ON THE NUMBER OF REPRESENTATIONS OF AN INTEGER BY CERTAIN QUADRATIC FORMS IN SIXTEEN VARIABLES

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ABSTRACT. We evaluate the convolution sums $\sum_{l,m\in\mathbb{N}, l+2m=n} \sigma_3(l)\sigma_3(m)$, $\sum_{l,m\in\mathbb{N}, l+3m=n} \sigma_3(l)\sigma_3(m)$, $\sum_{l,m\in\mathbb{N}, 2l+3m=n} \sigma_3(l)\sigma_3(m)$ and $\sum_{l,m\in\mathbb{N}, l+6m=n} \sigma_3(l)\sigma_3(m)$ for all $n \in \mathbb{N}$ using the theory of modular forms and use these convolution sums to determine the number of representations of a positive integer n by the quadratic forms $Q_8 \oplus Q_8$ and $Q_8 \oplus 2Q_8$, where the quadratic form Q_8 is given by

$$x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2.$$

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers. For $r, n \in \mathbb{N}$, let $\sigma_r(n) = \sum_{d|n, d\in\mathbb{N}} d^r$ be the divisor function. If n is not a positive integ er, set $\sigma_r(n) = 0$ and we write $\sigma(n)$ for $\sigma_1(n)$. For $a, b, r, s, n \in \mathbb{N}$, we define $W_{a,b}^{r,s}(n)$ by

$$W_{a,b}^{r,s}(n) := \sum_{\substack{l,m\in\mathbb{N} \\ al+bm=n}} \sigma_r(l)\sigma_s(m). \quad (1)$$

In [18] S. Ramanujan evaluated the convolution sums $W_{1,1}^{1,3}(n)$ and $W_{1,1}^{1,5}(n)$ explicitly and in [14], J. G. Huard et. al evaluated the convolution sums $W_{1,2}^{1,3}(n)$ and $W_{2,1}^{1,3}(n)$. In [11], N. Cheng and K. S. Williams explicitly evaluated the convolution sums $W_{1,2}^{1,5}(n)$ and $W_{2,1}^{1,5}(n)$. The convolution sums $W_{1,2}^{1,1}(n)$, $W_{1,4}^{1,1}(n)$, $W_{1,2}^{1,3}(n)$, $W_{2,1}^{1,3}(n)$, $W_{1,5}^{1,1}(n)$, $W_{1,5}^{1,2}(n)$ and $W_{1,5}^{2,1}(n)$ have been computed by E. Royer [19] and the convolution sums $W_{1,3}^{1,3}(n)$, $W_{1,3}^{3,1}(n)$ have been evaluated recently by O.X.M. Yao and E.X.W. Xia [24]. Convolution sums involving the divisor function $\sigma(n)$ have been extensively evaluated by K. S. Williams and his coauthors (see for example ([1]–[8], [10], [11], [14], [22], [23]). One of the reasons for the explicit evaluation of these convolution sums is to use them to obtain formulas for the number of representations of integers by certain types of quadratic forms. Let Q_4 denote the following quadratic form in 4 variables:

$$Q_4 : x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2. \quad (2)$$

Then the quadratic form Q_8 in 8 variables, given by $Q_8 = Q_4 \oplus Q_4$ is:

$$Q_8 = Q_4 \oplus Q_4 = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2. \quad (3)$$

2010 *Mathematics Subject Classification.* Primary 11A25, 11E25; Secondary 11E20, 11F11.
Key words and phrases. sum of divisor functions, convolution sums, modular forms of integral weight.

