RANKIN'S METHOD AND JACOBI FORMS OF SEVERAL VARIABLES

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Abstract. Following Rankin's method, D. Zagier computed the n-th Rankin-Cohen bracket of a modular form g of weight k_1 with the Eisenstein series of weight k_2 and then computed the inner product of this Rankin-Cohen bracket with a cusp form f of weight $k = k_1 + k_2 + 2n$ and showed that this inner product gives, upto a constant, the special value of the Rankin-Selberg convolution of f and g. This result was generalized to Jacobi forms of degree 1 by Y. Choie and W. Kohnen. In this paper, we generalize this result to Jacobi forms defined over $\mathcal{H} \times \mathbb{C}^{(g,1)}$.

1. Introduction

There are many interesting connections between differential operators and the theory of modular forms and many interesting results have been studied. In [10], R. A. Rankin gave a general description of the differential operators which send modular forms to modular forms. In [6], H. Cohen constructed bilinear operators and obtained elliptic modular form with interesting Fourier coefficients. In [12], D. Zagier studied the algebraic properties of these bilinear operators and called them as Rankin-Cohen brackets. In [13], following Rankin's method, Zagier computed the n-th Rankin-Cohen bracket of a modular form g of weight k_1 with the Eisenstein series of weight k_2 and then computed the inner product of this Rankin-Cohen bracket with a cusp form f of weight $k = k_1 + k_2 + 2n$ and showed that this inner product gives, upto a constant, the special value of the Rankin-Selberg convolution of f and g.

Rankin-Cohen brackets for Jacobi forms were studied by Y. Choie [2, 3] by using the heat operator. Following the work of Zagier mentioned in the above paragraph, Y. Choie and W. Kohnen [5] generalized the above result of Zagier to Jacobi forms. They computed the Petersson scalar product $\langle f, [g, E_{k_2,m_2}]_\nu \rangle$ of a Jacobi cusp form f of weight k, index m against the Rankin-Cohen bracket $[g, E_{k_2,m_2}]_\nu$ of a Jacobi form g of weight k_1, index m_1 and the Jacobi Eisenstein series E_{k_2,m_2} of weight k_2, index m_2, where $k = k_1 + k_2 + 2\nu$ and $m = m_1 + m_2$. Alhough the concept of Rankin-Selberg convolution has not been done yet in the case of Jacobi forms, the above mentioned work of Choie and Kohnen gives the inner product considered in terms of the special value of a kind of Rankin-Selberg type convolution of the Jacobi forms f and g.

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In this paper, we generalize the work of Choie and Kohnen to Jacobi forms defined over $\mathcal{H}\times\mathbb{C}^{(g,1)}$. Since the method is similar, we shall give only a brief sketch of the proof with the corresponding steps.

2. Preliminaries on Jacobi forms over $\mathcal{H} \times \mathbb{C}^{(g,1)}$

Let $g \geq 1$ be a fixed positive integer. The Jacobi group $\Gamma_{1,g}^J = \Gamma_1 \ltimes \left(\mathbb{Z}^{(g,1)} \times \mathbb{Z}^{(g,1)}\right)$ acts on $\mathcal{H} \times \mathbb{C}^{(g,1)}$ in the usual way by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \circ (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right),$$

where $\Gamma_1 = SL_2(\mathbb{Z})$ is the full modular group.

Let $k \in \mathbb{Z}$ and M be a positive definite symmetric half-integral matrix of size $g \times g$. If $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma_{1,g}^J$ and ϕ is a complex valued function on $\mathcal{H} \times \mathbb{C}^{(g,1)}$, then define

$$\phi|_{k,M}\gamma := (c\tau + d)^{-k} e(-c(c\tau + d)^{-1} M[z + \lambda\tau + \mu] + M[\lambda]\tau + 2\lambda^t M z)\phi(\gamma \cdot (\tau, z)),$$

where $A[B] = B^t AB$ with A, B matrices of appropriate size.

Let $J_{k,M}$ be the space of Jacobi forms of weight k and index M on $\Gamma_{1,g}^J$. That is, the space of holomorphic functions $\phi : \mathcal{H} \times \mathbb{C}^{(g,1)} \to \mathbb{C}$ satisfying $\phi|_{k,M}\gamma = \phi$, $\forall \gamma \in \Gamma_{1,g}^J$ and having a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n\in\mathbb{Z}, r\in\mathbb{Z}^g, 4n\geq M^{-1}[r^t]} c(n, r)e(n\tau + rz).$$

Further, we say that F is a cusp form if and only if $c(n, r) \neq 0$ implies $4n > M^{-1}[r^t]$. We denote the space of all Jacobi cusp forms by $J_{k,M}^{cusp}$.

