36}{5}\Delta_{4,15;2}(z) \\ & +\frac{28}{5}D\Phi_{1,15}(z) - 4D\Phi_{5,15}(z) + \frac{4}{5}D\Phi_{1,3}(z) + \frac{4}{5}DE_2(z). \end{aligned}$$

By comparing the n-th Fourier coefficients, we get the required the convolution sums. □

2.2. **Application to the number of representations.** In this section we apply the convolution sums $W_{15}(n)$ and $W_{3,5}(n)$ to derive the following theorem. Our method of proof is similar to that used by Alaca-Alaca-Williams (see for example [6, 1, 2]).

Theorem 2.3. *The number of representations of a positive integer n by the quadratic form $x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+5(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)$*

is equal to

$$\frac{12}{13}\sigma_3(n) + \frac{108}{13}\sigma_3\left(\frac{n}{3}\right) + \frac{300}{13}\sigma_3\left(\frac{n}{5}\right) + \frac{2700}{13}\sigma_3\left(\frac{n}{15}\right) + \frac{72}{91}\tau_{4,5}(n) + \frac{648}{91}\tau_{4,5}\left(\frac{n}{3}\right) + \frac{72}{7}\tau_{4,15;1}(n).$$

Proof. Let $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. For $l \in \mathbb{N}_0$, let

$$r(l) = \#\left\{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4 | x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2 = l\right\}$$

so that $r(0)=1$. For $l\in\mathbb{N}$, we know that (see [18])

$$r(l) = 12\sum_{\substack{d|l,\\ 3\nmid d}} d = 12\sigma(l) - 36\sigma\left(\frac{l}{3}\right).$$

Let $N(n)$ be the number of representations of the given quadratic form Q defined by (3). Then $N(n)$ is given by

$$\begin{aligned}
N(n) &= \sum_{\substack{l.m\in\mathbb{N}_0\\ l+5m=n}} \Big(\sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\\ x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2=l}} 1\Big)\Big(\sum_{\substack{(x_5,x_6,x_7,x_8)\in\mathbb{Z}^4\\ x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2=m}} 1\Big)\\
&= r(0)r\left(\frac{n}{5}\right) + r(n)r(0) + \sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}} r(l)r(m)\\
&= 12\sigma\left(\frac{n}{5}\right) - 36\sigma\left(\frac{n}{15}\right) + 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right)\\
&\quad + \sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}} \left(12\sigma(l) - 36\sigma\left(\frac{l}{3}\right)\right)\left(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)\right)\\
&= 12\sigma\left(\frac{n}{5}\right) - 36\sigma\left(\frac{n}{15}\right) + 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 144\sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}} \sigma(l)\sigma(m)\\
&\quad -432\sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) - 432\sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m)\\
&\quad +1296\sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}} \sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right)\\
&= 12\sigma\left(\frac{n}{5}\right) - 36\sigma\left(\frac{n}{15}\right) + 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right)\\
&\quad +144\, W_5(n) - 432\, W_{15}(n) - 432\, W_{3,5}(n) + 1296\, W_5\left(\frac{n}{3}\right).
\end{aligned}$$

Substituting the convolution sums using Theorem 2.1 and Theorem 2.2, we get the required formula for $N(n)$. □

2.3. **More applications.** Let Q_k be the quadratic form $x_1^2+x_2^2+x_3^2+x_4^2+k(x_5^2+x_6^2+x_7^2+x_8^2)$ and $N_k(n)$ be the number of representations of integers $n \geq 1$ by Q_k. In this section we use the convolution sums derived in [6, 3] to derive a formula for $N_6(n)$. We note that for $k=2,3,4$ similar formulas were obtained earlier by Williams [34], Alaca-Williams [7] and Alaca-Alaca-Williams [4] respectively. As mentioned in the introduction, we learnt from the referee that the evaluation of $N_6(n)$ has also been derived recently by Köklüce. To find $N_6(n)$ using our method, we need the convolution sums $W_6(n)$, $W_{2,3}$ and $W_{24}(n)$ which were derived by Alaca-Alaca-Williams and they are given in the following theorem.