The number of representations of a positive integer n by the quadratic form Q_4, denoted by $r_{Q_4}(n)$, is given by

$$r_{Q_4}(n) = 12\sigma(n) - 36\sigma(n/3). \tag{4}$$

(See [15, 14].) In [17], G. A. Lomadze obtained a formula for $r_{Q_8}(n)$, the number of representations of an integer n by the quadratic form Q_8, which is given below.

$$r_{Q_8}(n) = 24\sigma_3(n) + 216\sigma_3(n/3). \tag{5}$$

Using these formulas, Yao and Xia [24] obtained a formula for the number of representations of n by the quadratic form (in 12 variables) $Q_4 \oplus Q_8$. In order to obtain this formula, they used the convolution sums $W_{1,3}^{1,3}(n)$ and $W_{1,3}^{3,1}(n)$.

In this paper, we obtain a formula for the number of representations of an integer n by the quadratic form (in 16 variables) $Q_8 \oplus Q_8$ by using the convolution sums $W_{1,1}^{3,3}(n)$ and $W_{1,3}^{3,3}(n)$, which are proved using the theory of modular forms. We also compute the convolution sums $W_{1,2}^{3,3}(n)$, $W_{2,3}^{3,3}(n)$ and $W_{1,6}^{3,3}(n)$ (using the theory of modular forms) and use them to find a formula for the number of representations of a positive integer n by the quadratic form $Q_8 \oplus 2Q_8$.

2. Preliminaries and Statement of the Results

Let $M_k(\Gamma_0(N))$ be the space of modular forms of weight k for the congruence subgroup $\Gamma_0(N)$ and $S_k(\Gamma_0(N))$ the subspace of cusp forms of weight k for $\Gamma_0(N)$. For $k \geq 4$, let E_k be the normalized Eisenstein series of weight k in $M_k(\Gamma_0(1))$, given by

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where $q = e^{2i\pi z}$, $z \in \mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$, and B_k is the k-th Bernoulli number defined by $\displaystyle\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!}x^m$. The first few Eisenstein series are as follows:

$$E_4(z) = 1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n; \quad E_6(z) = 1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n; \quad E_8(z) = 1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n. \tag{6}$$

Since the space $M_8(\Gamma_0(1))$ is one-dimensional, we have the well-known identity

$$E_8(z) = E_4^2(z). \tag{7}$$

By comparing the n-th Fourier coefficients on both sides of (7), we have the identity

$$W_{1,1}^{3,3}(n) = \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n). \tag{8}$$

Let $\eta(z)$ be the Dedekind eta function defined by

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n), \ q = e^{2i\pi z}, z \in \mathbb{H}.$$

Define the functions $\Delta_{8,j}(z)$, $j = 2, 3, 6$ as follows:

$$\Delta_{8,2}(z) := \eta^8(z)\eta^8(2z) = \sum_{n=1}^{\infty} \tau_{8,2}(n) q^n, \tag{9}$$

$$\begin{aligned}\Delta_{8,3}(z) &:= \eta^{12}(z)\eta^4(3z) + 81\eta^6(z)\eta^4(3z)\eta^6(9z) + 18\eta^9(z)\eta^4(3z)\eta^3(9z)\\ &= \sum_{n=1}^{\infty} \tau_{8,3}(n) q^n,\end{aligned} \tag{10}$$

$$\Delta_{8,6}(z) := \frac{1}{240}(E_4(z)E_4(6z) - E_4(2z)E_4(3z)) = \sum_{n=1}^{\infty} \tau_{8,6}(n) q^n. \tag{11}$$

In [12], [13], conditions are given in order to determine the modularity of an eta-quotient (with weight, level, character). Using these conditions we see that $\Delta_{8,2}(z)$ and $\Delta_{8,3}(z)$ are cusp forms of weight 8 on $\Gamma_0(2)$ and $\Gamma_0(3)$ respectively. We show that these cusp forms are new forms in the respective spaces of cusp forms. A theorem of Sturm [21] states that the Fourier coefficients upto $\frac{k}{12} \times i_N$ determines a modular form of weight k on $\Gamma_0(N)$, where i_N is the index of the subgroup $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$. The first few Fourier coefficients of newforms of given weight and level are obtained using the database of $L-$functions, modular forms, and related objects (see [16]). Comparing the Fourier coefficients obtained from the database with the Fourier coefficients of the cusp forms defined in terms of eta-quotients, we conclude that $\Delta_{8,2}(z)$ and $\Delta_{8,3}(z)$ are newforms. By definition, the form $\Delta_{8,6}(z)$ is a cusp form in $S_8(\Gamma_0(6))$. By using the Sturm bound and using the first few coefficients of a newform of weight 8 on $\Gamma_0(6)$ from the database [16], we see that the form $\Delta_{8,6}$ is a newform. For basic details on modular forms and newforms, we refer to [20, 9].