For $F, G \in J_{k,M}$ with one of them a Jacobi cusp form, the Petersson inner product is defined as

$$\langle F, G\rangle = \int_{\Gamma_{1,g}^J \backslash \mathcal{H}\times\mathbb{C}^{(g,1)}} F(\tau, z)\overline{G(\tau, z)} v^k e(-4\pi M[y] \cdot v^{-1})\, dV_g^J,$$

where $\tau = u + iv, z = x + iy$, and $dV_g^J = v^{-g-2} dudvdxdy$ is the invariant measure. The space $(J_{k,M}^{cusp}, \langle, \rangle)$ is a finite dimensional Hilbert space. For more details on Jacobi forms on $\mathcal{H} \times \mathbb{C}^{(g,1)}$ we refer to [1, 14].

2.1. **Poincaré series.** Let $n \in \mathbb{Z}, r \in \mathbb{Z}^g$, with $4n > M^{-1}[r^t]$. For $k > g+2$ let $P_{k,M;(n,r)}$ be the (n,r)-th Poincaré series in $J_{k,M}^{cusp}$ characterized by

$$\langle \phi, P_{k,M;(n,r)} \rangle = \lambda_{k,M,D}\ c_\phi(n,r) \qquad \text{for all } \phi \in J_{k,M}^{cusp}, \tag{1}$$

where $c_\phi(n,r)$ denotes the (n,r)-th Fourier coefficient of ϕ and

$$\lambda_{k,M,D} := 2^{(g-1)(k-g/2-1)-g}\Gamma(k-g/2-1)\pi^{-k+g/2+1}(\det M)^{k-(g+3)/2}D^{-k+g/2+1},$$

$$D \ = \ \det(2T),\ T = \begin{pmatrix} n & r/2 \\ r^t/2 & M \end{pmatrix}.$$

The Poincaré series $P_{k,M;(n,r)}$ has the following Fourier expansion

$$P_{k,M;(n,r)}(\tau,z) = \sum_{n' \in \mathbb{Z}, r' \in \mathbb{Z}^g, 4n' > M^{-1}[r'^t]} (g_{k,M;(n,r)}(n',r') + (-1)^k g_{k,M;(n,r)}(n',-r'))\ e(n'\tau + r'z),$$

where

$$g_{k,M;(n,r)}(n',r') = \delta_M(n,r,n',r') + i^{-k}\pi 2^{1-g/2}(\det M)^{-1/2}$$
$$\cdot (D'/D)^{k/2-g/4-1/2} \sum_{c \geq 1} H_{M,c}(n,r,n',r') J_{k-g/2-1}\left(\frac{\pi\sqrt{D'D}}{2^{g-1}\det M \cdot c}\right),$$

where

$$D := \det 2 \begin{pmatrix} n' & r'/2 \\ r'^t/2 & M \end{pmatrix},$$

$$\delta_m(n,r,n',r') := \begin{cases} 1 & if D = D', r' \equiv r \pmod{\mathbb{Z}^g \cdot 2M} \\ 0 & otherwise \end{cases}$$

and

$$H_{M,c}(n,r,n',r') = c^{-g/2-1} \sum_{\substack{x (\mathrm{mod}\ c), \\ y (\mathrm{mod}\ c)^*}} e_c((M[x] + rx + n)y^{-1} + n'y + r'x)e_{2c}(r'M^{-1}r^t)$$

[In the above, x (resp. y) runs over a complete set of representatives for $\mathbb{Z}^{(g,1)}/c\mathbb{Z}^{(g,1)}$ (resp. $(\mathbb{Z}/c\mathbb{Z})^*$, and y^{-1} denotes an inverse of $y \pmod c$), $e_c(a) := e^{2\pi i a/c}$, $a \in \mathbb{Z}$, and $J_{k-g/2-1}$ denotes the Bessel function of order $k - g/2 - 1$.] For details we refer to [1, Lemma 1].

3. Generalized Heat Operator

For a positive definite symmetric half-integral matrix $M = (m_{ij})$ of size $g \times g$, we define the heat operator by

$$L_M := 8\pi i|M|\frac{\partial}{\partial \tau} - \sum_{1\le i,j\le g} M_{ij}\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}, \tag{2}$$

where $|M| = \det M$, M_{ij} is the cofactor of the entry m_{ij}, $\tau \in \mathcal{H}, z^t = (z_1, z_2, \cdots, z_g) \in \mathbb{C}^g$. Note that when $g = 1$ the above heat operator reduces to the classical heat operator, viz., $8\pi im\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}$.