Theorem 2.4. (cf. [6, 3])

$$\begin{aligned}
W_6(n) &= \frac{1}{120}\sigma_3(n)+\frac{1}{30}\sigma_3\left(\frac{n}{2}\right)+\frac{3}{40}\sigma_3\left(\frac{n}{3}\right)+\frac{3}{10}\sigma_3\left(\frac{n}{6}\right)\\
&\quad+\frac{1-n}{24}\sigma(n)+\frac{1-6n}{24}\sigma\left(\frac{n}{6}\right)-\frac{1}{120}c_6(n),\\
W_{2,3}(n) &= \frac{1}{120}\sigma_3(n)+\frac{1}{30}\sigma_3\left(\frac{n}{2}\right)+\frac{3}{40}\sigma_3\left(\frac{n}{3}\right)+\frac{3}{10}\sigma_3\left(\frac{n}{6}\right)\\
&\quad+\frac{1-2n}{24}\sigma\left(\frac{n}{2}\right)+\frac{1-3n}{24}\sigma\left(\frac{n}{3}\right)-\frac{1}{120}c_6(n),\\
W_{24}(n) &= \frac{1}{1920}\sigma_3(n)+\frac{1}{640}\sigma_3\left(\frac{n}{2}\right)+\frac{3}{640}\sigma_3\left(\frac{n}{3}\right)+\frac{1}{160}\sigma_3\left(\frac{n}{4}\right)\\
&\quad+\frac{9}{640}\sigma_3\left(\frac{n}{6}\right)+\frac{1}{30}\sigma_3\left(\frac{n}{8}\right)+\frac{9}{160}\sigma_3\left(\frac{n}{12}\right)+\frac{3}{10}\sigma_3\left(\frac{n}{24}\right)\\
&\quad+\frac{4-n}{96}\sigma(n)+\frac{1-6n}{24}\sigma\left(\frac{n}{24}\right)-\frac{61}{1920}c_{1,24}(n),
\end{aligned}$$

where $c_6(n)$ and $c_{1,24}(n)$ are the n-th Fourier coefficients of weight 4 normalized newforms which are given in [6, p. 492] *and* [3, p. 94] *respectively.*

In the following we use Theorem 2.4 to derive a formula for $N_6(n)$.

Theorem 2.5. *The number of representations of a positive integer n by the quadratic form Q_6 is given by*

$$\begin{aligned}
N_6(n) &= \frac{2}{5}\sigma_3(n)-\frac{2}{5}\sigma_3\left(\frac{n}{2}\right)+\frac{18}{5}\sigma_3\left(\frac{n}{3}\right)-\frac{8}{5}\sigma_3\left(\frac{n}{4}\right)-\frac{18}{5}\sigma_3\left(\frac{n}{6}\right)\\
&\quad+\frac{128}{5}\sigma_3\left(\frac{n}{8}\right)-\frac{72}{5}\sigma_3\left(\frac{n}{12}\right)+\frac{1152}{5}\sigma_3\left(\frac{n}{24}\right)-\frac{8}{15}c_6(n)\\
&\quad+\frac{32}{15}c_6\left(\frac{n}{2}\right)-\frac{128}{15}c_6\left(\frac{n}{4}\right)+\frac{122}{15}c_{1,24}(n).
\end{aligned}$$

Proof. For $l \in \mathbb{N}_0$, let

$$r_4(l) = \#\left\{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4 | x_1^2+x_2^2+x_3^2+x_4^2=l\right\}$$

so that $r(0) = 1$. For $l \in \mathbb{N}$, we know the formula due to Jacobi (see [17])

$$r_4(l) = 8 \sum_{\substack{d|l,\\ 4\nmid d}} d = 8\sigma(l) - 32\sigma\left(\frac{l}{4}\right).$$