Theorem 2.1. *Let $n \in \mathbb{N}$, and let $\tau_{8,j}(n)$, $j = 2, 3, 6$ be the n-th Fourier coefficients of the newforms $\Delta_{8,j}(z)$ defined in* (9), (10) *and* (11) *respectively. Then*

$$W_{1,2}^{3,3}(n) = -\frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7\left(\frac{n}{2}\right) + \frac{1}{272}\tau_{8,2}(n).$$

$$W_{1,3}^{3,3}(n) = -\frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{9840}\sigma_7(n) + \frac{81}{9840}\sigma_7\left(\frac{n}{3}\right) + \frac{1}{246}\tau_{8,3}(n).$$

$$\begin{aligned}W_{2,3}^{3,3}(n) = &-\frac{1}{240}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{167280}\sigma_7(n) + \frac{1}{10455}\sigma_7\left(\frac{n}{2}\right)\\ &+\frac{27}{55760}\sigma_7\left(\frac{n}{3}\right) + \frac{27}{3485}\sigma_7\left(\frac{n}{6}\right) + \frac{7}{8160}\tau_{8,2}(n) + \frac{189}{2720}\tau_{8,2}\left(\frac{n}{3}\right)\\ &+\frac{1}{820}\tau_{8,3}(n) + \frac{4}{205}\tau_{8,3}\left(\frac{n}{2}\right) - \frac{1}{480}\tau_{8,6}(n).\end{aligned}$$

$$\begin{aligned}W_{1,6}^{3,3}(n) = &-\frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{6}\right) + \frac{1}{167280}\sigma_7(n) + \frac{1}{10455}\sigma_7\left(\frac{n}{2}\right)\\ &+\frac{27}{55760}\sigma_7\left(\frac{n}{3}\right) + \frac{27}{3485}\sigma_7\left(\frac{n}{6}\right) + \frac{7}{8160}\tau_{8,2}(n) + \frac{189}{2720}\tau_{8,2}\left(\frac{n}{3}\right)\\ &+\frac{1}{820}\tau_{8,3}(n) + \frac{4}{205}\tau_{8,3}\left(\frac{n}{2}\right) + \frac{1}{480}\tau_{8,6}(n).\end{aligned}$$

Let $N_1(n)$ be the number of representations of the positive integer n by the quadratic form $Q_{16} = Q_8 \oplus Q_8$, where Q_8 is defined by (3). Note that Q_{16} is the quadratic form $x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2 + x_9^2 + x_9x_{10} + x_{10}^2 + x_{11}^2 + x_{11}x_{12} + x_{12}^2 + x_{13}^2 + x_{13}x_{14} + x_{14}^2 + x_{15}^2 + x_{15}x_{16} + x_{16}^2$.

Theorem 2.2. *Let $n \in \mathbb{N}$. Then,*

$$N_1(n) = \frac{240}{41}\sigma_7(n) + \frac{19440}{41}\sigma_7\left(\frac{n}{3}\right) + \frac{1728}{41}\tau_{8,3}\left(n\right), \tag{12}$$

where $\tau_{8,3}(n)$ is given by (10).