Let $r^t = (r_1, r_2, \cdots, r_g)$. Then using the fact that

$$\begin{aligned} \frac{\partial}{\partial \tau}(e(n\tau + rz)) &= 2\pi in\; e(n\tau + rz),\\ \frac{\partial}{\partial z_\alpha}(e(n\tau + rz)) &= 2\pi i\; r_\alpha\; e(n\tau + rz), \quad 1 \le \alpha \le g,\\ \frac{\partial}{\partial z_\alpha}\frac{\partial}{\partial z_\beta}(e(n\tau + rz)) &= (2\pi i)^2\; r_\alpha r_\beta\; e(n\tau + rz), \quad 1 \le \alpha, \beta \le g, \end{aligned}$$

we get

$$\begin{aligned} L_M(e(n\tau + rz) =&\; 8\pi i|M| \cdot 2\pi in \cdot e(n\tau + rz) - \sum_{1\le\alpha,\beta\le g} M_{\alpha\beta}(2\pi i)^2 r_\alpha r_\beta\; e(n\tau + rz)\\ =&\; (2\pi i)^2(4n|M| - \tilde{M}[r^t])\; e(n\tau + rz), \end{aligned} \tag{3}$$

where $\tilde{A}$ denotes the matrix (A_{ij}) with A_{ij} being the cofactor of the ij-th entry of the symmetric matrix A.

We obtain the action of the heat operator on Jacobi forms in the following lemma.

Lemma 3.1. *Let $F \in J_{k,M}$. Then*

$$(L_M F)\big|_{k+2,M} A = L_M(F\big|_{k,M} A) + (8\pi i|M|)\left(k - \frac{g}{2}\right)\left(\frac{\gamma}{\gamma\tau + \delta}\right)(F\big|_{k,M} A), \tag{4}$$

for all $A = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma_1$. *In general, for any integer* $\nu \ge 0$,

(5)

$$(L_M^\nu F)|_{k+2\nu,M} A$$
$$= \sum_{l=0}^{\nu}\binom{\nu}{l}(8\pi i|M|)^{\nu-l}\frac{(k - g/2 + \nu - 1)!}{(k - g/2 + l - 1)!}\left(\frac{\gamma}{\gamma\tau + \delta}\right)^{\nu-l} L_M^l(F|_{k,M}A).$$

Moreover, for all $\lambda, \lambda' \in \mathbb{Z}^g$,

$$L_M(F|_M[\lambda, \lambda']) = (L_M F)|_M[\lambda, \lambda']. \tag{6}$$

Proof. Though our L_M operator differs (slightly) from the operator defined in [4], the proof goes along the same lines of the proof of Lemma 3.3 of [4]. □

We now define the Rankin-Cohen bracket for Jacobi forms on $\mathcal{H} \times \mathbb{C}^{(g,1)}$.

Definition: Let $\nu \geq 0$ be an integer and let $F \in J_{k_1,M_1}, G \in J_{k_2,M_2}$, where k_1, k_2 are positive integers and M_1 and M_2 are positive-definite, symmetric half-integer matrices of size $g \times g$. Define the ν-th Rankin-Cohen bracket of F and G by

$$[F,G]_\nu = \sum_{l=0}^{\nu} (-1)^l \binom{k_1 - \frac{g}{2} + \nu - 1}{\nu - l} \binom{k_2 - \frac{g}{2} + \nu - 1}{l} |M_1|^{\nu-l} |M_2|^l L_{M_1}^l(F) L_{M_2}^{\nu-l}(G) \tag{7}$$

Using Lemma 3.1 we show that the Rankin-Cohen bracket $[\ ,\ ]_\nu$ gives a bilinear map from $J_{k_1,M_1} \times J_{k_2,M_2}$ to $J_{k_1+k_2+2\nu,M_1+M_2}$ (in fact, to $J^{cusp}_{k_1+k_2+2\nu,M_1+M_2}$ if $\nu > 0$).

Proposition 3.2. *Let $\nu \geq 0$ be an integer and let $F \in J_{k_1,M_1}, G \in J_{k_2,M_2}$. Then $[F,G]_\nu \in J_{k_1+k_2+2\nu,M_1+M_2}$. If $\nu > 0$, then $[F,G]_\nu \in J^{cusp}_{k_1+k_2+2\nu,M_1+M_2}$.*