Then N_6 is given by

$$\begin{aligned}
N_6(n) &= \sum_{\substack{l.m\in\mathbb{N}_0\\ l+6m=n}} \Big(\sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\\ x_1^2+x_2^2+x_3^2+x_4^2=l}} 1\Big)\Big(\sum_{\substack{(x_5,x_6,x_7,x_8)\in\mathbb{Z}^4\\ x_5^2+x_6^2+x_7^2+x_8^2=m}} 1\Big)\\
&= r_4(0)r_4\left(\frac{n}{6}\right) + r_4(n)r_4(0) + \sum_{\substack{l,m\in\mathbb{N}\\ l+6m=n}} r_4(l)r_4(m)\\
&= 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{6}\right) - 32\sigma\left(\frac{n}{24}\right)\\
&\quad + \sum_{\substack{l,m\in\mathbb{N}\\ l+6m=n}} \left(8\sigma(l) - 32\sigma\left(\frac{l}{4}\right)\right)\left(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)\right)\\
&= 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{6}\right) - 32\sigma\left(\frac{n}{24}\right) + 64 \sum_{\substack{l,m\in\mathbb{N}\\ l+6m=n}} \sigma(l)\sigma(m)\\
&\quad -256 \sum_{\substack{l,m\in\mathbb{N}\\ l+6m=n}} \sigma(l)\sigma\left(\frac{m}{4}\right) - 256 \sum_{\substack{l,m\in\mathbb{N}\\ l+6m=n}} \sigma\left(\frac{l}{4}\right)\sigma(m)\\
&\quad +1024 \sum_{\substack{l,m\in\mathbb{N}\\ l+6m=n}} \sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right)\\
&= 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{6}\right) - 32\sigma\left(\frac{n}{24}\right) + 64\, W_6(n)\\
&\quad +1024\, W_6\left(\frac{n}{4}\right) - 256\, W_{24}(n) - 256\, W_{2,3}\left(\frac{n}{2}\right).
\end{aligned}$$

Substituting the convolution sums from Theorem 2.4 in the above gives the required formula for $N_6(n)$. □

Remark 2.1. The representation numbers $N_k(n)$ for $k = 2, 4$ were obtained by Williams [34] and by Alaca-Alaca-Williams [4] using the convolution sums $W_{1,8}(n)$, $W_{1,16}(n)$ and for $k = 3$ it was derived by Alaca-Williams [7] as a consequence of the representation of positive integers by certain octonary quadratic forms. Note that $N_3(n)$ can also be obtained in a similar way as done in the cases $k = 2, 4$. In fact,

$$\begin{aligned}
N_3(n) &= 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{3}\right) - 32\sigma\left(\frac{n}{12}\right) \qquad (12)\\
&\quad +64\, W_3(n) + 1024\, W_3\left(\frac{n}{4}\right) - 256\, W_{12}(n) - 256\, W_{3,4}(n).
\end{aligned}$$

Using the convolution sums $W_3(n)$, $W_{3,4}(n)$ and $W_{12}(n)$ obtained in [18, 1], we have the following formula for $N_3(n)$:

$$\begin{aligned} N_3(n) \quad = \quad & \frac{8}{5}\,\sigma_3(n) - \frac{16}{5}\,\sigma_3\left(\frac{n}{2}\right) + \frac{72}{5}\,\sigma_3\left(\frac{n}{3}\right) + \frac{128}{5}\,\sigma_3\left(\frac{n}{4}\right) \qquad (13) \\ & -\frac{144}{5}\,\sigma_3\left(\frac{n}{6}\right) + \frac{1152}{5}\,\sigma_3\left(\frac{n}{12}\right) + \frac{88}{15}\,c_{1,12}(n) + \frac{8}{15}\,c_{3,4}(n). \end{aligned}$$

The difference between the formula given in [7, Theorem 1.1 (ii)] and (13) is due to different cusp forms used. In [7] coefficients of the newform of weight 4 and level 6 appear while in the above formula Fourier coefficients of two cusp forms of weight 4 and level 12 appear.

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