In [17, Formula (VII), p.12], Lomadze proved the following formula for $N_1(n)$:
For $n \in \mathbb{N}$,

$$N_1(n) = \frac{240}{41}\left(\sigma_7(n) + 81\sigma_7(n/3)\right) + \frac{16}{41}\sum_{\substack{(x_1,\ldots,x_8)\in\mathbb{Z}^8\\ x_1^2+x_1x_2+x_2^2+\ldots+x_7^2+x_7x_8+x_8^2=n}} (45x_1^4 - 30nx_1^2 + 2n^2). \tag{13}$$

Comparing the above two expressions for $N_1(n)$ (equations (12) and (13)), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$, we have*

$$\tau_{8,3}(n) = \frac{1}{108}\sum_{\substack{(x_1,\ldots,x_8)\in\mathbb{Z}^8\\ x_1^2+x_1x_2+x_2^2+\ldots+x_7^2+x_7x_8+x_8^2=n}} (45x_1^4 - 30nx_1^2 + 2n^2). \tag{14}$$

Let $N_2(n)$ be the number of representations of n by the quadratic form $Q'_{16} = Q_8 \oplus 2Q_8$ defined by:
$Q'_{16}(x_1,\ldots,x_{16}) = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2 + 2(x_9^2 + x_9x_{10} + x_{10}^2 + x_{11}^2 + x_{11}x_{12} + x_{12}^2 + x_{13}^2 + x_{13}x_{14} + x_{14}^2 + x_{15}^2 + x_{15}x_{16} + x_{16}^2)$.

Theorem 2.4. *Let $n \in \mathbb{N}$. Then,*

$$\begin{aligned} N_2(n) \ &= \frac{240}{697}\sigma_7(n) + \frac{3840}{697}\sigma_7\left(\frac{n}{2}\right) + \frac{19440}{697}\sigma_7\left(\frac{n}{3}\right) + \frac{311040}{697}\sigma_7\left(\frac{n}{6}\right) \\ &\quad + \frac{936}{85}\tau_{8,2}(n) + \frac{75816}{85}\tau_{8,2}\left(\frac{n}{3}\right) + \frac{2592}{205}\tau_{8,3}(n) + \frac{41472}{205}\tau_{8,3}\left(\frac{n}{2}\right), \end{aligned} \tag{15}$$

where $\tau_{8,2}(n)$ and $\tau_{8,3}(n)$ are given by (9) *and* (10) *respectively.*

3. Proofs

3.1. **Proof of Theorem 2.1.** The vector space $M_8(\Gamma_0(2))$ has dimension 3 with a basis $\{E_8(z), E_8(2z), \Delta_{8,2}(z)\}$, where $\Delta_{8,2}(z)$ is the unique normalized newform of weight 8 and level 2. Now $E_4(z)E_4(2z) \in M_8(\Gamma_0(2))$ and writing it as a linear combination of the above basis, we get

$$E_4(z)E_4(2z) = \frac{1}{17}E_8(z) + \frac{16}{17}E_8(2z) + \frac{360}{17}\Delta_{8,2}(z).$$

The dimension of the space $M_8(\Gamma_0(3))$ is 3 with basis $\{E_8(z), E_8(3z), \Delta_{8,3}(z)\}$, where $\Delta_{8,3}(z)$ is the unique normalized newform of weight 8 and level 3. Now $E_4(z)E_4(3z) \in M_8(\Gamma_0(3))$. Writing it as a linear combination of the above basis, we obtain

$$E_4(z)E_4(3z) = \frac{1}{82}E_8(z) + \frac{81}{82}E_8(3z) + \frac{9600}{41}\Delta_{8,3}(z).$$