Proof. By (6) we see that the action of the heat operator on Jacobi forms is invariant under the lattice action and so the invariance with respect to the lattice action of the Rankin-Cohen bracket follows from the definition. It remains to show that the Rankin-Cohen bracket is invariant, under the stroke operation, with respect to the group action. Making use of (5), we see that for $A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_1$

$$\begin{aligned}
&[F,G]_\nu \big|_{k_1+k_2+2\nu,M_1+M_2} A \\
&= \sum_{l=0}^{\nu} (-1)^l \binom{k_1'+\nu}{\nu-l} \binom{k_2'+\nu}{l} |M_1|^{\nu-l}|M_2|^l \ L_{M_1}^l(F)\big|_{k_1+2l,M_1} A \ L_{M_2}^{\nu-l}(G)\big|_{k_2+2(\nu-l),M_2} A \\
&= \sum_{l=0}^{\nu} (-1)^l \binom{k_1'+\nu}{\nu-l} \binom{k_2'+\nu}{l} \Bigg(\sum_{u=0}^{l} \sum_{v=0}^{\nu-l} \binom{l}{u} \binom{\nu-l}{v} (8\pi i)^{\nu-u-v} |M_1|^{\nu-u} |M_2|^{\nu-v} \\
&\qquad \times \frac{(k_1'+l)!}{(k_1'+u)!} \frac{(k_2'+\nu-l)!}{(k_2'+v)!} \left(\frac{c}{c\tau+d}\right)^{\nu-u-v} L_{M_1}^u(F)\ L_{M_2}^v(G) \Bigg),
\end{aligned} \tag{8}$$

where $k_j' = k_j - \frac{g}{2} - 1,\ j = 1, 2$. When $u + v = \nu$, the right-hand side becomes $[F,G]_\nu$, and so it remains to show that the terms corresponding to $u + v < \nu$ vanish. It is easy to see that the coefficient corresponding to $L_{M_1}^u(F)\ L_{M_2}^v(G)$, with $u + v \leq \nu - 1,\ u \leq v$ is given by

$$\left(\frac{8\pi ic}{c\tau+d}\right)^{\nu-u-v} |M_1|^{\nu-u}|M_2|^{\nu-v} \ \frac{(k_1'+\nu)!}{u!(k_1'+u)!} \frac{(k_2'+\nu)!}{v!(k_2'+v)!} \sum_{l=u}^{\nu-v} \frac{(-1)^l}{(l-u)!(\nu-v-l)!}$$

and the sum in the last expression is equal to zero. This completes the proof. □

We shall now state the main theorem of this paper.

Theorem 3.3. *Let $F \in J_{k,M}^{cusp}$ with Fourier coefficients $a(n,r)$ and $G \in J_{k_1,M_1}$ with Fourier coefficients $b(n,r)$. Let E_{k_2,M_2} be the Jacobi Eisenstein series in J_{k_2,M_2} such that $k = k_1 + k_2 + 2\nu$ with $\nu \geq 0, M = M_1 + M_2$ and $k_1 > g+2, k_2 > k_1 + g + 2$. Then*

$$\langle F, [G, E_{k_2,M_2}]_\nu \rangle = c_{k,k_2,M,M_2,g;\nu} \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^g,\\ 4n\geq M_1^{-1}[r^t]}} \frac{(4n|M_1| - \tilde{M}_1[r^t])^\nu \ a(n,r)\overline{b(n,r)}}{(4n|M| - \tilde{M}[r^t])^{k-g/2-1}}, \tag{9}$$

where

$$c_{k,k_2,M,M_2,g;\nu} = 2^{k'(g-1)-g-2\nu}\pi^{-k'-2\nu}|M|^{k'-1/2}|M_2|^{-\nu}\Gamma(k')\frac{\nu!\ k_2'!}{(k_2'+\nu)!}, \tag{10}$$

with $k' = k - g/2 - 1$, $k_2' = k_2 - g/2 - 1$.

The rest of this section is devoted to a proof of Theorem 3.3.

3.1. **Action of heat operator on Eisenstein series.** Let M be as before and let $E_{k,M}$ be the Jacobi Eisenstein series of weight k and index M defined by

$$E_{k,M} = \sum_{\gamma\in\Gamma_{1,g,\infty}^J\backslash\Gamma_{1,g}^J} 1\big|_{k,M}\gamma, \tag{11}$$

where $\Gamma_{1,g,\infty}^J = \left\{ \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, (0,\mu) \right) \big| a \in \mathbb{Z}, \mu \in \mathbb{Z}^g \right\}$.