The space $M_8(\Gamma_0(6))$ is 9 dimensional with basis

$$\{E_8(z), E_8(2z), E_8(3z), E_8(6z), \Delta_{8,2}(z), \Delta_{8,2}(3z), \Delta_{8,3}(z), \Delta_{8,3}(2z), \Delta_{8,6}(z)\},$$

where $\Delta_{8,6}(z)$ is the unique normalized newform of weight 8 and level 6. Now the functions $E_4(2z)E_4(3z)$ and $E_4(z)E_4(6z)$ are modular forms in $M_8(\Gamma_0(6))$. Writing each of them in terms of the above basis, we have the modular identities

$$\begin{aligned} E_4(2z)E_4(3z) &= \frac{1}{1394}E_8(z) + \frac{8}{697}E_8(2z) + \frac{81}{1394}E_8(3z) + \frac{648}{697}E_8(6z) + \frac{840}{17}\Delta_{8,2}(z) \\ &\quad + \frac{68040}{17}\Delta_{8,2}(3z) + \frac{2880}{41}\Delta_{8,3}(z) + \frac{46080}{41}\Delta_{8,3}(2z) - 120\Delta_{8,6}(z) \end{aligned}$$

and

$$\begin{aligned} E_4(z)E_4(6z) &= \frac{1}{1394}E_8(z) + \frac{8}{697}E_8(2z) + \frac{81}{1394}E_8(3z) + \frac{648}{697}E_8(6z) + \frac{840}{17}\Delta_{8,2}(z) \\ &\quad + \frac{68040}{17}\Delta_{8,2}(3z) + \frac{2880}{41}\Delta_{8,3}(z) + \frac{46080}{41}\Delta_{8,3}(2z) + 120\Delta_{8,6}(z). \end{aligned}$$

By comparing the n-th Fourier coefficients, we obtain the required convolution sums.

3.2. **Proof of Theorem 2.2.** For $n \in \mathbb{N}$, let $r_{Q_8}(n)$ be given by (5) and we let $r_{Q_8}(0) = 1$. Then $N_1(n)$ is given by

$$\begin{aligned} N_1(n) &= \sum_{\substack{a,b\in\mathbb{N}\cup\{0\} \\ a+b=n}} \left(\sum_{\substack{(x_1,\ldots,x_8)\in\mathbb{Z}^8 \\ Q_8(x_1,\ldots,x_8)=a}} 1 \right) \times \left(\sum_{\substack{(x_9,\ldots,x_{16})\in\mathbb{Z}^8 \\ Q_8(x_9,\ldots,x_{16})=b}} 1 \right) \\ &= 2r_{Q_8}(n)r_{Q_8}(0) + \sum_{\substack{a,b\in\mathbb{N} \\ a+b=n}} r_{Q_8}(a)r_{Q_8}(b) \\ &= 48\sigma_3(n) + 432\sigma_3\left(\frac{n}{3}\right) + \sum_{\substack{a,b\in\mathbb{N} \\ a+b=n}} \left(24\sigma_3(a) + 216\sigma_3\left(\frac{a}{3}\right)\right)\left(24\sigma_3(b) + 216\sigma_3\left(\frac{b}{3}\right)\right) \\ &= 48\sigma_3(n) + 432\sigma_3\left(\frac{n}{3}\right) + 576\sum_{\substack{a,b\in\mathbb{N} \\ a+b=n}} \sigma_3(a)\sigma_3(b) + 5184\sum_{\substack{a,b\in\mathbb{N} \\ a+b=n}} \sigma_3(a)\sigma_3\left(\frac{b}{3}\right) \\ &\quad + 5184\sum_{\substack{a,b\in\mathbb{N} \\ a+b=n}} \sigma_3\left(\frac{a}{3}\right)\sigma_3(b) + 46656\sum_{\substack{a,b\in\mathbb{N} \\ a+b=n}} \sigma_3\left(\frac{a}{3}\right)\sigma_3\left(\frac{b}{3}\right) \\ &= 48\sigma_3(n) + 432\sigma_3\left(\frac{n}{3}\right) + 576\, W_{1,1}^{3,3}(n) + 10368\, W_{1,3}^{3,3}(n) + 46656\, W_{1,1}^{3,3}\left(\frac{n}{3}\right). \end{aligned}$$

Substituting the formulas for the convolution sum $W_{1,1}^{3,3}$ from (8) and the convolution sum $W_{1,3}^{3,3}(n)$ from Theorem 2.1, we get the required formula for $N_1(n)$.

3.3. **Proof of Theorem 2.4.** As before, set $r_{Q_8}(0)=1$. Then using (5), we see that $N_2(n)$ $(n\in\mathbb{N})$ is given by

$$
\begin{aligned}
N_2(n) &= \sum_{\substack{a,b\in\mathbb{N}\cup\{0\}\\ a+2b=n}} \left(\sum_{\substack{(x_1,\ldots,x_8)\in\mathbb{Z}^8\\ Q_8(x_1,\ldots,x_8)=a}} 1 \right) \times \left(\sum_{\substack{(x_9,\ldots,x_{16})\in\mathbb{Z}^8\\ Q_8(x_9,\ldots,x_{16})=b}} 1 \right)\\
&= r_{Q_8}\left(\frac{n}{2}\right) r_{Q_8}(0) + r_{Q_8}(n) r_{Q_8}(0) + \sum_{\substack{a,b\in\mathbb{N}\\ a+2b=n}} r_{Q_8}(a) r_{Q_8}(b)\\
&= 24\sigma_3\left(\frac{n}{2}\right) + 216\sigma_3\left(\frac{n}{6}\right) + 24\sigma_3(n) + 216\sigma_3\left(\frac{n}{3}\right)\\
&\quad + \sum_{\substack{a,b\in\mathbb{N}\\ a+2b=n}} \left(24\sigma_3(a) + 216\sigma_3\left(\frac{a}{3}\right)\right)\left(24\sigma_3(b) + 216\sigma_3\left(\frac{b}{3}\right)\right)\\
&= 24\sigma_3\left(\frac{n}{2}\right) + 216\sigma_3\left(\frac{n}{6}\right) + 24\sigma_3(n) + 216\sigma_3\left(\frac{n}{3}\right) + 576 \sum_{\substack{a,b\in\mathbb{N}\\ a+2b=n}} \sigma_3(a)\sigma_3(b)\\
&\quad +5184 \sum_{\substack{a,b\in\mathbb{N}\\ a+2b=n}} \sigma_3(a)\sigma_3\left(\frac{b}{3}\right) + 5184 \sum_{\substack{a,b\in\mathbb{N}\\ a+2b=n}} \sigma_3\left(\frac{a}{3}\right)\sigma_3(b) + 46656 \sum_{\substack{a,b\in\mathbb{N}\\ a+2b=n}} \sigma_3\left(\frac{a}{3}\right)\sigma_3\left(\frac{b}{3}\right)\\
&= 24\sigma_3\left(\frac{n}{2}\right) + 216\sigma_3\left(\frac{n}{6}\right) + 24\sigma_3(n) + 216\sigma_3\left(\frac{n}{3}\right)\\
&\quad +576\, W_{1,2}^{3,3}(n) + 5184\, W_{1,6}^{3,3}(n) + 5184\, W_{2,3}^{3,3}(n) + 46656\, W_{1,2}^{3,3}\left(\frac{n}{2}\right).
\end{aligned}
$$

Substituting the formulas for the convolution sums $W_{1,2}^{3,3}(n)$, $W_{2,3}^{3,3}(n)$ and $W_{1,6}^{3,3}(n)$ given in Theorem 2.1, we get the required formula for $N_2(n)$.

Acknowledgements. The research was carried out during the second author's visit to the Harish-Chandra Research Institute (HRI), Allahabad. He would like to thank the institute for the hospitality during his stay. We thank the referee for a careful reading of the paper and pointing out important corrections in the statement of Theorem 2.1 and also about the formula for $N_1(n)$ obtained by Lomadze.

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