Lemma 3.4. *For a positive integer ν, we have*

$$\begin{aligned}(L_M^\nu E_{k,M})(\tau,z) \ &= \ (-4)^\nu \frac{\Gamma(k-g/2+\nu)}{\Gamma(k-g/2)}|M|^\nu \\ &\sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_{1,g,\infty}^J/\Gamma_{1,g}^J \lambda\in\mathbb{Z}} (2\pi i c)^\nu (c\tau+d)^{-k-\nu} e\left(M[\lambda]\frac{a\tau+b}{c\tau+d} + \frac{2\lambda^t M z}{c\tau+d} - \frac{cM[z]}{c\tau+d}\right)\end{aligned} \tag{12}$$

Proof. Using the definition of the Eisenstein series, we have

$$L_M^\nu E_{k,M} = \sum_{\gamma\in\Gamma_{1,g,\infty}^J/\Gamma_{1,g}^J} L_M^\nu \left(1\big|_{k,M}\gamma\right).$$

By taking $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (a\lambda, b\lambda)\right)$ as a set of coset representatives in the above sum, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1, \lambda \in \mathbb{Z}^g$, we have

$$L_M^\nu E_{k,M} = \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \lambda \in \mathbb{Z}^g.} L_M^\nu \left(1\big|_{k,M} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \big|_M (a\lambda, b\lambda).$$

It is easy to see that

$$\begin{aligned} L_M \left(1\big|_{k,M} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= L_M \left((c\tau + d)^{-k} e\left(\frac{-c\, M[z]}{c\tau + d}\right)\right) \\ &= -8\pi i c|M|(k - g/2)(c\tau + d)^{-k-1} e\left(\frac{-c\, M[z]}{c\tau + d}\right), \end{aligned}$$

where we have used the fact that

$$\sum_{1\le \alpha,\beta\le g} M_{\alpha\beta} \left(\sum_{1\le i\le g} m_{i\beta} z_i\right)\left(\sum_{1\le i\le g} m_{i\alpha} z_i\right) = |M| \sum_{1\le\alpha,\beta\le g} m_{\alpha\beta} z_\alpha z_\beta,$$
$$\sum_{1\le\alpha,\beta\le g} M_{\alpha\beta} m_{\alpha\beta} = g|M|$$

Therefore,

$$\begin{aligned} &L_M^\nu \left((c\tau + d)^{-k} e\left(\frac{-c\, M[z]}{c\tau + d}\right)\right) \\ &\quad = (-4)^\nu \frac{\Gamma(k - g/2 + \nu)}{\Gamma(k - g/2)} |M|^\nu (2\pi i c)^\nu (c\tau + d)^{-k-\nu} e\left(\frac{-c\, M[z]}{c\tau + d}\right). \end{aligned} \tag{13}$$

Since,

$$\begin{aligned} &(c\tau + d)^{-k-\nu} e\left(\frac{-c\, M[z]}{c\tau + d}\right) \big|_{k,M} [a\lambda, b\lambda] \\ &\quad = (c\tau + d)^{-k-\nu}\; e\left(M[\lambda]\frac{a\tau + b}{c\tau + d} + \frac{2\lambda^t M z}{c\tau + d} - \frac{cM[z]}{c\tau + d}\right), \end{aligned} \tag{14}$$

the required result follows. □

3.2. **Representation of $[G, E]_\nu$ in terms of the Poincaré series.** We first obtain a growth estimate for the Fourier coefficients of a Jacobi form.

Lemma 3.5. *Let $k > g+2$ and $F \in J_{k,M}$ with Fourier coefficients $c(n, r)$. Put $D_1 = \sum_{i,j} M_{ij} r_i r_j - 4n|M|$. Then*

$$c(n, r) \ll |D_1|^{k-g/2-1}, \qquad \text{if } D_1 < 0. \tag{15}$$

Moreover, if F is a cusp form, then

$$c(n,r) \ll |D_1|^{k/2-g/2}. \tag{16}$$

In the above the $\ll$ constants depend only on k, g and $|M|$.

Proof. If F is a cusp form, then the required estimate was proved by Böcherer and Kohnen [1]. If F is not a cusp form, then it can be written as a linear combination of the Eisenstein series $E_{k,M}$ and a cusp form. We now show that $e_{k,M}(n,r)$, the (n,r)-th Fourier coefficient of $E_{k,M}$, satisfies the estimate (15), from which the lemma follows.

Taking the same set of coset representatives as in the proof of the above lemma, we get

$$E_{k,M} = \frac{1}{2}\sum_{c,d\in\mathbb{Z},(c,d)=1}\sum_{\lambda\in\mathbb{Z}^g} 1|_{k,M}\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}, (a\lambda, b\lambda)\right]$$

$$= \frac{1}{2}\sum_{c,d\in\mathbb{Z},(c,d)=1}\sum_{\lambda\in\mathbb{Z}^g}(c\tau+d)^{-k} e\left(\frac{-c}{c\tau+d}M[z+a\lambda\tau+b\lambda] + M[a\lambda]\tau + 2a\lambda^t Mz\right)$$

$$= \frac{1}{2}\sum_{c,d\in\mathbb{Z},(c,d)=1}\sum_{\lambda\in\mathbb{Z}^g}(c\tau+d)^{-k} e\left(M[\lambda]\frac{a\tau+b}{c\tau+d} + \frac{2\lambda^t Mz}{c\tau+d} - c\frac{M[z]}{c\tau+d}\right)$$

Proceeding in the usual way by splitting the sum into two parts as $c=0$ and $c\neq 0$, we see that

$$E_{k,M}(\tau,z) = \sum_{\lambda\in\mathbb{Z}^g} e(M[\lambda]\tau + 2\lambda^t Mz) + \sum_{c=1}^{\infty} c^{-k}\sum_{d(c),(d,c)=1}\sum_{\lambda(c)} e(\frac{a}{c}M[\lambda])F_{k,M}(\tau+d/c, z-\lambda/c),$$

where $F_{k,M}(\tau,z) = \sum_{p\in\mathbb{Z},q\in\mathbb{Z}^g}(\tau+p)^{-k}e\left(\frac{-M[z+q]}{\tau+p}\right)$. Using the Poisson summation formula the (n,r)-th Fourier coefficient of $F_{k,M}(\tau,z)$ is given by

$$\gamma(n,r) = \begin{cases} 0 & \text{if } \tilde{M}[r^t] \geq 4n|M|, \\ \alpha_{k,g}|M|^{-1/2}\left(\frac{2\pi i}{4|M|}(4n|M| - \tilde{M}[r^t])\right)^{k-g/2-1} & \text{if } \tilde{M}[r^t] < 4n|M|. \end{cases}$$

where $\alpha_{k,g} = \left(\frac{1}{2i}\right)^{g/2}\dfrac{\pi\csc(\pi(k-g/2))}{\Gamma(k-g/2)}$. Plugging in this Fourier coefficient and estimating the Gauss sum we get

$$e_{k,M}(n,r) \ll D_1^{k-g/2-1},$$

where the $\ll$ constant depends only on k, g and $|M|$. □

We need the following lemma which gives the absolute convergence of a series which is required to get an expression of the Rankin-Cohen bracket of F with the Eisenstein series in terms of the Poincaré series.

Lemma 3.6. *The series*

$$v^k e^{-2\pi M[y]/v} \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^g, 4n\geq M_1^{-1}[r^t],\\ \gamma\in\Gamma^J_{1,\infty}/\Gamma^J_1}} (4n|M_1| - \tilde{M_1}[r^t]) e(n\tau + rz)|_{k,M}\gamma$$

$(\tau = u+iv, z_j = x_j + iy_j, y = (y_1, y_2, \cdots, y_g)^t)$ *is absolutely uniformly convergent on the subsets* $V_{\epsilon,C} = \{(\tau, z) \in \mathcal{H}\times\mathbb{C}^g | v \geq \epsilon, |y_j v^{-1}| \leq C, |x_j| \leq 1/\epsilon, u \leq 1/\epsilon, \forall j = 1, 2, \cdots, g\}$ *for given* $\epsilon > 0, C > 0$

Proof. Using Lemma 3.5, it is sufficient to prove the uniform convergence of the series

$$v^k e^{-2\pi M[y]/v} \sum_{n\in\mathbb{Z}, r\in\mathbb{Z}^g, 4n\geq M_1^{-1}[r^t], \gamma\in\Gamma^J_{1,\infty}/\Gamma^J_1} (4n|M_1| - \tilde{M_1}[r^t])^{\nu+k_2-g/2-1} |e(n\tau + rz)|_{k,M}\gamma(\tau, z)|$$

in the given ranges.

Let $\tau' \in \mathcal{H}_g$ such that $Z := \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_{g+1}$. Also let $T = \begin{pmatrix} n & r^t/2 \\ r/2 & M \end{pmatrix}$. Note that by the assumption that $4n \geq M_1^{-1}[r^t]$, we see that T is positive semi-definite. Now we embed $\Gamma^J_{1,g} = SL_2(\mathbb{Z}) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$ into Γ_{g+1} (denoted by $\gamma \mapsto \gamma^*$) defined by combining the following two embeddings:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \mapsto \left(\begin{pmatrix} a & 0 & b & 0 \\ 0 & I_{g-1} & 0 & 0_{g-1} \\ c & 0 & d & 0 \\ 0 & 0_{g-1} & 0 & I_{g-1} \end{pmatrix}, (\lambda, \mu)\right)$$

and

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu), \right) \mapsto \begin{pmatrix} A & 0 & B & \mu' \\ \lambda & 1 & \mu & 0 \\ C & 0 & D & -\lambda' \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(\lambda'^t, \mu'^t) = (\lambda, \mu) \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}$. We have

$$(e^{n,r}|_{k,M}\gamma)(\tau, z) = e^{-2\pi i m\tau'} (e^T|_k \gamma^*)(Z)$$

where $e^T(Z) := e^{2\pi i tr(TZ)}$ and $|_k$ on functions $F : H_{g+1} \mapsto \mathbb{C}$

We can view the sum of absolute terms of the (n, r)-th Poincaré series as a sub-series of the sum of absolute terms of the T-th Poincaré series on Γ_{g+1}

Let $(\tau, z) \in V_{\epsilon,C}$ for some $\epsilon > 0$ and $C > 0$. Then by taking $\tau' = i\ min_{1\leq j\leq g}\{(\frac{y_j^2}{v} + \delta)\}$ with $\delta > 0$, we see that $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in H_{g+1}$ with $Y = Im\ Z > \epsilon'/2$ for some ϵ' depending on ϵ, C and δ. Since the sum of the absolute terms of the T-th Poincaré series on the subsets

$Y \geq \epsilon' I_g, tr(X'X) \leq \frac{1}{\epsilon'}$ (up to some constants) is majorized by that sum evaluated at an arbitrary single point Z_0, say $Z_0 = iI_g$, (cf. [9]) so it is sufficient to prove the convergence of above series at $(\tau, z) = (i, 0, \cdots, 0)$ i.e the convergence of the series (using the the coset representation)

$$\sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^g, 4n>M_1^{-1}[r^t],\\ (c,d)=1, \lambda\in\mathbb{Z}^g}} (4n|M_1| - \tilde{M_1}[r^t])^{\nu+k_2-g/2-1} |(ci+d)^{-k}|$$
$$|e^{2\pi i(\frac{-c}{ci+d}M[a\lambda i+b\lambda]+M[a\lambda])}||e^{2\pi i(n\frac{ai+b}{ci+d}+r.\frac{a\lambda i+b\lambda}{ci+d})}|$$

Now proceeding as in [5] we get the required convergence with the assumption that $k_2 > k_1 + g + 2$. □

Proposition 3.7. *Let k, k_1, k_2, M, M_1, M_2 be as in Theorem 3.3. Let $G \in J_{k_1,M_1}$ with Fourier expansion*

$$G(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^g,\\ 4n\geq M_1^{-1}[r^t]}} b(n,r) e(n\tau + rz).$$

Then

$$[G, E_{k_2,M_2}]_\nu = c_{k_1,k_2,M_1,M_2,g;\nu} \sum_{n\in\mathbb{Z}, r\in\mathbb{Z}^g, 4n\geq M_1^{-1}[r^t]} (4n|M_1| - \tilde{M_1}[r^t])^\nu \; b(n,r) \; P_{k,M;(n,r)}, \tag{17}$$

where $c_{k_1,k_2,M_1,M_2,g;\nu} = (2\pi)^{-2\nu}|M_2|^{-\nu}\dfrac{v!(k_2-g/2-1)!}{(k_2-g/2+\nu-1)!}$

Proof. Using the definition of the Poincaré series, the action of the heat operator on Fourier coefficients, and by the absolute convergence (obtained in Lemma 3.6) we see that the series on the right-hand side of (17) can be written as

$$\sum_{\gamma\in\Gamma^J_{1,g,\infty}\backslash\Gamma^J_{1,g}} \left(1\big|_{k,M}\gamma\right)(\tau,z) \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^g,\\ 4n\geq M_1^{-1}[r^t]}} (4n|M_1| - \tilde{M_1}[r^t])^\nu \; b(n,r) \; e^{n,r}(\gamma\circ(\tau,z))$$
$$= \sum_{\gamma\in\Gamma^J_{1,g,\infty}\backslash\Gamma^J_{1,g}} \left(1\big|_{k,M}\gamma\right)(\tau,z)\cdot(2\pi i)^{-2\nu}(L^\nu_{M_1}G)(\gamma\circ(\tau,z))$$
$$= (2\pi i)^{-2\nu} \sum_{\gamma\in\Gamma^J_{1,g,\infty}\backslash\Gamma^J_{1,g}} \left(1\big|_{k_2,M_2}\gamma\right)(\tau,z)(L^\nu_{M_1}G)\big|_{k_1+2\nu,M_1}\gamma(\tau,z).$$

We taking the same set of representatives as in the proof of Lemma 3.4 for the sum over γ and using the fact that $G \in J_{k,M_1}$, we get

$$\begin{aligned}
&(2\pi i)^{-2\nu} \sum_{\gamma \in \Gamma^J_{1,g,\infty}\backslash \Gamma^J_{1,g}} \left(1\big|_{k_2,M_2}\gamma\right)(\tau,z)(L^\nu_{M_1}G)\big|_{k_1+2\nu,M_1}\gamma(\tau,z) \\
&= (2\pi i)^{-2\nu} \sum_{l=0}^{\nu} 4^{\nu-l}\frac{(k_1-g/2+\nu-1)!}{(k_1-g/2+l-1)!}\binom{\nu}{l}|M_1|^{\nu-l}L^l_{M_1}(G)(\tau,z) \\
&\times \sum_{\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \lambda\in\mathbb{Z}^g} \left(\frac{\pi i c}{c\tau+d}\right)^{\nu-l}(c\tau+d)^{-k_2} e\left(\frac{-c\, M_2[z+a\lambda\tau+b\lambda]}{c\tau+d}+M_2[a\lambda]+2(a\lambda)^t M_2 z\right)
\end{aligned}$$

Using Lemma 3.4, the inner sum in the above expression is equal to

$$(-4)^{-(\nu-l)}\frac{(k_2-g/2-1)!}{(k_2-g/2+\nu-l-1)!}|M_2|^l L^{\nu-l}_{M_2}E_{k_2,M_2}(\tau,z),$$

therefore, we finally find that the sum on the right-hand side of (17) equals

$$\begin{aligned}
(2\pi)^{-2\nu}|M_2|^{-\nu}\sum_{l=0}^{\nu}(-1)^l\frac{(k_1-g/2+\nu-1)!(k_2-g/2-1)!}{(k_1-g/2+l-1)!(k_2-g/2+\nu-l-1)!}\binom{\nu}{l}|M_1|^{\nu-l}|M_2|^l \\
\times L^l_{M_1}(G)(\tau,z)L^{\nu-l}_{M_2}E_{k_2,M_2}(\tau,z).
\end{aligned}$$

The proof is now complete. □

3.3. **Proof of Theorem 3.3.** We first observe that by Lemma 3.5 the series on the right hand side of (9) is absolutely convergent and hence is majorized by

$$\sum_{\substack{n\geq 1, r\in\mathbb{Z}^g \\ 4n\geq M_1^{-1}[r^t]}} \frac{\left(4n|M_1|-\tilde{M}_1[r^t]\right)^{k_1-g/2-1+\nu}}{\left(4n|M|-\tilde{M}[r^t]\right)^{k/2-1}} \ll \sum_{n\geq 1}\frac{n^{g/2}.n^{k_1-g/2-1+\nu}}{n^{k/2-1}} = \sum_{n\geq 1}\frac{1}{n^{\frac{k_2-k_1}{2}}} < \infty$$

after putting $k = k_1 + k_2 + 2\nu$ and by our assumption that $k_2 > k_1 + g + 2$.

The standard fundamental domain for the action of $\Gamma^J_{1,g}$ on $\mathcal{H}\times\mathbb{C}^{(g,1)}$ is contained in one of the sets $V_{\epsilon,C}$ occurring in the statement of Lemma 3.6, for appropriate ϵ and C. Therefore, using Lemma 3.6 we deduce from Lemma 3.7 that

$$\langle F, [G, E_{k_2,M_2}]_\nu\rangle = c_{k_1,k_2,M_1,M_2,g;\nu}\sum_{n\in\mathbb{Z}, r\in\mathbb{Z}^g, 4n\geq M_1^{-1}[r^t]}(4n|M_1|-\tilde{M}_1[r^t])^\nu\overline{b(n,r)}\langle F, P_{k,M;(n,r)}\rangle.$$

where $c_{k_1,k_2,M_1,M_2,g;\nu}$ is defined as in (17).

Note that $4n > M_1^{-1}[r^t]$ implies $4n > M^{-1}[r^t]$ and hence the Poincaré series $P_{k,M;(n,r)}$ are all cusp forms. On the other hand, if $4n = M_1^{-1}[r^t]$, $4n \geq M^{-1}[r^t]$ implies $r = 0$ and $n = 0$, in which case one has the Eisenstein series $E_{k,M}$. Since F is cusp form, $\langle F, E_{k,M}\rangle$ is zero. From (1), we obtain

$$\langle F, [G, E_{k_2,M_2}]_\nu \rangle = c_{k,k_2,M,M_2,g;\nu} \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^g, \\ 4n \geq M_1^{-1}[r^t]}} \frac{(4n|M_1| - \tilde{M}_1[r^t])^\nu \; a(n,r)\overline{b(n,r)}}{(4n|M| - \tilde{M}[r^t])^{k-g/2-1}},$$

where $c_{k,k_2,M,M_2,g;\nu}$ is defined as in (10). □

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